PORT-ESTIMATION OF A SHAPE SECOND-ORDER PARAMETER

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Abstract:
- In this paper we study, under a semi-parametric framework and for heavy right tails, a class of location invariant estimators of a shape second-order parameter, ruling the rate of convergence of the normalised sequence of maximum values to a non-degenerate limit. This class is based on the PORT methodology, with PORT standing for peaks over random thresholds. Asymptotic normality of such estimators is achieved under a third-order condition on the right-tail of the underlying model $F$ and for suitable large intermediate ranks. An illustration of the finite sample behaviour of the estimators is provided through a small-scale Monte-Carlo simulation study.

Key-Words:
- asymptotic properties; location/scale invariant estimation; Monte-Carlo simulation; PORT methodology; sample of excesses; semi-parametric estimation; shape second-order parameters; statistics of extremes; third-order framework.

AMS Subject Classification:
- 62G32, 62E20; 65C05.
1. INTRODUCTION AND MOTIVATION

Let $X_n = (X_1, ..., X_n)$ denote a random sample of $n$ independent, identically distributed (i.i.d.) random variables (r.v.’s) with distribution function (d.f.) $F$. We are interested in heavy-tailed models, i.e. in d.f.’s with a regularly varying right-tail. This means that, for $\xi > 0$, the right tail-function

$$F := 1 - F$$

is such that

$$\lim_{t \to \infty} \frac{F(tx)/F(t)}{x^{-1/\xi}} = 1,$$

for all $x > 0$.

We then say that $F$ is of regular variation at infinity with an index equal to $-1/\xi$, and define

$$G_\xi(x) := \begin{cases} \exp\left(-(1 + \xi x)^{-1/\xi}\right), & 1 + \xi x > 0, \text{ if } \xi \neq 0 \\ \exp(-\exp(-x)), & x \in \mathbb{R}, \text{ if } \xi = 0 \end{cases}$$

the general extreme-value (EV) distribution function. If (1.1) holds, we are in the domain of attraction for maxima of $G_\xi$, with $\xi > 0$, and we write $F \in \mathcal{D}_M(G_\xi > 0)$, meaning that it is possible to find sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the maximum $X_{n:n} := \max(X_1, ..., X_n)$, linearly normalized, i.e. $(X_{n:n} - b_n)/a_n$, converges in distribution to a non-degenerate r.v. with d.f. $G_\xi(x)$, in (1.2), with $\xi > 0$. This type of heavy-tailed models arises often in practice, in fields like telecommunication traffic, finance, insurance, economics, ecology and biometry, among others. The parameter $\xi$, in (1.2), is the extreme-value index (EVI), one of the primary parameters of extreme events.

Let $F^{-}$ denote the generalised inverse function of $F$, defined by

$$F^{-}(t) := \inf \{x : F(x) \geq t\},$$

and let $U$ be the associated (reciprocal) quantile function, defined as

$$U(t) := F^{-}(1 - 1/t), \quad t \geq 1.$$

1.1. First and second-order conditions for heavy-tailed models

In a heavy-tailed framework, i.e. if (1.1) holds, with the usual notation $RV_a$ for the class of regularly varying functions at infinity with an index $a \in \mathbb{R}$, and on the basis of the results in Gnedenko (1943), for the right-tail function $F = 1 - F$,
and in de Haan (1984), for \( U \) in (1.4), the following first-order conditions are equivalent,

\[
F \in D_M(G \xi > 0) \iff \bar{F} \in RV_{-1/\xi} \iff U \in RV_\xi.
\]

Now we need to say something about the rate of convergence in (1.5), and assume that the following limiting relation holds for every \( x > 0 \),

\[
\lim_{t \to \infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \begin{cases} 
\frac{x^\rho - 1}{\rho}, & \text{if } \rho < 0 \\
\ln x, & \text{if } \rho = 0,
\end{cases}
\]

where \( |A| \) must then be in \( RV_\rho \) (Geluk and de Haan, 1987). The second-order parameter \( \rho \leq 0 \) rules the rate of convergence provided by (1.6), which increases with \( |\rho| \). Note further that in the scope of applications, the most common models depend on a location or shift parameter \( s \in \mathbb{R} \), not necessarily null, i.e. \( F(x) \equiv F_s(x) = F_0(x-s) \). Then, \( U(t) \equiv U_s(t) = U_0(t) + s \) and also both \( A \) and \( \rho \) depend obviously on \( s \), i.e. \( A = A_s \) and \( \rho = \rho_s \), with

\[
\rho_s := \begin{cases} 
-\xi, & \text{if } \xi + \rho_0 < 0 \land s \neq 0 \\
\rho_0, & \text{otherwise}.
\end{cases}
\]

Among the literature specifically devoted to the estimation of the second-order parameter \( \rho \), in (1.6), we mention Gomes et al. (2002), Fraga Alves et al. (2003a), and the more recent articles by Goegebeur et al. (2008; 2010), Ciperca and Mercadier (2010) and Caeiro and Gomes (2012a,b). Indeed, most of the research devised to improve the classical EVI-estimators tries to reduce the dominant component of their asymptotic bias, deals with second-order reduced-bias (SORB) EVI-estimators, and an adequate estimation of \( \rho \) is needed, for an adequate reduction of the bias. Some of the pioneering papers in the area of SORB-estimation are the ones by Beirlant et al. (1999), Feuerverger and Hall (1999), Gomes et al. (2000) and Gomes and Martins (2001; 2002). More recently, the minimum-variance reduced-bias (MVRB) EVI-estimators, studied in Caeiro et al. (2005), Gomes et al. (2007) and Gomes et al. (2008c), among others, also call for an adequate estimation of \( \rho \). An overview of the subject can be found in Chapter 6 of the book by Reiss and Thomas (2007). See also Gomes et al. (2008a) and Beirlant et al. (2012) in this respect. However, despite of scale-invariant, all these MVRB EVI-estimators are not location-invariant.

### 1.2. The PORT methodology

Let \( X_{i,n}, 1 \leq i \leq n, \) be the o.s.’s associated with the random sample \( X_n = (X_1, \ldots, X_n) \) with common d.f. \( F_0 \). The class of estimators suggested here is a
function of the sample of excesses over a random threshold $X_{n_q:n}$, with $n_q = \lfloor nq \rfloor + 1$, where $\lfloor x \rfloor$ stands for the integer part of $x$. Such a sample is denoted by

$$X^{(q)}_n := (X_{n:n} - X_{n_q:n}, X_{n-1:n} - X_{n_q:n}, ..., X_{n_q+1:n} - X_{n_q:n})$$

where, we can have

- $0 < q < 1$, for any $F_0 \in D_M(G_{\xi > 0})$ (the random threshold, $X_{n_q:n}$, is an empirical quantile);
- $q = 0$, for d.f.’s with a finite left endpoint $x_F := \inf\{x : F_0(x) > 0\}$, (the random threshold is the minimum, $X_{1:n}$).

