
SKEWNESS INTO THE PRODUCT OF TWO NORMALLY DISTRIBUTED VARIABLES AND THE RISK CONSEQUENCES

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Abstract:

- The analysis of skewness is an essential tool for decision-making since it can be used as an indicator on risk assessment. It is well known that negative skewed distributions lead to negative outcomes, while a positive skewness usually leads to good scenarios and consequently minimizes risks. In this work the impact of skewness on risk analysis will be explored, considering data obtained from the product of two normally distributed variables. In fact, modelling this product using a normal distribution is not a correct approach once skewness in many cases is different from zero. By ignoring this, the researcher will obtain a model understating the risk of highly skewed variables and moreover, for too skewed variables most of common tests in parametric inference cannot be used. In practice, the behaviour of the skewness considering the product of two normal variables is explored as a function of the distributions parameters: mean, variance and inverse of the coefficient variation. Using a measurement error model, the consequences of skewness presence on risk analysis are evaluated by considering several simulations and visualization tools using R software ([10]).

Key-Words:

- *product of normal variables; inverse coefficient of variation; skewness; probability risk analysis; measurement error model.*

AMS Subject Classification:

- 62E17, 62E10.

1. INTRODUCTION

Consequences for the presence of skewness are very important, especially in risk analysis. When distribution of the expected value is skewed produces distortions on the decisions of the risk-neutral decision maker. In Mumpower and McClelland ([9]) authors analyse the consequences on a model of random measurement error. Another example, in valuation risk of assets, the risk-averse investors prefer positive skewness Krans and Litzenberg ([8], Harvey and Siddique [6]) and the effect of skewness on the R^2 of the model has influenced over the predictability of the model of assets Cochran ([4]).

Our objective in this paper is to study the relation between skewness of the distribution of a product of two normal variables and the parameters of these normal distributions. Our work has two focus: from a theoretical point of view using the moment-generating function, and through several simulations, using Monte-Carlo methods we estimate the skewness of the product of two variables.

Distribution of the product of normal variables is an open problem in statistics. First work has been undertaken by Craig ([3]), in his early paper, who was actually the first to determinate the algebraic form of the moment-generating function of the product. In Aroian and Taneja ([2]) proved the approximation of the product using the standardized Pearson type III distribution. But nowadays, the problem is not closed; although the product of two normal variables is not, in general, normally distributed; however, under some conditions, it is showed that the distribution of the product can be approximated by means of a Normal distribution Aroian and Taneja ([2]). The presence of the product of normal variables is well-known in Risk analysis Hayya and Ferrara ([7]), where functional relationships concerning two normally distributed variables (correlated or non-correlated) are encountered.

There are several test to estimate the normality of a sample, but for large size sample results are not always correct Deb and Sefton ([5]). The most accurate test for large size is skewness test. In this paper, we use the moment-generating function for analysing the value of skewness for a product of two normally distributed variables. We considered the influence of three parameters from the two distributions: mean, variance and correlation. Using the formula for skewness, we can calculate the value of the skewness for the product as a function of two set of parameters: First, where the mean, the variance and correlation between the two distributions are used for calculations. The second one is formed the inverse of the coefficient of variation for each distribution and the correlation.

At section 2, the moment-generating function for a product of two normally distributed variables is introduced. The formulas for three parameters of the product: mean, variance (standard deviation) and skewness are studied and

the evolution of skewness for the product of two normal variables is analysed in Section 3. Several cases are considered: taking into account first, the presence of correlation between both variables is assumed; second, the two normally distributed variables are uncorrelated. The influence of the parameters, mean and standard deviation of the two variables is analysed. The graphical visualization of the results is incorporated. In Section 4, an analysis of the effect of skewness for a model of random measurement error is introduced. Finally, Section 5 contains conclusions of the paper.

2. MOMENTS OF THE PRODUCT OF TWO NORMAL VARIABLES

Let X and Y be two normal probability functions, with means μ_x and μ_y and standard deviations σ_x and σ_y , respectively, r the coefficient of correlation and the inverses of the coefficient of variation, for the two variables, are: $\rho_x = \frac{\mu_x}{\sigma_x}$ and $\rho_y = \frac{\mu_y}{\sigma_y}$.

Craig ([3]) determined the moments, seminvariants, and the moment generating function of $z = \frac{xy}{\sigma_x \sigma_y}$. The moment generating function of z , $M_z(t)$ is:

$$(2.1) \quad M_z(t) = \frac{\exp \frac{(\rho_x^2 + \rho_y^2 - 2r\rho_x\rho_y)t^2 + 2\rho_x\rho_y t}{2(1-(1+r)t)(1+(1-r)t)}}{\sqrt{(1-(1+r)t)(1+(1-r)t)}}$$

where t is the order of the moment.

Let μ_z and σ_z be the mean and the standard deviation of z . Values of mean and standard deviation and skewness of z are calculated as (see ([3]) and ([1]):

$$(2.2) \quad \mu_z = \rho_x \rho_y + r ,$$

$$(2.3) \quad \sigma_z = \sqrt{\rho_x^2 + \rho_y^2 + 2r\rho_x\rho_y + 1 + r^2} ,$$

$$(2.4) \quad \alpha_3 = \frac{2 \left(3\rho_x\rho_y + r^3 + 3\rho_x\rho_y r^2 + 3r(\rho_x^2 + \rho_y^2 + 1) \right)}{(\rho_x^2 + \rho_y^2 + r^2 + 2\rho_x\rho_y r + 1)^{3/2}} .$$

An alternative approach, without using the inverse of the coefficient of variation, can be obtained.

Proposition 2.1. *Let $x \sim N(\mu_x, \sigma_x^2)$ and $y \sim N(\mu_y, \sigma_y^2)$ be two normal variables with correlation r . Defining $x = x_0 + z_1$ and $y = x_0 + z_2$, where*

$$(2.5) \quad \begin{bmatrix} x_0 \\ z_1 \\ z_2 \end{bmatrix} \sim N \left(\begin{bmatrix} 0 \\ \mu_x \\ \mu_y \end{bmatrix} \begin{bmatrix} r\sigma_x\sigma_y & 0 & 0 \\ 0 & \sigma_x^2 - r\sigma_x\sigma_y & 0 \\ 0 & 0 & \sigma_y^2 - r\sigma_x\sigma_y \end{bmatrix} \right) ,$$

the two variables x and y are decomposed into independent summands, one of which is shared between them. Then, the moment-generating function of $z = xy = (x_0 + z_1)(x_0 + z_2)$ is

$$(2.6) \quad M_z[t] = \frac{\exp \frac{t(t\mu_y^2\sigma_x^2 + t\mu_x^2\sigma_y^2 - 2\mu_x\mu_y(-1 + tr\sigma_x\sigma_y))}{2 + 2t\sigma_x\sigma_y(-2r + t(-1 + r^2)\sigma_x\sigma_y)}}{\sqrt{1 + t\sigma_x\sigma_y(-2r + t(-1 + r^2)\sigma_x\sigma_y)}}.$$