Any statistical inference methodology based on the sample of excesses $X^{(q)}_n$, in (1.8), will be called a PORT-methodology, with PORT standing for peaks over random thresholds, a term coined by Araújo Santos et al. (2006). This methodology enabled the introduction and study of classical location/scale invariant EVI-estimators, like the PORT-Hill and the PORT-Moment estimators, studied for finite-samples in Gomes et al. (2008b). This methodology was also applied in the estimation of high quantiles in Henriques-Rodrigues and Gomes (2009).

Such a methodology leads to location-invariant estimation, where the unshifted model $F_0$ thus plays a central role. In what follows, we use the notation $\chi_q$ for the $q$-quantile of the d.f. $F_0$, i.e. the value

$$\chi_q := F_0^{-1}(q)$$

(by convention $\chi_0 := x_F$, the left endpoint of $F_0$), with $F^{-1}(\cdot)$ defined in (1.3). Since $n_q/n \to q$, as $n \to \infty$, we then know that the o.s. $X_{n_q:n}$, associated with a sample from $F_0$, is a consistent estimator for $F_0^{-1}(q)$ (Mosteller, 1946, under stronger assumptions on $F$; van der Vaart, 1998, p.308), i.e. we have the following convergence in probability:

$$X_{n_q:n} \overset{p}{\to} \chi_q = F_0^{-1}(q), \quad \text{for} \quad 0 \leq q < 1 \quad (\chi_0 = x_F).$$

### 1.3. Scope of the paper

We shall make use of the above-mentioned PORT methodology for heavy tails. Henceforth $\xi > 0$ denotes the first-order parameter of the model underlying the available data, $\rho_0 \leq 0$ is the second-order parameter of the associated unshifted model, and $\chi_q$ has been provided in the limit of (1.10), in order to introduce a class of location-invariant semi-parametric estimators of the so-called PORT-$\rho$ second-order parameter,

$$\rho_q := \begin{cases} -\xi, & \text{if } \xi + \rho_0 < 0 \land \chi_q \neq 0 \\ \rho_0, & \text{otherwise.} \end{cases}$$
Note that when applying the PORT-methodology, we are working with the sample of excesses in (1.8), and we can assume that we are dealing with an unshifted d.f. $F_0$ underlying the r.v. $X_0$, to which we are inducing a random shift, strictly related to $\chi_q$, in (1.9). More precisely, we have a shift $s = -\chi_q$, i.e. we are working with $X_q := X_0 - \chi_q$, and use the simpler notation $\rho_q$ for $\rho - \chi_q$, with $\rho$ defined in (1.7). Hence $\rho_q = -\xi \neq \rho_0$ if and only if $\chi_q \neq 0$ and the underlying model is such that $\xi + \rho_0 < 0$, just as written in (1.11), i.e. $\rho_q \neq \rho_0$ if and only if $s = 0$, $\chi_q \neq 0$ and $\xi + \rho_0 < 0$.

The main motivation for a class of estimators of the shape second-order parameter $\rho_q$, in (1.11), is related to its possible use, concomitantly with a class of PORT estimators of the functional $A$, in (1.6), or at least of an adequate location-invariant estimator of the scale parameter of such a $A$-function, in the building of second-order PORT-MVRB EVI-estimators, invariant for changes in location. The study of the asymptotic behaviour of such EVI-estimators is a challenging theoretical open subject, out of the scope of this paper, but already dealt with by Monte-Carlo simulation, in Gomes et al. (2011, 2013).

The building block of our estimators of the shape second-order parameter $\rho_q$, defined in (1.11) are of the same kind as the statistics used in Dekkers et al. (1989), Gomes et al. (2002), Fraga Alves et al. (2003a) and Caeiro and Gomes (2006), among others, i.e. for $\alpha > 0$ we consider the moment statistics

\begin{equation}
M_{n,k}^{(\alpha)} = M_{n,k}^{(\alpha)}(X_n) := \frac{1}{k} \sum_{i=1}^{k} (\ln X_{n-i+1:n} - \ln X_{n-k:n})^\alpha,
\end{equation}

but now applied to the sample of excesses $X_n^{(q)}$, $0 \leq q < 1$, in (1.8). For an intermediate $k$-sequence, i.e. a sequence $k = k_n$ of positive integers such that

\begin{equation}
k = k_n \to \infty \quad \text{and} \quad k = o(n) \quad \text{as} \quad n \to \infty,
\end{equation}

we shall thus consider the location and scale-invariant statistics,

\begin{equation}
M_{n,k}^{(\alpha,q)} = M_{n,k}^{(\alpha,q)}(X_n^{(q)}) := \frac{1}{k} \sum_{i=1}^{k} \left( \frac{\ln X_{n-i+1:n} - X_{n-k:n}}{X_{n-k:n} - X_{n-k-nq:n}} \right)^\alpha,
\end{equation}

defined for $k < n - n_q$, with $M_{n,k}^{(\alpha,q)}(X_n)$ given in (1.12), $\alpha > 0$.

Regarding the tuning parameters $\tau_q \in \mathbb{R}$, $\alpha, \theta_1, \theta_2 \in \mathbb{R}^+$, $\theta_1, \theta_2 \neq 1$ and $\theta_1 < \theta_2$, we shall consider the PORT-versions of the statistics used in Fraga Alves et al. (2003a) for the estimation of $\rho$, in (1.6), i.e.

\begin{equation}
T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q-q)} := \frac{M_{n,k}^{(\alpha,q)}}{\Gamma(\alpha+1)} \tau_q - \frac{M_{n,k}^{(\alpha,\theta_1,q)}}{\Gamma(\alpha\theta_1+1)} \tau_{\theta_1} - \frac{M_{n,k}^{(\alpha,\theta_2,q)}}{\Gamma(\alpha\theta_2+1)} \tau_{\theta_2} =: \frac{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q-q)}(\xi)}{D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q-q)}(\cdot)},
\end{equation}
with $\Gamma(t)$ denoting the complete Gamma function. As detailed in Section 3.1, under adequate conditions upon the growth of $k = k_n$, $T^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n, k}$ converges in probability to

$$
(1.16) \quad t_{\alpha, \theta_1, \theta_2}(\rho_q) := \frac{\theta_2 (\theta_1 - 1) (1 - \rho_q)^{\alpha \theta_2 - \theta_1} - \theta_1 (1 - \rho_q)^{\alpha \theta_2 - \alpha - (1 - \rho_q)} + (1 - \rho_q)^{\alpha (\theta_2 - \theta_1)}}{(\theta_2 - \theta_1) (1 - \rho_q)^{\alpha \theta_2 - \theta_2 (1 - \rho_q)^{\alpha (\theta_2 - \theta_1)} + \theta_1}}.
$$

**Remark 1.1.** Note that the function $t_{\alpha, \theta_1, \theta_2}(\rho_q)$, defined for $\rho_q \leq 0, \alpha > 0, \theta_1, \theta_2 \in \mathbb{R}^+ \setminus \{1\}$, $\theta_1 < \theta_2$, is a decreasing function of $\rho_q$ if $\theta_1, \theta_2 > 1$ or $\theta_1, \theta_2 < 1$ and increasing otherwise. Since $t_{\alpha, \theta_1, \theta_2}(\rho_q)$ is always monotone continuous then it is invertible. Moreover,

$$
\lim_{\rho_q \to -\infty} t_{\alpha, \theta_1, \theta_2}(\rho_q) = \frac{\theta_2 (\theta_1 - 1)}{\theta_2 - \theta_1} \quad \text{and} \quad \lim_{\rho_q \to 0} t_{\alpha, \theta_1, \theta_2}(\rho_q) = \frac{\theta_1 - 1}{\theta_2 - \theta_1}.
$$

The general class of consistent $\rho_q$-estimators, invariant for changes in location, already introduced and validated under a second-order framework in Henriques-Rodrigues and Gomes (2012), and named PORT-$\rho$ class of estimators, it is now written as

$$
\hat{\rho}_{n, k|T}^{(\alpha, \theta_1, \theta_2, \tau_q, q)} := - \left| t_{\alpha, \theta_1, \theta_2}(T^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n, k}) \right|,
$$

with $T^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n, k}$ given in (1.15).

The simplest choice of tuning control parameters suggested in Fraga Alves et al. (2003a) for the classical $\rho$-estimators, $(\alpha, \theta_1, \theta_2) = (1, 2, 3)$, gives rise to an explicit $\rho$-estimator, denoted $\hat{\rho}^{(\tau)}_k$ in the aforementioned paper, and leads us to a simpler class of PORT-$\rho$ estimators of the shape second-order parameter $\rho_q$, because it only depends on the tuning parameter $\tau_q$. With $\rho_q$ defined in (1.11), we have that

$$
t(\rho_q) = t_{1, 2, 3}(\rho_q) = \frac{3(1 - \rho_q)}{3 - \rho_q} = \begin{cases} 
\frac{3(1 + \xi)}{3 + \xi}, & \text{if } \xi + \rho_0 < 0 \land \chi_q \neq 0, \\
\frac{3(1 - \rho_0)}{3 - \rho_0}, & \text{otherwise}.
\end{cases}
$$

Thus the PORT-$\rho$ estimator associated with $(\alpha, \theta_1, \theta_2) = (1, 2, 3)$ is explicitly given by

$$
\hat{\rho}^{(\tau_q, q)}_k \equiv \hat{\rho}^{(1, 2, 3, \tau_q, q)}_{n, k|T} := - \frac{3(T^{(1, 2, 3, \tau_q, q)}_{n, k})^{-1}}{T^{(1, 2, 3, \tau_q, q)}_{n, k} - 3},
$$

where

$$
T^{(1, 2, 3, \tau_q, q)}_{n, k} = \frac{(M^{(1, q)}_{n, k})^{\tau_q} - (M^{(2, q)}_{n, k}/2)^{\tau_q/2}}{(M^{(2, q)}_{n, k}/2)^{\tau_q/2} - (M^{(3, q)}_{n, k}/6)^{\tau_q/3}},
$$

for any $\tau_q \in \mathbb{R}$, with $M^{(\alpha, q)}_{n, k}$ given in (1.14). The notation $a^{\ln b} = b \ln a$ is used for $\tau_q = 0$. 

PORT-Estimation of a Shape Second-Order Parameter
In Section 2 of this paper we present preliminary asymptotic results related to the PORT-methodology. In Section 3 we justify the class of PORT-\(\rho\) estimators of the shape second-order parameter \(\rho_q\), in (1.11), addressing the possibility of shifted heavy-tailed models, and refer the conditions required for their consistency and asymptotic normality. In Section 4, we illustrate the finite sample behaviour of the new estimators through a small-scale Monte-Carlo simulation study. Finally, in Section 5, we present the proofs of the results in Section 3.

2. TECHNICAL RESULTS RELATED TO THE PORT-METHODOLOGY

2.1. The second-order PORT-framework for heavy-tailed models

Under the aforementioned set-up in Section 1.3, the transformed r.v., \(X_q = X_0 - \chi_q\), has an associated quantile function given by \(U_q(t) = U_0(t) - \chi_q\). The second-order condition in (1.6) translates as

\[
\lim_{t \to \infty} \frac{\ln U_q(tx) - \ln U_q(t) - \xi \ln x}{A_q(t)} = \begin{cases} \frac{x^{\rho_q} - 1}{\rho_q}, & \text{if } \rho_q < 0 \\ \ln x, & \text{if } \rho_q = 0, \end{cases}
\]

for all \(x > 0\). Moreover, \(|A_q| \in RV_{\rho_q}; \rho_q \leq 0\), and \(A_q\) relates to \(A_0\) according to the following lemma.

**Lemma 2.1.** Assume \(U_0 \in RV_{\xi}\) satisfies the second order condition in (1.6) with \(\rho = \rho_0\) and \(A = A_0\). Then \(U_q(t) = U_0(t) - \chi_q\), with \(\chi_q\) defined in (1.9), is such that \(U_q \in RV_{\xi}\) and (2.1) holds with \(\rho_q\) given in (1.11) and

\[
A_q(t) := \begin{cases} \frac{\xi \chi_q}{U_0(t)}, & \text{if } \xi + \rho_0 < 0 \land \chi_q \neq 0 \\ A_0(t), & \text{if } \xi + \rho_0 > 0 \lor \chi_q = 0 \\ A_0(t) + \frac{\xi \chi_q}{U_0(t)}, & \text{if } \xi + \rho_0 = 0 \land \chi_q \neq 0. \end{cases}
\]