Proof: The moment-generating function of $z = xy$ is the same that $z = (x_0 + z_1)(x_0 + z_2)$, that is the product of two independent variables, then we know that

$$(2.7) \quad M_z[t] = \int_{-\infty}^{\infty} e^{tz} f(z) dz.$$

The joint probability density function (pdf) $f(z)$ could be written as the product of the independent three marginal pdf of the variables,

$$(2.8) \quad f(z) = f_{x_0}(x_0) f_{z_1}(z_1) f_{z_2}(z_2) = \frac{\exp\left(-\frac{x_0^2}{2r\sigma_x\sigma_y} - \frac{(z_1 - \mu_x)^2}{2(\sigma_x^2 - r\sigma_x\sigma_y)} - \frac{(z_2 - \mu_x)^2}{2(\sigma_y^2 - r\sigma_x\sigma_y)}\right)}{2\sqrt{2}\pi^{3/2}\sqrt{r\sigma_x\sigma_y}\sqrt{\sigma_x^2 - r\sigma_x\sigma_y}\sqrt{\sigma_y^2 - r\sigma_x\sigma_y}}.$$

Then,

$$(2.9) \quad \begin{aligned} M_z[t] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{tz} f_{x_0}(x_0) f_{z_1}(z_1) f_{z_2}(z_2) dx_0 dz_1 dz_2 \\ &= \frac{1}{2\sqrt{2}\pi^{3/2}\sqrt{r\sigma_x\sigma_y}\sqrt{\sigma_x^2 - r\sigma_x\sigma_y}\sqrt{\sigma_y^2 - r\sigma_x\sigma_y}} \\ &\quad \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t(x_0+z_1)(x_0+z_2) - \frac{x_0^2}{2r\sigma_x\sigma_y} - \frac{(z_1 - \mu_x)^2}{2(\sigma_x^2 - r\sigma_x\sigma_y)} - \frac{(z_2 - \mu_x)^2}{2(\sigma_y^2 - r\sigma_x\sigma_y)}} dx_0 dz_1 dz_2 \end{aligned}$$

where we have the following assumptions: $t \in \mathbb{Z}$ non negative, $\sigma_x > 0$, $\sigma_y > 0$, $\sigma_x, \sigma_y, \mu_x, \mu_y$ all of them real numbers and $-1 \leq r \leq 1$. Solving the integral (2.9) results for these assumptions,

$$(2.10) \quad \frac{\sqrt{\sigma_y(-r\sigma_x - \sigma_y)}\sqrt{-r\sigma_y(r\sigma_y - \sigma_x)} \exp\left(\frac{t(\mu_x^2\sigma_y^2 t - 2\mu_x\mu_y(r\sigma_x\sigma_y t - 1) + m\mu_y^2\sigma_x^2 t)}{2((r-1)\sigma_x\sigma_y t - 1)((r+1)\sigma_x\sigma_y t - 1)}\right)}{\sqrt{\sigma_y(\sigma_y - r\sigma_x)}\sqrt{r\sigma_y(\sigma_x - r\sigma_y)}\sqrt{(r^2 - 1)\sigma_x^2\sigma_y^2 t^2 - 2r\sigma_x\sigma_y t + 1}}.$$

Then, simplifying this expression in (2.10) results (2.6), as the expression of the moment-generating function of the product $z = xy$. \square

Derivatives of order i of (2.6) provides moments of order i , for $i = 1, 2, \dots$. The moments of the distribution of the product of two normal variables are calculated:

1. Mean: First derivative of the moment generating (2.6) function respect t for $t = 0$:

$$(2.11) \quad E[z] = \mu_z = \mu_x \mu_y + r \sigma_x \sigma_y ;$$

2. Variance: Difference between second moment and first moment up to 2

$$(2.12) \quad \text{Var}[z] = \sigma_z^2 = \mu_y^2 \sigma_x^2 + 2 \mu_x \mu_y r \sigma_x \sigma_y + (\mu_x^2 + (1 + r^2) \sigma_x^2) \sigma_y^2 ;$$

3. Skewness: Quotient between the third central moment and the second central moment up to the $3/2$.

$$(2.13) \quad \alpha_3[z] = \frac{\left(6 \mu_y^2 r \sigma_x^3 \sigma_y + 6 \mu_x \mu_y (1 + r^2) \sigma_x^2 \sigma_y^2 + 2r (3 \mu_x^2 + (3 + r^2) \sigma_x^2) \sigma_x^3 \sigma_y\right) \sigma_y^2}{\left(\mu_y^2 \sigma_x^2 + 2 \mu_x \mu_y r \sigma_x \sigma_y + (\mu_x^2 + (1 + r^2) \sigma_x^2) \sigma_y^2\right)^{3/2}}.$$

Attention will be paid to the evolution of skewness. Skewness for a normal distributed variable would be zero. For a product of two normal variables, skewness zero would be a proof of normality, other values should be carefully analysed.

3. EVOLUTION OF SKEWNESS OF THE PRODUCT OF TWO NORMAL VARIABLES

In the previous section the formula for skewness of the product of two normally distributed variables was introduced (see equation 2.13). Obviously, there are five factors that have a certain influence into the value of skewness: mean and standard deviation of each of the variables into product and the correlation between them.

3.1. Product of two correlated normally distributed variables

According Proposition 1, we can formulated several specific cases for the product of two normally distributed variables, and study evolution of the skewness.

Corollary 3.1. *For the product of two correlated normally distributed variables, three cases are considered:*

- a) *Product of two standard normal distributions $N(0, 1)$ with $r = 1$. This a very special case, the product of this two variables follows a Chi-Square with one degree of freedom. In this case, skewness of product is $2\sqrt{2}$ than is bigger than zero and equals the theoretical value of skewness for the Chi-Square distribution with 1 degree of freedom ($\sqrt{8}$).*

- b) Two normal variables with same mean $\mu_x = \mu_y = \mu$ and same unit standard deviation $\sigma_x = \sigma_y = 1$. In this case we considered different values for correlation r :

$$(3.1) \quad \alpha_3[z] = \frac{2(3\mu^2 r + 3\mu^2(1+r^2) + r(3 + 3\mu^2 + r^2))}{(1 + 2\mu^2 + 2\mu^2 r + r^2)^{3/2}}.$$

In Figure 1, evolution of the skewness is represented for two factors: mean and correlation. When the mean of two variables is zero, skewness is a increasing function of the correlation. When $\mu = 0$ and $r = 1$ we have the Chi-Square case. When r tends to zero, then $\alpha_3[z]$ tends 0 when ratio (inverse of the coefficient of variation) $\frac{\mu}{\sigma} = 0$, but as the ratio increases the skewness rises rapidly, until it is at its maximum when the inverse of the coefficient of variation is one, then $\mu = \sigma = 1$.