2.2. Third-order framework and asymptotic behaviour of auxiliary statistics

Next, we present the asymptotic behaviour of the statistics \(M_{n,k}^{(\alpha,q)}\) defined in (1.14), based on the sample of excesses \(X_n^{(q)}\), \(0 \leq q < 1\), defined in (1.8). This requires a third-order framework because we further need to know the rate of convergence in (1.6). It is common to assume a third-order condition that rules such
a rate of convergence through the shape third-order parameter \( \rho' \leq 0 \), assuming that for all \( x > 0 \),

\[
\lim_{t \to \infty} \frac{\ln U(t x) - \ln U(t) - \xi \ln x - \frac{x^\rho}{\rho} - 1}{\rho + \rho'} = \frac{x^\rho' - 1}{\rho + \rho'},
\]

with \( |A| \in RV_\rho \) and \( |B| \in RV_{\rho'} \). For technical simplicity, we shall assume that \( \rho, \rho' < 0 \), i.e., we assume to be working in a class \( \mathcal{H} \) of heavy-tailed models, such that, as \( t \to \infty \),

\[
U(t) = C t^\xi \left\{ 1 + D_1 t^\rho + D_2 t^{\rho + \rho'} + o(t^{\rho + \rho'}) \right\},
\]

where \( C > 0 \). Details on the third-order condition in (2.3) can be found in Fraga Alves et al. (2003b, 2006) and more generally in Wang and Cheng (2005).

Note that the statistics \( M_{n,k}^{(\alpha,q)} \), in (1.14), depend on \( q \) through \( \chi_q \), in (1.9) (see also (1.10)), but are obviously independent on any shift \( s \) imposed to the data. We can thus assume throughout that \( s = 0 \).

Let \( \mathbb{E} \) and \( \text{Var} \) denote the mean value and variance operators, respectively, and let \( E \) denote a unit exponential random variable. For any real \( \alpha > 0 \), with \( \xi > 0 \) and \( \rho < 0 \), let us define

\[
\mu_\alpha^{(1)}(\xi) := \mathbb{E}\left(E^\alpha e^{-\xi E}\right) = \frac{\Gamma(\alpha + 1)}{(1 + \xi)^{\alpha + \tau}}, \quad \mu_\alpha^{(1)}(0) = \Gamma(\alpha + 1),
\]

\[
\sigma_\alpha^{(1)} := \sqrt{\text{Var}(E^\alpha)} = \sqrt{\Gamma(2\alpha + 1) - \Gamma^2(\alpha + 1)},
\]

\[
\mu_\alpha^{(2)}(\xi, \rho) := \mathbb{E}(E^{\alpha - 1} e^{-\xi E} (e^{\rho E} - 1)/\rho) = \frac{\Gamma(\alpha)}{\rho} \left( \frac{(1 + \xi)^\alpha - (1 + \xi - \rho)^\alpha}{(1 + \xi)^\alpha - (1 + \xi - \rho)^\alpha(1 + \xi)^\alpha} \right),
\]

\[
\sigma_\alpha^{(2)}(\rho) := \sqrt{\text{Var}(E^{\alpha - 1}(e^{\rho E} - 1)/\rho)} = \sqrt{\mu_\alpha^{(3)}(\rho) - \left( \mu_\alpha^{(2)}(\rho) \right)^2},
\]

\[
\eta_\alpha^{(3)}(\xi, \rho) := \mathbb{E}\left(E^{\alpha - 2} \left( (e^{-\xi E} - 1)/(-\xi) \right) \right) \left((e^{\rho E} - 1)/\rho\right)
\]

\[
= \begin{cases} 
- \frac{1}{\xi \rho} \ln \frac{(1 + \xi)(1 - \rho)}{1 + \xi - \rho}, & \text{if } \alpha = 1 \\
- \frac{\Gamma(\alpha)}{\xi \rho (\alpha - 1)} \left( \frac{1}{(1 + \xi - \rho)^\alpha} - \frac{1}{(1 + \xi)^\alpha} \right) + 1, & \text{if } \alpha \neq 1,
\end{cases}
\]

and

\[
\mu_\alpha^{(3)}(\rho) := \mathbb{E}\left(E^{\alpha - 2} \left((e^{\rho E} - 1)/\rho\right)^2\right)
\]

\[
= \begin{cases} 
\frac{1}{\rho^2} \ln \frac{(1 - \rho)^2}{2\rho}, & \text{if } \alpha = 1 \\
\frac{\Gamma(\alpha)}{\rho^2 (\alpha - 1)} \left( \frac{1}{(1 - 2\rho)^\alpha} - \frac{2}{(1 - \rho)^\alpha} \right) + 1, & \text{if } \alpha \neq 1.
\end{cases}
\]
Let us further introduce the notations:

\[(2.7) \quad \overline{p}_\alpha^{(j)}(\rho) := \frac{\mu^{(j)}_\alpha(\rho)}{\mu^{(1)}_\alpha}, \quad j = 2, 3, \quad \overline{p}_\alpha^{(2)}(\xi, \rho) := \frac{\mu^{(2)}_\alpha(\xi, \rho)}{\mu^{(1)}_\alpha}, \]

\[(2.8) \quad \overline{\eta}_\alpha^{(3)}(\xi, \rho) := \frac{\mu^{(3)}_\alpha(\xi, \rho)}{\mu^{(1)}_\alpha}, \]

\[(2.9) \quad \sigma_\alpha^{(1)} := \frac{\sigma^{(1)}_\alpha}{\mu^{(1)}_\alpha}, \quad \sigma_\alpha^{(2)}(\rho) := \frac{\sigma^{(2)}_\alpha(\rho)}{\mu^{(1)}_\alpha}, \]

and for any \(\theta_1, \theta_2 > 0\), define

\[(2.10) \quad d_{\alpha, \theta_1, \theta_2}(\rho) := \overline{p}_{\alpha \theta_1}^{(2)}(\rho) - \overline{p}_{\alpha \theta_2}^{(2)}(\rho). \]

Recall that \(E_i, \ i \geq 1\), are i.i.d. unit exponential r.v.'s, and, with \(\sigma_\alpha^{(1)}\) given in (2.6), define the asymptotically standard normal r.v.'s

\[(2.11) \quad Z_k^{(\alpha)} := \sqrt{k}\left(\frac{1}{k} \sum_{i=1}^{k} E_i^a - \Gamma(\alpha + 1)\right)/\sigma_\alpha^{(1)}. \]

Now, together with (2.9), we can combine these as follows:

\[(2.12) \quad W_k^{(\alpha, \theta_1, \theta_2)} := \overline{\sigma}_\alpha^{(1)} Z_k^{(\alpha, \theta_1)}/\theta_1 - \overline{\sigma}_\alpha^{(1)} Z_k^{(\alpha, \theta_2)}/\theta_2. \]