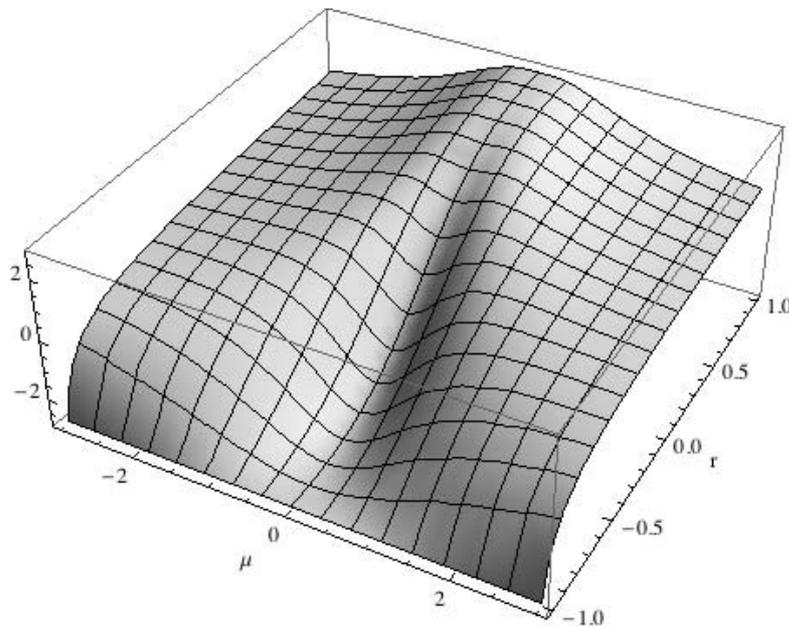


Figure 1: Skewness for product two normal variables same mean and standard deviation equals to 1.

- c) Two normal variables with the same mean $\mu_x = \mu_y = \mu$ and the same standard deviation $\sigma_x = \sigma_y = \sigma$. Considering two distribution with the same parameters (mean and standard deviation). In general, skewness tends to zero as ratio tends to infinity. The closer $|r|$ is to one the slower the approach of skewness to zero. When $r = 1$ then we have skewness of a Chi-Square Distribution ($2\sqrt{2}$) (see Figure 2).

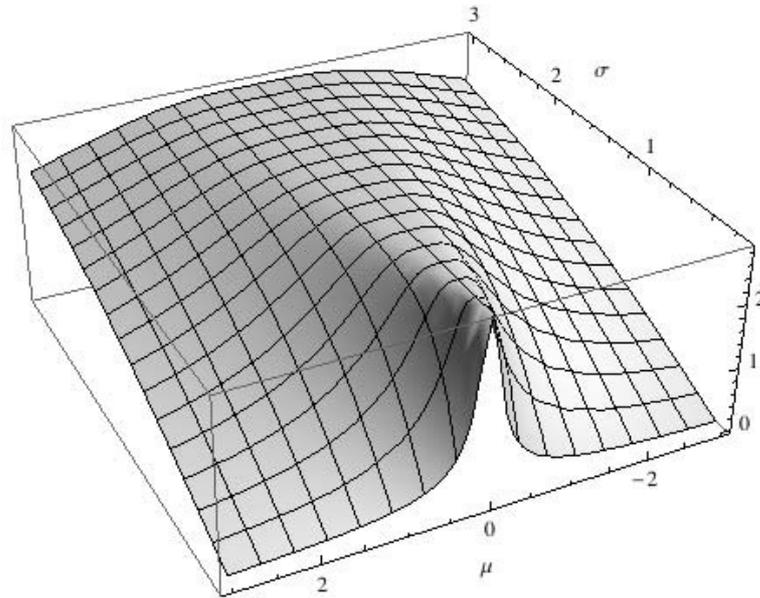


Figure 2: Skewness for product two normal variables $r = 1$.

When considering different mean or different standard deviation, the evolution of skewness presents more variability as a function of the values of the parameters, but in a rough inspection of the graphics we are identify several aspects:

1. If we consider the same standard deviation and different values for the mean, skewness zero is very common when σ tends to zero and the means are different.
2. When standard deviation increases the skewness increases for the same values of correlation and mean.
3. If we consider the same mean, skewness is very common, and only when the standard deviation are lower, skewness zero exists.
4. In general, the presence of correlation has effect on the presence of skewness, skewness zero or lower is more common when r tends to zero.

Now, we are going to study the skewness of uncorrelated distributions ($r = 0$).

3.2. Product of two uncorrelated normal distributions

Two uncorrelated normal variables ($r = 0$) are now considered. When the two variables are uncorrelated $r = 0$ and the value of skewness is a function of

only 4 parameters (means and variances), then:

$$(3.2) \quad \alpha_3[z] = \frac{6 \mu_x \mu_y \sigma_x^2 \sigma_y^2}{(\mu_y^2 \sigma_x^2 + (\mu_x^2 + \sigma_x^2) \sigma_y^2)^{3/2}}.$$

3.2.1. Influence of inverse of the coefficient of variation

For the moment-generating function of the product of these two variables equation (2.6) is used, considering the inverse of the coefficient of variation $\rho_x = \frac{\mu_x}{\sigma_x}$ and $\rho_y = \frac{\mu_y}{\sigma_y}$. Now, skewness of the product of variables $z = xy$ will be:

$$(3.3) \quad \alpha_3[z] = \frac{6 \rho_x \rho_y}{(\rho_y^2 + \rho_x^2 + 1)^{3/2}}.$$

Skewness depends on ratios ρ_x and ρ_y . In Figure 3, the skewness as a function of this ratios is illustrated.

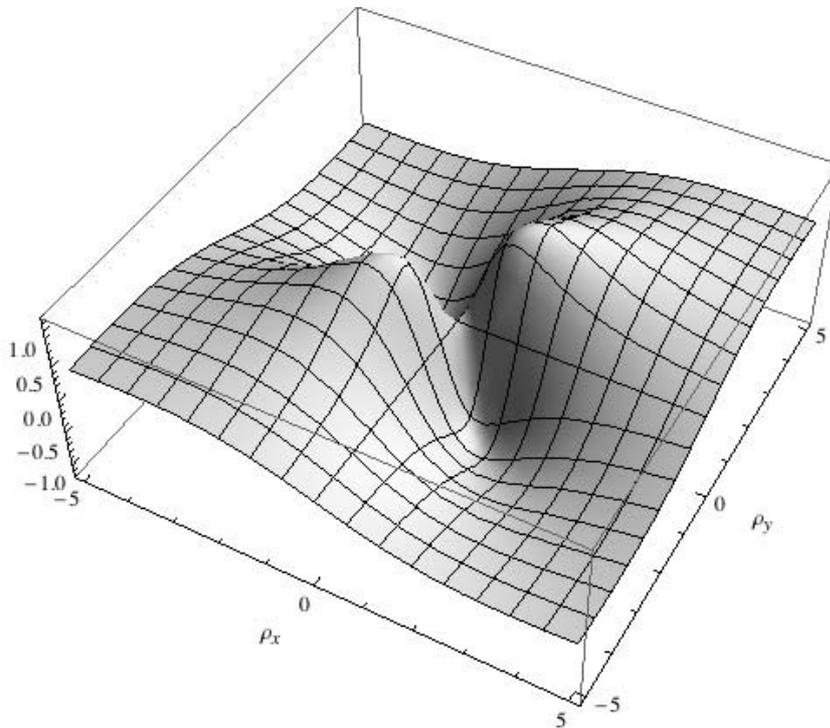


Figure 3: Skewness as a function of ratios ρ_x and ρ_y .

When $\rho_x \rightarrow 0$ and $\rho_y \rightarrow 0$ then $\alpha_3[z] \rightarrow 0$, that is true too, when one of them tends to zero and the other doesn't tend to infinity. The approach to the normal distribution for the product of two variables will be influenced by the existence

of large ρ_x, ρ_y values. Obviously, as the skewness decreases the approximation to normal distribution improves. Differentiating $\alpha_3[z]$ with respect to two variables, ρ_x and ρ_y , equals zero and solving we find the extreme points of skewness:

$$(3.4) \quad \left(\frac{\partial \alpha_3[z]}{\partial \rho_x}, \frac{\partial \alpha_3[z]}{\partial \rho_y} \right) = \left(\frac{6\rho_y(1 - 2\rho_x^2 + \rho_y^2)}{(1 + \rho_x^2 + \rho_y^2)^{5/2}}, \frac{6\rho_x(1 - 2\rho_y^2 + \rho_x^2)}{(1 + \rho_x^2 + \rho_y^2)^{5/2}} \right).$$

There are five points: $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$. Looking for the extreme values we are considering the Hessian matrix $H(\rho_x, \rho_y)$ say equals to:

$$(3.5) \quad \begin{pmatrix} \frac{18\rho_x\rho_y(2\rho_x^2 - 3(1 + \rho_y^2))}{(1 + \rho_x^2 + \rho_y^2)^{7/2}} & -\frac{6(1 - 2\rho_x^4 + \rho_y^2 + 2\rho_y^4 + \rho_x^2(1 - 11\rho_y^2))}{(1 + \rho_x^2 + \rho_y^2)^{7/2}} \\ -\frac{6(1 - 2\rho_x^4 + \rho_y^2 + 2\rho_y^4 + \rho_x^2(1 - 11\rho_y^2))}{(1 + \rho_x^2 + \rho_y^2)^{7/2}} & -\frac{18\rho_x\rho_y(3 + \rho_x\rho_y(3 + 3\rho_x^2 - 2\rho_y^2))}{(1 + \rho_x^2 + \rho_y^2)^{7/2}} \end{pmatrix}.$$

Then, we have: in $H(0, 0)$ there is a saddle point; in $H(1, 1)$ and $H(-1, -1)$ there are maxima with value $\frac{2}{\sqrt{3}}$ and in $H(-1, 1)$ and $H(1, -1)$ there minima with value $-\frac{2}{\sqrt{3}}$. Thus, the skewness of z is largest when $|\mu_x| = \sigma_x$ and $|\mu_y| = \sigma_y$. If x and y have equal standard deviation (σ), the skewness of the product will be largest when $\mu_x = \mu_y = \sigma$. Figure 4 represents different values of skewness for combination of points for ρ_x and ρ_y from -5 until 5 , these values are represented at "x"-axe. (At the center of graph are values with $\rho_x = \rho_y = 0$).

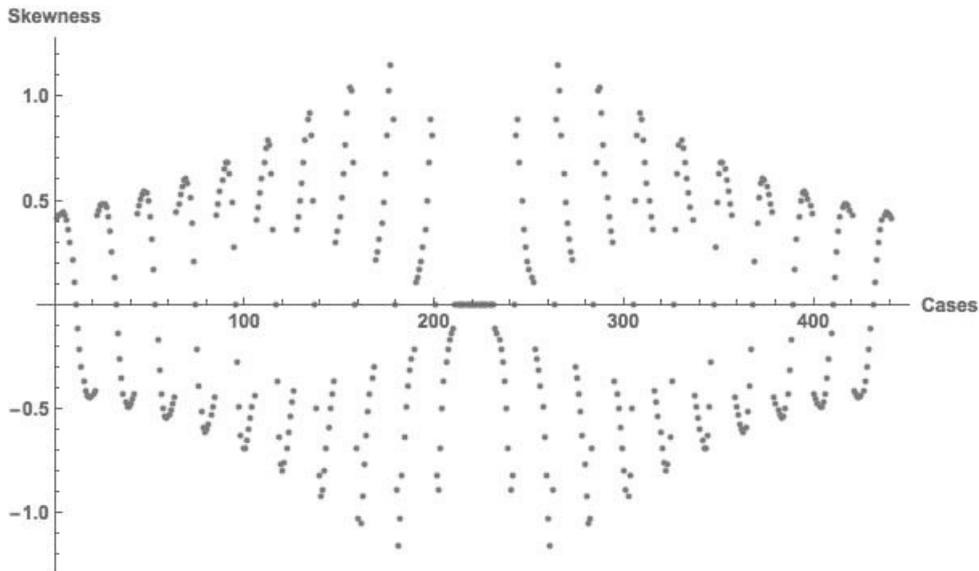


Figure 4: Skewness combination ρ_x and ρ_y in $[-5, 5]$.

Table 1 resumes values of skewness for several products of two normal variables. The theoretical value, according to (3.3) and value for a simulation using Monte-Carlo Simulation for the product of two normal variables with 1.000.000 points is presented:

Table 1: Skewness for product of two normal variables.

Parameters	Theoretical Skewness	Skewness
$\mu_x = 1, \mu_y = 0.5, \sigma = 1$	0.88889	0.883946
$\mu_x = 5, \mu_y = 0.5, \sigma = 1$	0.111531	0.109204
$\mu = 1, \sigma_x = 1, \sigma_y = 0.5$	0.816497	0.811504
$\mu = 1, \sigma_x = 1, \sigma_y = 5$	0.411847	0.404888

The effect of the inverse of the coefficient of variation is correct for the two first rows in Table 1. In the first row we have two distributions with $\rho_x = 1, \rho_y = 0.5$ and skewness is high (0.88); in the second one the ratios for two distributions are higher ($\rho_x = 5, \rho_y = 0.5$) and skewness is lower (0.11). But this tendency is not correct when we consider examples in row 3 and 4 in Table 1. In row 3, we have $\rho_x = 1, \rho_y = 2$, these values are higher than $\rho_x = 1, \rho_y = 0.2$, values in row 4, but the evolution of skewness is inverse. Then, there is a inverse dependency between ρ and skewness but influence of μ is very important and may change the tendency.

A previous analysis of the influence of the value of mean and variance and their effect over skewness was considered. In next section, a more exhaustive study of this influence is explored.