Finally, for \(\tau \in \mathbb{R}, \alpha, \theta > 0\), and with \((\overline{p}_\alpha^{(2)}(\rho), \overline{p}_\alpha^{(2)}(\xi, \rho))\) and \(\overline{\eta}_\alpha^{(3)}(\xi, \rho)\) defined in (2.7) and (2.8), respectively, we define

\[(2.13) \quad c_{\alpha, \theta, \tau}(\rho) := (\alpha \theta - 1)\overline{p}_{\alpha \theta}^{(3)}(\rho) + \alpha(\tau - \theta)(\overline{p}_{\alpha \theta}^{(2)}(\rho))^2, \]

\[(2.14) \quad g_{\alpha, \theta, \tau}(\xi, \rho) := \overline{p}_{\alpha \theta}^{(2)}(\xi, \rho) + (\alpha \theta - 1)\overline{\eta}_{\alpha \theta}^{(3)}(\xi, \rho) + \alpha(\tau - \theta)\overline{p}_{\alpha \theta}^{(2)}(\rho)\overline{p}_{\alpha \theta}^{(2)}(-\xi), \]

\[(2.15) \quad h_{\alpha, \theta, \tau}(\xi) := 2\overline{p}_{\alpha \theta}^{(2)}(-2\xi) + (\alpha \theta - 1)\overline{p}_{\alpha \theta}^{(3)}(-\xi) + \alpha(\tau - \theta)\overline{p}_{\alpha \theta}^{(2)}(-\xi)^2. \]

We first state Proposition 2.1, related to the behaviour of \(M_{n,k}^{(\alpha)}\), in (1.12), now needed only for \(s = 0\ (\rho = \rho_0)\), proved in Gomes et al. (2002), also under a third-order framework.

Proposition 2.1 (Gomes et al., 2002). Under the third-order condition (2.3), with \(\rho_0, \rho_0' < 0\), for intermediate sequences \(k = k_n\), i.e. sequences of positive integers such that (1.13) holds, and with \(M_{n,k}^{(\alpha)}, \mu_\alpha^{(1)}, \overline{p}_\alpha^{(2)}(\rho), j = 2, 3, \sigma_\alpha^{(1)}\) and \(Z_k^{(\alpha)}\) defined in (1.12), (2.5), (2.7), (2.9) and (2.11), respectively,

\[
M_{n,k}^{(\alpha)} = \frac{d}{\alpha} \left\{ \xi^{\alpha} \mu_\alpha^{(1)} \left\{ 1 + \frac{\sigma_\alpha^{(1)} Z_k^{(\alpha)}}{k} + \frac{\sigma_\alpha^{(2)}(\rho_0) A_0(n/k)}{\sqrt{k}^2} \right. \right.
+ \left. \left( \frac{\alpha^{(\alpha - 1)}}{2^{\alpha - 1}} \right) \overline{p}_\alpha^{(3)}(\rho_0) A_0^2(n/k) + \frac{\alpha^{(\alpha - 2)}}{2^{\alpha - 2}} \overline{p}_\alpha^{(2)}(\rho_0 + \rho_0') A_0(n/k) B_0(n/k) \right\} (1 + o_p(1)).
\]

We next provide, under the third-order framework in (2.3), the behaviour of \(M_{n,k}^{(\alpha, \theta)}\), in (1.14).
Proposition 2.2. Let us assume that \((1.13)\) holds, as well as the third-order condition in (2.3), with \(\rho_0, \rho'_0 < 0\). We then get for \(M^{(\alpha,q)}_{n,k}\), in (1.14), \(\alpha > 0\),
\(k < n - n_q\), with \(X_q\) and \(M^{(\alpha)}\) (for \(s = 0\)), given in (1.10) and (1.12), respectively,
\(\mu^{(1)}_1\) and \((\mu^{(j)}_1(\rho), \sigma^{(2)}_\alpha(\xi, \rho), \rho^{(3)}_\alpha(\rho))\) and \(\eta^{(3)}_\alpha(\xi, \rho)\) respectively given in (2.5), (2.7) and (2.8), the distributional representation,

\[
\begin{align*}
M^{(\alpha,q)}_{n,k} & = M^{(\alpha)}_{n,k} + \alpha^{\alpha_0} \mu^{(1)}_1 \frac{\chi_q}{U_0(n/k)} \left[ \sigma^{(2)}_\alpha(\xi) - \eta^{(3)}_\alpha(\xi, \rho) \right] A_0(n/k)(1 + o_p(1)) \\
& + \left\{ \frac{\alpha \sigma^{(2)}_\alpha(\xi, \rho_0) A_0(n/k)}{U_0(n/k)} \right\} \left( 1 + o_p(1) \right) \\
& + \left\{ \frac{\alpha \sigma^{(2)}_\alpha(\xi, \rho_0) A_0(n/k)}{U_0(n/k)} \right\} \left( 1 + o_p(1) \right).
\end{align*}
\]

3. ASYMPTOTIC BEHAVIOUR OF THE PORT-\(\rho\) ESTIMATORS

3.1. Consistency of the PORT-\(\rho\) estimators

For \(\alpha > 0\), let us consider the statistics \(M^{(\alpha,q)}_{n,k} = M^{(\alpha,q)}_{n,k}(X^{(q)}_n)\), in (1.14),
defined for \(k < n - n_q\), with \(X^{(q)}_n\) the sample of excesses in (1.8). Under the third-order framework in (2.3), if (1.13) holds, on the basis of the results in Propositions 2.1 and 2.2, similarly to the developments in Fraga Alves et al. (2003a), and for real tuning parameters \(\tau_q \in \mathbb{R}\) and \(\theta \neq 0\),

\[
\begin{align*}
\left( \frac{M^{(\alpha,q)}_{n,k}}{\mu^{(1)}_1} \right) & \rightarrow \frac{\alpha \tau_q}{\xi} \left( 1 + \frac{\tau_q}{\theta} \frac{\sigma^{(1)}_\alpha}{\sqrt{k}} Z^{(\alpha)}_k \right) + \frac{\alpha \tau_q \sigma^{(2)}_\alpha(\rho_0) A_0(n/k)}{U_0(n/k)} + \frac{\alpha \sigma^{(2)}_\alpha(\xi, \rho_0) A_0(n/k)}{U_0(n/k)} (1 + o_p(1)) \\
& + \left\{ \frac{\alpha \sigma^{(2)}_\alpha(\xi, \rho_0) A_0(n/k)}{U_0(n/k)} \right\} \left( 1 + o_p(1) \right),
\end{align*}
\]

i.e.