Graph evolution of skewness as a function of mean and standard deviation is represented in Figure 5. In the first graph, we consider $\mu_x = 1, \mu_y, \sigma_y = 1$ and we depict skewness *vs.* σ_x , for three different values of $\mu_y = 0.5, 1, 5$. When σ_x increase, skewness increase until a point where decrease, higher values of μ_y produce faster decreasing for skewness. In this cases, there is a direct relation between skewness and ratio $\frac{\mu}{\sigma}$. At the second one, we consider $\mu_y = 1, \sigma_x = 1, \sigma_y$ and we depict skewness *vs.* μ_x , for three different values of $\sigma_y = 0.2, 1, 2$. The effect is very similar to the latter, but now: when σ is lower the decreasing of skewness is smaller than for higher values of σ , that is, the inverse effect between skewness and ratio inverse of the coefficient of variation $\frac{\mu}{\sigma}$.

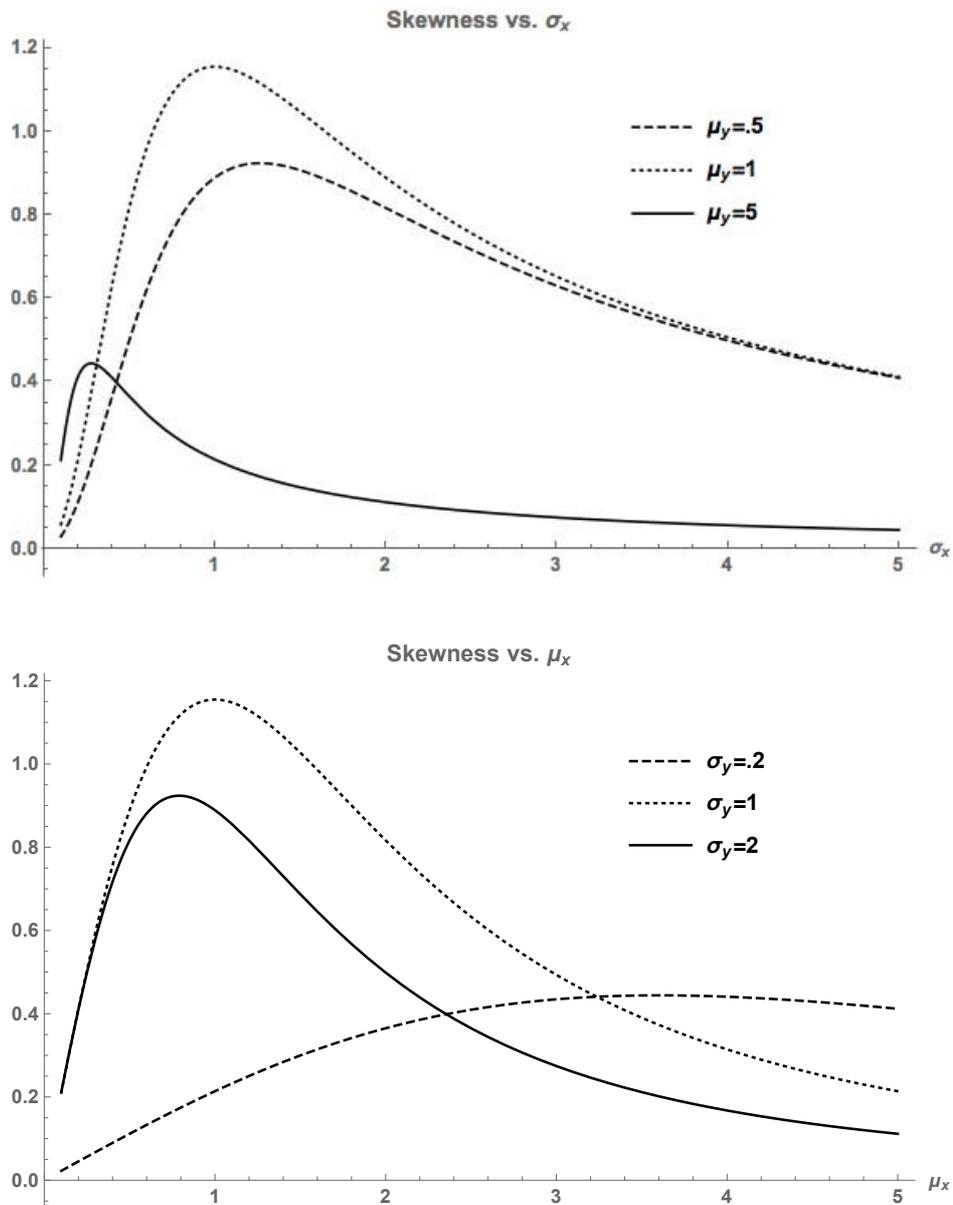


Figure 5: Skewness.

3.2.2. Influence of parameters: mean and variance

Skewness for the product of two normal functions is a function of the values of parameters of the distributions. In the previous section it was shown that the influence of the ratio (inverse of the coefficient of variation) was direct, so that the higher the value of it, there was a lower value of skewness, but this influence

appeared nuanced if we considered the particular values for the parameters of the distribution. There is a strong dependence on the value of the standard deviation, so that the higher the value faster the skewness approaches zero, which contradicts, in part, that a greater ratio lower value of skewness. On the other hand also the average value had influence that the higher the average value also decreases the value of skewness.

From the moment-generating function as in (2.6), skewness (Sk) of a product of two normally distributed variables $x \sim N(\mu_x, \sigma_x)$ and $y \sim N(\mu_y, \sigma_y)$ is a function of the parameters of the two uncorrelated distributions:

$$(3.6) \quad Sk = \frac{6\mu_x \mu_y \sigma_x^2 \sigma_y^2}{(\mu_y^2 \sigma_x^2 + (\sigma_x^2 + \mu_x^2) \sigma_y^2)^{3/2}}.$$

When this value is zero, then skewness is zero, as a normal distribution. In the following cases, skewness of the product of two normal variables is zero, $Sk = 0$:

$$(3.7) \quad \mu_x \mu_y \sigma_x \neq 0 \quad \text{and} \quad \sigma_y = 0;$$

$$(3.8) \quad \mu_x \mu_y \neq 0, \quad \sigma_y \neq 0 \quad \text{and} \quad \sigma_x = 0;$$

$$(3.9) \quad -2\sigma_x \sqrt{\mu_y^2 + \sigma_y^2} = 0, \quad \mu_y^2 \sigma_x + \sigma_x \sigma_y^2 \neq 0 \quad \text{and} \quad \mu_x = 0;$$

$$(3.10) \quad \mu_x \neq 0, \quad \mu_x^2 \sigma_y + \sigma_x^2 \sigma_y \neq 0 \quad \text{and} \quad \mu_y = 0.$$

Equations (3.8) and (3.9) represent Y and X as constants, then both cases are not normal variables. For the other two cases, we can use the normal distribution as an approach of the product. For another situations, the product of two normal variables is skewed and normal distribution doesn't represent a good approach for the product.

The evolution of Sk (3.6) follows the evolution of the equation (3.3). There are three saddle point at $(\sigma_x = 0)$, $(\sigma_y = 0)$ and $(\mu_x = \mu_y = 0)$, two minima at $(\mu_x = \sigma_x, \mu_y = -\sigma_y)$ and $(\mu_x = -\sigma_x, \mu_y = \sigma_y)$, and two maxima at $(\mu_x = \sigma_x, \mu_y = \sigma_y)$ and $(\mu_x = -\sigma_x, \mu_y = -\sigma_y)$. Range for skewness is in $\left[-\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right]$.

Figure 6 represents values of skewness for several combinations of values of parameters, the central part of the graph corresponds to values of μ near zero for both distributions

The same effect appears in Figure 7. The product of two normal variables, where $\mu \in [-2, 2]$ and $\sigma \in [0.1, 5]$ was considered here. Upper figure represents values from μ_x and μ_y , down figure represents values from σ_x and σ_y . Skewness tends to zero when both means (μ_x, μ_y) are lower or when at least one of the variances is small.

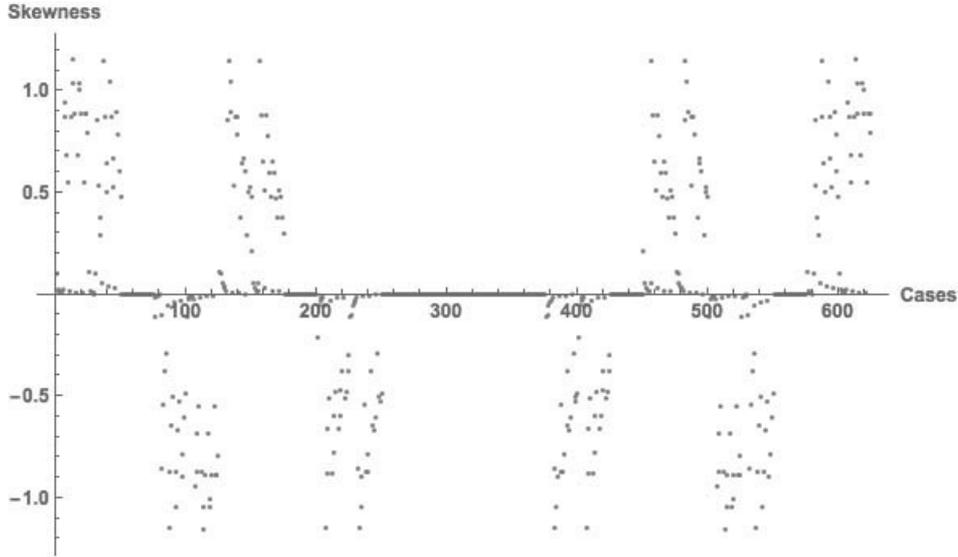


Figure 6: Skewness (Sk) for the product of two normal variables with $\mu \in \{-2, -1, 0, 1, 2\}$ and $\sigma \in \{0.1, 1.1, 2.1, 3.1, 4.1\}$.

In Figure 8, different values of parameters are considered and evolution of skewness is represented: we observe the existence of a tendency, when parameters are high values (>1), the skewness tends to zero. Values for distribution represented in this figures are: $\sigma_x \in \{0.25, 1, 1.75, 2.5\}$, $\mu_x \in [0, 2]$, $\mu_y \in [0, 2]$, $\sigma_y \in \{0.25, 1.25, 2.25\}$. In columns is represented evolution of σ_x and σ_y in rows.

So far, the evolution of the skewness from a theoretical point of view according to the formula obtained through the moment generating function was analysed. In this last part, a number of real examples calculated using the Monte-Carlo method to simulate two uncorrelated normal distributions was considered. A total of 1,000,000 points were generated for each and their respective products were obtained. With the data thus generated, were calculated: the mean, variance and skewness of the product distribution. The results are shown in Figure 9.

The distributions used in the simulation had the following characteristics: $X \sim (\mu_x, \sigma_x)$ with values for $\mu_x = \{0, 0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, 2.25, 2.5\}$ and $\sigma_x = \{0.25, 0.5, 0.75, 1, 1.25, 1.5, 1.75, 2, 2.25\}$ and $Y \sim (\mu_y, \sigma_y)$ for the same values for the parameters that variable X . In the Figure 9, evolution of the graph represent the evolution of the parameters, at first cases we have small values for the four parameters and then, they grow following the sequence: σ_y , μ_y , σ_x and μ_x .