\[
\begin{align*}
\left( \frac{M^{(\alpha,q)}_{n,k}}{\mu^{(1)}_1} \right) & \rightarrow \frac{\alpha \tau_q}{\xi} \left( 1 + \frac{\tau_q}{\theta} \frac{\sigma^{(1)}_\alpha}{\sqrt{k}} Z^{(\alpha)}_k \right) + \frac{\alpha \tau_q \sigma^{(2)}_\alpha(\rho_0) A_0(n/k)}{U_0(n/k)} + \frac{\alpha \sigma^{(2)}_\alpha(\xi, \rho_0) A_0(n/k)}{U_0(n/k)} (1 + o_p(1)) \\
& + \left\{ \frac{\alpha \sigma^{(2)}_\alpha(\xi, \rho_0) A_0(n/k)}{U_0(n/k)} \right\} \left( 1 + o_p(1) \right),
\end{align*}
\]

with \(M^{(\alpha,q)}_{n,k}, \mu^{(1)}_1(\rho), \sigma^{(2)}_\alpha(\xi, \rho), \sigma^{(2)}_\alpha(\rho), \sigma^{(3)}_\alpha(\xi, \rho), \sigma^{(3)}_\alpha(\xi, \rho)\) and \(h^{(\alpha,q)}(\xi, \rho)\)
given in (1.14), (2.5), (2.7), (2.9), (2.11), (2.13), (2.14) and (2.15), respectively.
Let us next introduce the notations,

\[(3.2) \quad u_{\alpha,\theta_1,\theta_2,\tau}(\rho) := \left\{ c_{\alpha,\theta_1,\tau}(\rho) - c_{\alpha,\theta_2,\tau}(\rho) \right\} / (2\xi),\]

\[(3.3) \quad v_{\alpha,\theta_1,\theta_2}(\rho, \rho') := \overline{p}_{\alpha,\theta_1}^{(2)}(\rho + \rho') - \overline{p}_{\alpha,\theta_2}^{(2)}(\rho + \rho') \equiv d_{\alpha,\theta_1,\theta_2}(\rho + \rho'),\]

\[(3.4) \quad w_{\alpha,\theta_1,\theta_2,\tau}(\xi, \rho) := \{ g_{\alpha,\theta_1,\tau}(\xi, \rho) - g_{\alpha,\theta_2,\tau}(\xi, \rho) \} / \xi,\]

\[(3.5) \quad y_{\alpha,\theta_1,\theta_2,\tau}(\xi) := \{ h_{\alpha,\theta_1,\tau}(\xi) - h_{\alpha,\theta_2,\tau}(\xi) \} / 2,\]

with \(d_{\alpha,\theta_1,\theta_2}(\rho), \ c_{\alpha,\theta_1,\tau}(\rho), \ g_{\alpha,\theta_1,\tau}(\xi, \rho)\) and \(h_{\alpha,\theta_1,\tau}(\xi)\) defined in (2.10), (2.13), (2.14) and (2.15), respectively. On the basis of (3.1), using the notation \(W_k^{[\alpha,\theta_1,\theta_2]}\) in (2.12), and with \(D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi)\) defined in (1.15), we can write

\[(3.6) \quad D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi) \equiv \xi^{\alpha,\theta_q} \left( \frac{\tau_q}{\xi} W_k^{(\alpha,\theta_1,\theta_2)} + \frac{\alpha\tau_q}{\xi} A_0(n/k) \right) \left\{ d_{\alpha,\theta_1,\theta_2}(\rho_0) + u_{\alpha,\theta_1,\theta_2,\tau}(\rho_0) A_0(n/k)(1 + o_p(1)) + v_{\alpha,\theta_1,\theta_2}(\rho_0, \rho'_0) B_0(n/k)(1 + o_p(1)) \right\} + \frac{\alpha\tau_q}{U_0(n/k)} \left\{ d_{\alpha,\theta_1,\theta_2}(-\xi) + w_{\alpha,\theta_1,\theta_2,\tau}(\xi, \rho_0) A_0(n/k)(1 + o_p(1)) + \frac{\chi_q}{U_0(n/k)} (1 + o_p(1)) \right\},\]

i.e.

\[D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}(\xi) \equiv D_{n,k}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi) + \frac{\alpha\tau_q}{U_0(n/k)} \left\{ d_{\alpha,\theta_1,\theta_2}(-\xi) + w_{\alpha,\theta_1,\theta_2,\tau}(\xi, \rho_0) A_0(n/k)(1 + o_p(1)) + \frac{\chi_q}{U_0(n/k)} (1 + o_p(1)) \right\}.\]

The dominant component of the right hand-side of (3.6) depends on the relative behaviour of the functions \(A_0(t)\) and \(1/U_0(t)\). We shall thus consider three different regions related to \(\chi_q\), in (1.9), and the vector \((\xi, \rho_0)\) of the unshifted model \(F_0\) associated with the available data:

- \(R_1 := \{ F_0 : \xi + \rho_0 < 0 \land \chi_q \neq 0 \}\),
- \(R_2 := \{ F_0 : \xi + \rho_0 > 0 \lor \chi_q = 0 \}\),
- \(R_3 := \{ F_0 : \xi + \rho_0 = 0 \land \chi_q \neq 0 \}\).

We now state the following:

**Theorem 3.1** (Henriques-Rodrigues and Gomes, 2013, Theorem 1). **Under the validity of the second-order condition in (1.6), with** \(\rho = \rho_0 < 0\), \(\rho_q\) **defined in** (1.11), \(\overline{\rho}_{n,k/T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)}\) **defined in (1.17), and with an explicit expression given in** (1.18) **for the particular case** \((\alpha, \theta_1, \theta_2) = (1, 2, 3)\), **is consistent for the estimation of** \(\rho_q\), **i.e.**

\[\overline{\rho}_{n,k/T}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} \xrightarrow{p} \rho_q, \quad n \to \infty.\]
for any real $\alpha > 0$, $\tau_q \in \mathbb{R}$, $\theta_1, \theta_2 \in \mathbb{R}^+ \setminus \{1\}$, $\theta_1 < \theta_2$ and $0 < q < 1$ or $q = 0$ if $\chi_0 = x_r$, the left endpoint of the underlying parent, is finite, provided that $k$ is an intermediate sequence, and moreover, with $A_q$ defined in (2.2),

\[
\sqrt{k}A_q(n/k) \to \infty, \text{ as } n \to \infty.
\]

**Remark 3.1.** Note that when we consider models $F_0 \in \mathcal{R}_1$, $A_0(t) = o(1/U_0(t))$ and with $A_q(t) = \xi_q/U_0(t)$, by (2.2), condition (3.7) corresponds to $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$. For models $F_0 \in \mathcal{R}_2$, $1/U_0(t) = o(A_0(t))$ and since $A_q(t) = A_0(t)$, condition (3.7) is equivalent to $\sqrt{k}A_0(n/k) \to \infty$, as $n \to \infty$. Finally, for models $F_0 \in \mathcal{R}_3$, $1/U_0(t) = O(A_0(t))$ and since $A_q(t) = A_0(t) + \xi_q/U_0(t)$, condition (3.7) is equivalent to $\sqrt{k}A_0(n/k) \to \infty$ or $\sqrt{k}/U_0(n/k) \to \infty$, as $n \to \infty$.