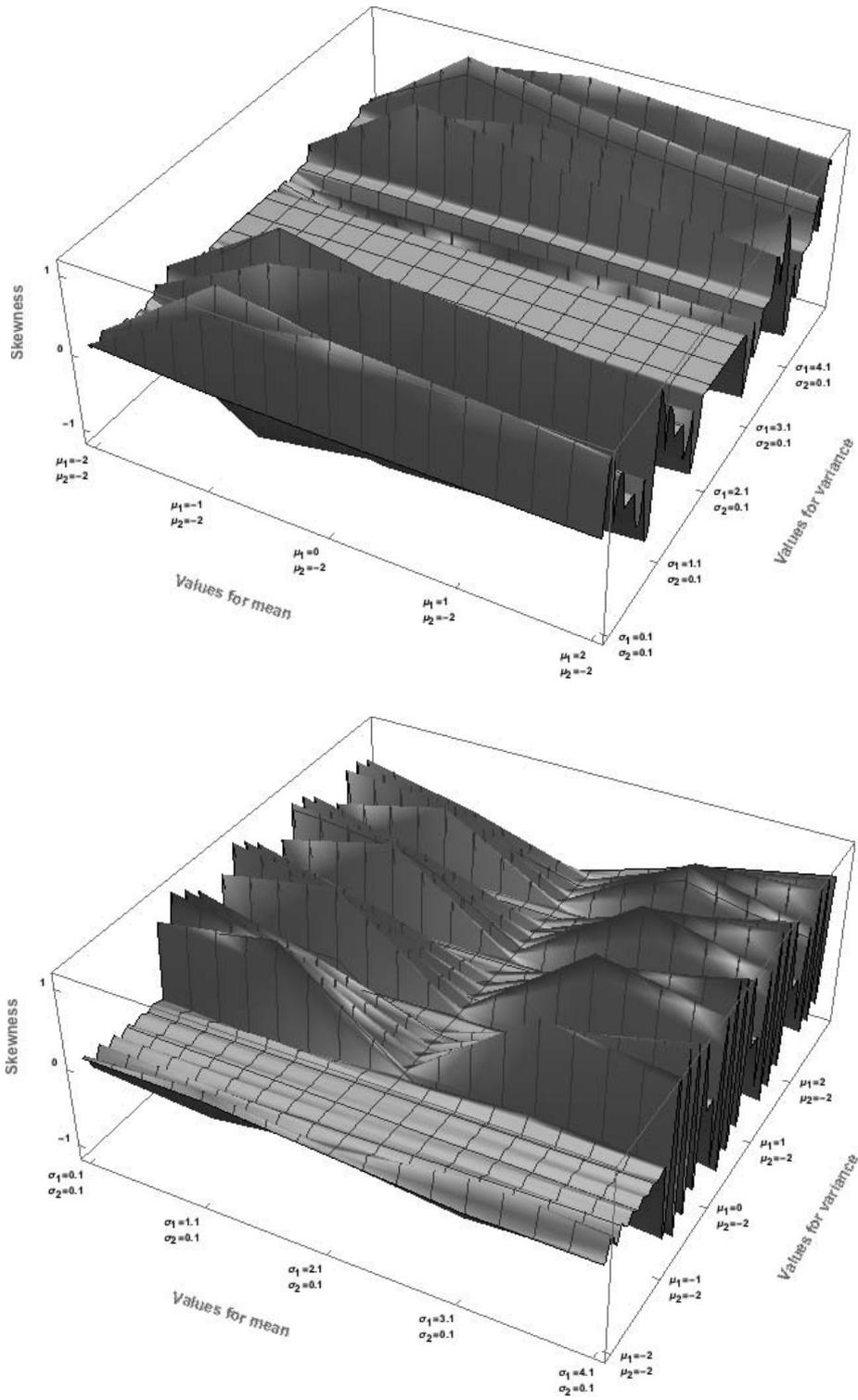


Figure 7: Skewness product of two normal variables.

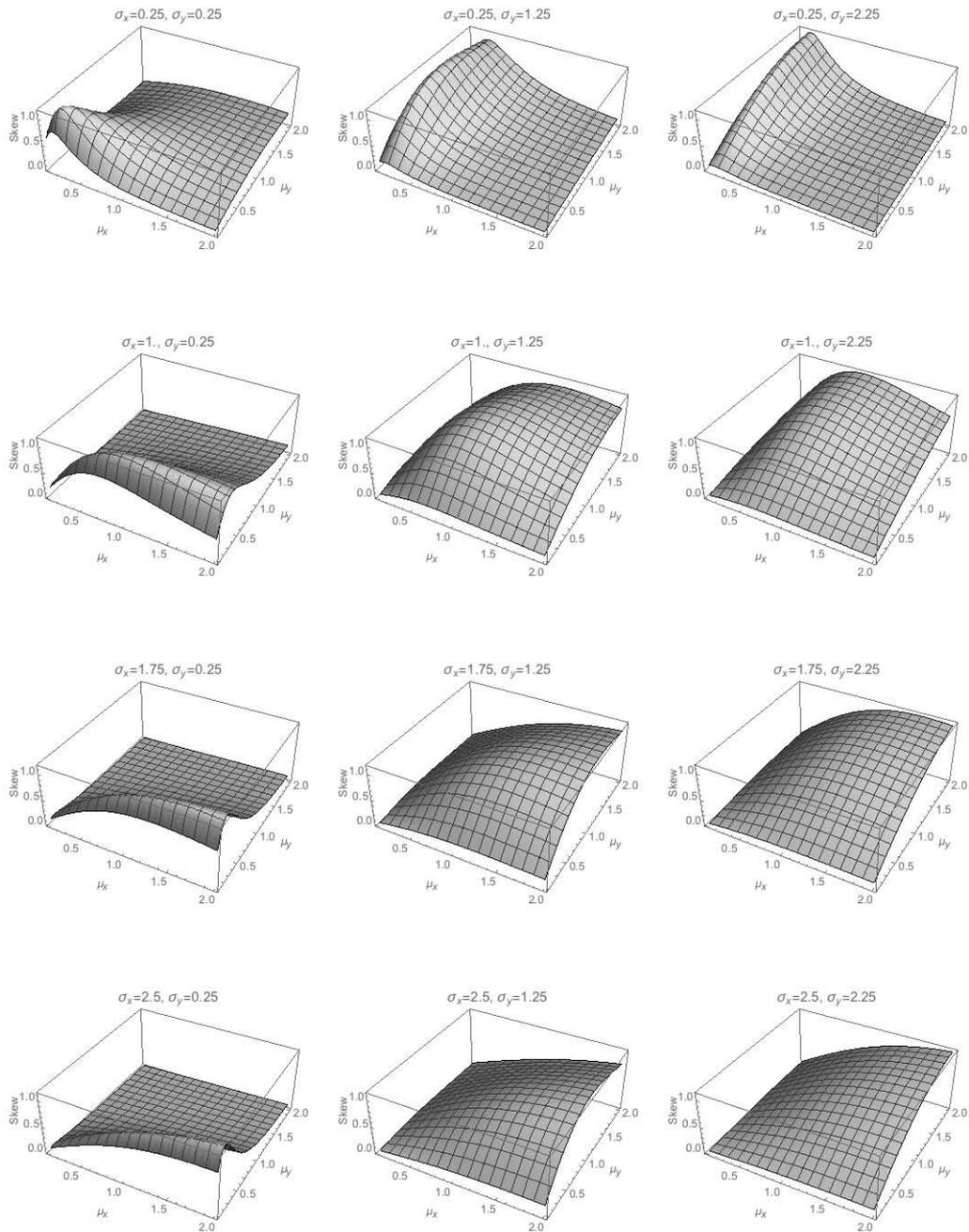


Figure 8: Evolution of skewness for the product of two normal variables.

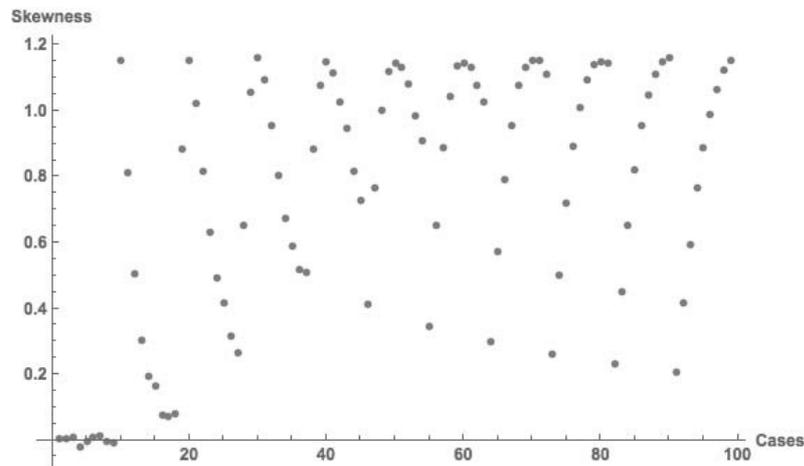


Figure 9: Evolution of skewness for the product of two normal variables
— Monte-Carlo simulation.

Skewness distribution is slightly different from the expected distribution, and there are only few products where the value is zero or less than 0.1. It appears to be a tendency that the value of skewness is small when all parameters of the distributions are small and when there exists some high values for some parameter. However, when all the parameters are big, skewness grow up.

4. SKEWNESS. CONSEQUENCES ON RISK

As it is referred in Mumpower and McClelland ([9]) skewed distributions of risk estimates amplify the “winner’s curse” so that the estimated risk premium for low-probability events is likely to be lower than the normative value. An application of this, is the Probabilistic risk analysis (PRA), where the probability of an event is based on frequency data, physical measurement or expert judgement. In such situation the existence of some type of error is obvious. A simple decision analysis situation could be a lottery of the form pay-off V' with probability p' , otherwise 0 with probability $(1 - p')$. Assume that the decision-maker is an unbiased, valid estimator of p and V ; however, those estimates are not perfectly reliable, there is an associate noise or random error. Then we define:

$$(4.1) \quad p = p' + e_p, \quad V = V' + e_V,$$

where e_V represents the error of the variable.

So thus, p and V are the observed values for decision-maker, if the error term is symmetrically distributed so that, both variables could follow a normal distribution with mean V' and variance S_V^2 and p' and S_p^2 , respectively.

At this situation, the decision-maker has two branches into the tree: one is the lottery with expected value: $EV = pV$ and the other is a constant value C . Let $C = EV$.

We consider an example Mumpower and McClelland ([9]) involving a decision point with two options. The first consists of a simple lottery of the payoff $V \sim N(4, 1)$ with probability $p \sim N(0.4, 0.01)$; the second one consists of a certainty $C = 1.6$. We define $EV = p * V$, as a product of two uncorrelated normal distributions. Applying the formulas for the moments we have:

1. Mean: $E[EV] = 0.44 = 1.6 = C$;
2. Variance: $V(EV) = 0.33$;
3. Skewness: $\alpha_3(EV) = 0.506$.

Then, the distribution of the product is positively skewed and as a consequence, even though the expected value of EV and the value of C are the same, it is more likely that any randomly drawn single estimate, EV will be lower in value than C . At this situation, a risk-neutral decision-making who bases the choice by comparing both values, will select C with a probability greater than 0.5 since $p(C > EV) > 0.5$. In our example, this value is exactly $p(C > EV) = 0.5339$. Value of the median of the EV distribution is $1.55122 < 1.6$, lower that its expectation value.

Skewness is determined by the inverse of the coefficient of variation of the two variables V and p . When the inverse of the coefficient de variation is reduced then skewness will be high, and increasing the inverse of the coefficient of variation a lower value for skewness is obtained. Then, decreasing the level of measurement error this will not necessarily reduce the level of skewness.

From (3.3), value of skewness as a function of the inverse of the coefficient of variation is

$$(4.2) \quad \alpha_3(\rho_p, \rho_V) = \frac{6 \rho_p \rho_V}{(\rho_p^2 + \rho_V^2 + 1)^{3/2}} .$$

For our example, value of $\rho_V = \frac{4}{1} = 4$ that is equal to $\rho_p = \frac{0.4}{0.1} = 4$. Considering $\rho_V = 4$ as constant, varying ρ_p and using first derivative of (4.2) we obtain:

$$(4.3) \quad \frac{\partial \alpha_3}{\partial \rho_p} = \frac{24}{(\rho_p^2 + 17.)^{3/2}} - \frac{72 \cdot \rho_p^2}{(\rho_p^2 + 17.)^{5/2}} .$$

Skewness (4.2) has two maxima at $\rho_p = 2.91548$ and $\rho_p = -2.91548$, then is an increasing function in $(0, 2.91548)$ and then it decreases. Thus if the ρ_p increases, skewness could be increased.

For ρ_V conclusions are the same for the symmetry of the expression (3.3) with respect to both values of the inverse of the coefficient of variation.

These results are in the following proposition.

Proposition 4.1. *Let $V \sim N(\mu_V, \sigma_V)$ and $p \sim N(\mu_p, \sigma_p)$ be two normal uncorrelated distributions. We consider the variable product $EV = pV$, the skewness of EV is a increasing function for ρ_p and for ρ_V , where $\rho_p = \frac{\mu_p}{\sigma_p}$ and $\rho_V = \frac{\mu_V}{\sigma_V}$ are the inverse of the coefficient of variation.*

Skewness in the product of normal distributions has several implications on the risk analysis:

- The mean value is not the mode of the distribution.
- Risk-neutral decision-making choice using the median value of the product distribution and then, the probability of choice C as greater than 0.5.
- Decreasing the level of measurement error will not necessarily reduce the level of skewness.
- For the product of two uncorrelated normal variables there exists a maximum value for skewness $\left(\left|\frac{2}{\sqrt{3}}\right|\right)$.

5. CONCLUSIONS

The product of two normally distributed variables is an open question. This product could be considered a normally distributed variable in specific circumstances. Craig ([3]) and other authors later calculate the moment-generating function for the product and consider that the product is normally distributed when the inverse of coefficient of variation is high-valued.

In this paper we calculate the moment-generating function and we have used it in order to estimate the skewness of product distribution. When the value of the mean of at least one variable is high (> 1) then skewness is low, and when the value of mean is low, skewness is high. This relation lead to several questions that we have shown.

When the variables are correlated skewness is very common and normality of the product is not hold. When the variables are uncorrelated, the existence of skewness is a function of mean and variance of the two distributions. When the inverse of the coefficient of variation is high, skewness is low, but only when the variance is not low; the existence of a variable with a low value for variance produces the existence of skewness into the product of variables.

Skewed distributions of joint probability estimates and expected value estimates can affect the risk and consequently the choices of decision-makers. We have shown the effects in a particular probability risk analysis model.

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REFERENCES

- [1] AROIAN, L.A. (1947). The probability function of a product of two normal distributed variables, *Annals of Mathematical Statistics*, **18**, 256–271.
- [2] AROIAN, L.A.; TANEJA, V.S. and CORNWELL, L.W. (1978). Mathematical forms of the distribution of the product of two normal variables, *Communication in Statistics – Theory and Methods*, **7**(2), 164–172.
- [3] CRAIG, CECIL C. (1936). On the frequency function of xy , *Annals of Mathematical Society*, **7**, 1–15.
- [4] COCHRANE, J.H. (2006). The Dog that Did not Bark: A Defense of Return Predictability. Working Paper, University of Chicago.
- [5] DEB, P. and SEFTON, M. (1996). The distribution of a Lagrange multiplier test of normality, *Economic Letters*, **51**, 123–130.
- [6] HARVEY, C.R. and SIDDIQUE, A. (2000). Conditional skewness in asset pricing tests, *Journal of Finance*, **55**, 1263–1295.
- [7] HAYYA, JACK C. and FERRARA, WILLIAM L. (1972). On normal approximations of the frequency functions of standard forms where the main variables are normally distributed, *Management Science*, **13**(2), 173–186.
- [8] KRAUS, A. and LITZENBERG, R.H. (1975). Measurement error, skewness and risk analysis: coping with the long tail of the distribution, *Journal of Finance*, **31**, 1085–1099.
- [9] MUMPOWER, JERRY L. and MCCLELLAND, GARY (2002). Skewness preference and the valuation of risk assets, *Risk Analysis*, **22**(2), 277–290.
- [10] R CORE TEAM (2015). A language and environment for statistical computing, *R Foundation for Statistical Computing*, Vienna, Austria.