### 3.2. Non-degenerate asymptotic behaviour of the PORT-$\rho$ estimators

In this section, and under a third-order framework, we derive the non-degenerate asymptotic properties of the PORT-$\rho$ classes of estimators introduced with all the generality in (1.17), and particularised in (1.18). We first state the following result:

**Proposition 3.1** (Fraga Alves et al., 2003). Under the validity of the second-order condition in (1.6), with $\rho < 0$, if (1.13) holds and $\sqrt{k}A(n/k) \to \infty$, as $n \to \infty$, the asymptotic variance of $W_k^{(\alpha,\theta_1,\theta_2)}$, in (2.12), is

\[
\sigma^2_{W_{|\alpha,\theta_1,\theta_2}} = \frac{2}{\alpha} \left( \frac{\Gamma(2\alpha\theta_1)}{\theta_1^2\Gamma^2(\alpha\theta_1)} + \frac{\Gamma(2\alpha\theta_2)}{\theta_2^2\Gamma^2(\alpha\theta_2)} - \frac{(\theta_1+\theta_2)\Gamma(\alpha(\theta_1+\theta_2))}{\theta_1^2\Gamma^2(\alpha\theta_1)\Gamma(\alpha\theta_2)} \right) - \left( \frac{1}{\theta_1} - \frac{1}{\theta_2} \right)^2,
\]

and the asymptotic covariance of $(W_k^{(\alpha,\theta_1,\theta_2)}, W_k^{(\alpha,\theta_1,\theta_2)})$ is given by

\[
\sigma_{W_{|\alpha,\theta_1,\theta_2}} = \frac{1}{\alpha} \left( \frac{(\theta_1+1)\Gamma(\alpha(\theta_1+1))}{\theta_1^2\Gamma(\alpha\theta_1)} - \frac{(\theta_2+1)\Gamma(\alpha(\theta_2+1))}{\theta_2^2\Gamma(\alpha\theta_2)} - \frac{2\Gamma(2\alpha\theta_1)}{\theta_1^2\Gamma^2(\alpha\theta_1)} \right) + \left( \frac{1}{\theta_1} - \frac{1}{\theta_2} \right) \left( \frac{1}{\theta_1} - \frac{1}{\theta_2} \right) \left( \frac{1}{\theta_1} - \frac{1}{\theta_2} \right).
\]

Note that $t'_{\alpha,\theta_1,\theta_2}(\rho) := dt_{\alpha,\theta_1,\theta_2}(\rho)/d\rho$, with $t_{\alpha,\theta_1,\theta_2}(\rho)$ defined in (1.16), is given by

\[
t'_{\alpha,\theta_1,\theta_2}(\rho)(1-\rho) \left( (\theta_2-\theta_1)(1-\rho)^{\alpha\theta_2} - \theta_2(1-\rho)^{\alpha(\theta_2-\theta_1)} + \theta_1 \right)^2
\]

\[
= \alpha\theta_1\theta_2 \left\{ \theta_1(\theta_2 - 1)(1-\rho)^{\alpha(\theta_2-\theta_1)} + (1,(1-\rho)^{\alpha(\theta_2-\theta_1)+1}) \right. \\
- (\theta_2-\theta_1)(1-\rho)^{\alpha(\theta_2-\theta_1)} \left( 1 + (1-\rho)^{\alpha(\theta_2-\theta_1)+1} \right) \\
- \theta_2(\theta_1-1)(1-\rho)^{\alpha\theta_2} \left( 1 + (1-\rho)^{\alpha(\theta_2-\theta_1)+1} \right) \right\}.
\]
Let us further use the notations,

\[ y_{T}^{(a, b_1, b_2, \tau)}(\xi, \rho) := \frac{w_{a, 1, b_1, \tau}(\xi) - t_{a, b_1, b_2}(\rho) u_{a, 1, b_2, \tau}(\xi)}{d_{a, 1, b_2}(\rho)}, \]

\[ y_{\rho|T}^{(a, b_1, b_2, \tau)}(\xi, \rho) := \frac{w_{a, 1, b_1, \tau}(\xi, \rho)}{d_{a, 1, b_2}(\rho)}, \]

\[ z_{T}^{(a, b_1, b_2)}(\xi, \rho) := \frac{d_{a, 1, b_1}(\rho) - t_{a, b_1, b_2}(\rho) d_{a, 1, b_2}(\rho)}{\xi d_{a, 1, b_2}(\rho)}, \]

\[ z_{\rho|T}^{(a, b_1, b_2)}(\xi, \rho) := \frac{\xi}{d_{a, 1, b_2}(\rho)}, \]

\[ u_{T}^{(a, b_1, b_2, \tau)}(\rho) := \frac{u_{a, 1, b_1}(\rho) - t_{a, b_1, b_2}(\rho) u_{a, 1, b_2}(\rho)}{d_{a, 1, b_2}(\rho)}, \]

\[ u_{\rho|T}^{(a, b_1, b_2, \tau)}(\rho) := \frac{\xi}{d_{a, 1, b_2}(\rho)}, \]

\[ v_{T}^{(a, b_1, b_2, \tau)}(\rho, \rho') := \frac{v_{a, 1, b_1}(\rho, \rho') - t_{a, b_1, b_2}(\rho) v_{a, 1, b_2}(\rho, \rho')}{d_{a, 1, b_2}(\rho)}, \]

\[ v_{\rho|T}^{(a, b_1, b_2, \tau)}(\rho, \rho') := \frac{v_{a, 1, b_1}(\rho, \rho')}{d_{a, 1, b_2}(\rho)}, \]

\[ f_{T}^{(a, b_1, b_2)}(\xi, \rho) := \xi \frac{d_{a, 1, b_1}(\rho) - t_{a, b_1, b_2}(\rho) d_{a, 1, b_2}(\rho)}{d_{a, 1, b_2}(\rho)}, \]

\[ f_{\rho|T}^{(a, b_1, b_2)}(\xi, \rho) := \frac{\xi}{d_{a, 1, b_2}(\rho)}, \]

\[ g_{T}^{(a, b_1, b_2, \tau)}(\xi, \rho) := \frac{w_{a, 1, b_1, \tau}(\xi, \rho) - t_{a, b_1, b_2}(\rho) w_{a, 1, b_2, \tau}(\xi, \rho)}{d_{a, 1, b_2}(\rho)}, \]

\[ g_{\rho|T}^{(a, b_1, b_2, \tau)}(\xi, \rho) := \frac{w_{a, 1, b_1, \tau}(\xi, \rho)}{d_{a, 1, b_2}(\rho)}, \]

with \( t_{a, b_1, b_2}(\rho), d_{a, b_1, b_2}(\rho), u_{a, b_1, b_2, \tau}(\rho), v_{a, b_1, b_2, \tau}(\rho, \rho'), w_{a, b_1, b_2, \tau}(\xi, \rho), y_{a, b_1, b_2, \tau}(\xi) \) and \( t'_{a, b_1, b_2}(\rho) \) given in (1.16), (2.10), (3.2), (3.3), (3.4), (3.5) and (3.10), respectively.

We can finally derive the non-degenerate asymptotic behaviour of the class of PORT-\( \rho \) estimators, in (1.17).

\textbf{Theorem 3.2.} Let us assume that the third-order condition in (2.3) holds, with \( \rho_0, \rho'_0 < 0 \) and consider the PORT-\( \rho \) class of estimators, \( \rho_{(a, b_1, b_2, \tau, \eta, q)} \), defined in (1.17), with \( \rho_q \) given in (1.11). Then, with \( \theta_1 < \theta_2 \), real numbers different from 1, \( \alpha > 0, \tau_q \in \mathbb{R} \) and \( 0 < q < 1 \) or \( q = 0 \) provided that \( \chi_0 = x_\nu \) is finite, and intermediate sequences of positive integers \( k = k_n \), as in (1.13), such that (3.7) holds, we have:
i) In $\mathcal{R}_1$, let us consider the regions

$\mathcal{R}_{11} := \{\rho_0 < -2\xi \land \chi_q \neq 0\}$,

$\mathcal{R}_{12} := \{\rho_0 = -2\xi \land \chi_q \neq 0\}$

and $\mathcal{R}_{13} := \{-2\xi < \rho_0 < -\xi \land \chi_q \neq 0\}$. If we further assume that $\lim_{n \to \infty} \sqrt{k} A_0(n/k) = \lambda$ and $\lim_{n \to \infty} \sqrt{k}/U_0^2(n/k) = \lambda_Y$, we get

$$
\frac{\sqrt{k}}{U_0(n/k)} \left( \rho_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)} - \rho_q \right) \xrightarrow{d} n \to \infty \mathcal{N} \left( \mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)}, \sigma_{\rho_0|T,\alpha,\theta_1,\theta_2,\tau_\gamma,q}^2 \right),
$$

with

$$
\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)} = \begin{cases} 
\chi_q \lambda_Y y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\xi,-\xi), & \text{in } \mathcal{R}_{11} \\
\frac{\lambda}{\chi_q} y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\xi,\rho_0) + \frac{\lambda}{\chi_q} y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\xi,-\xi), & \text{in } \mathcal{R}_{12} \\
\frac{\lambda}{\chi_q} y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\xi,\rho_0), & \text{in } \mathcal{R}_{13},
\end{cases}
$$

$y_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ and $z_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau)}(\xi,\rho)$ defined in (3.11) and (3.12), respectively. Moreover, we have

$$
\sigma_{\rho_0|T,\alpha,\theta_1,\theta_2,\tau_\gamma,q}^2 = \sigma_{\rho_0|T,\alpha,\theta_1,\theta_2}^2 \left( \star_{\alpha,\theta_1,\theta_2} (\xi) \right)^2,
$$

where

$$
\sigma_{\rho_0|T,\alpha,\theta_1,\theta_2}^2 = \frac{1}{\alpha_0 \alpha_1 \alpha_2} \text{Var} \left( W_k^{(\alpha,\theta_1,\theta_2)}(\theta_1) - t_{\alpha,\theta_1,\theta_2}(-\xi) W_k^{(\alpha,\theta_1,\theta_2)}(\theta_1) \right)
$$

with $\sigma_{W_k^{(\alpha,\theta_1,\theta_2)}}^2, \sigma_{W_k^{(\alpha,\theta_1,\theta_2)}}^2, \sigma_{W_k^{(\alpha,\theta_1,\theta_2)}}^2, \sigma_{W_k^{(\alpha,\theta_1,\theta_2)}}^2$ and $\star_{\alpha,\theta_1,\theta_2}(\rho)$ given in (3.8), (3.9) and (3.10), respectively.

ii) In $\mathcal{R}_2$, let us consider the regions

$\mathcal{R}_{21} := \{-\xi < \rho_0 < -\frac{\xi}{2} \land \chi_q \neq 0\}$,

$\mathcal{R}_{22} := \{\rho_0 = -\frac{\xi}{2} \land \chi_q \neq 0\}$

and $\mathcal{R}_{23} := \{\frac{\xi}{2} < \rho_0 < 0 \land \chi_q = 0\}$. If we further assume that $\lim_{n \to \infty} \sqrt{k} A_0(n/k) = \lambda_A$, $\lim_{n \to \infty} \sqrt{k} A_0(n/k) B_0(n/k) = \lambda_B$ and $\lim_{n \to \infty} \sqrt{k}/U_0(n/k) = \lambda_X$, we get

$$
\sqrt{k} A_0(n/k) \left( \rho_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)} - \rho_q \right) \xrightarrow{d} n \to \infty \mathcal{N} \left( \mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)}, \sigma_{\rho_0|T,\alpha,\theta_1,\theta_2,\tau_\gamma,q}^2 \right),
$$

where with

$$
\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)} := u_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\rho_0) \lambda_A + v_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\rho_0, \rho_0') \lambda_B,
$$

and $u_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\rho)$, $v_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\rho, \rho')$ and $w_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma)}(\xi, \rho)$ given in (3.13), (3.14) and (3.15), respectively,

$$
\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)} = \begin{cases} 
\chi_q \lambda_X f_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi, \rho_0), & \text{in } \mathcal{R}_{21} \\
\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)} + \chi_q \lambda_X f_{\rho_0|T}^{(\alpha,\theta_1,\theta_2)}(\xi, \rho_0), & \text{in } \mathcal{R}_{22} \\
\mu_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau_\gamma,q)}, & \text{in } \mathcal{R}_{23}.
\end{cases}
$$
Additionally,
\[ \sigma_{\rho_0|T,\alpha,\theta_1,\theta_2=q}^2 = \sigma_{\rho_0|T,\alpha,\theta_1,\theta_2=q}^2 = \left\{ \sigma_{T|\alpha,\theta_1,\theta_2}/t_{\alpha,\theta_1,\theta_2}(\rho_0) \right\}^2, \]
with \( \sigma_{T|\alpha,\theta_1,\theta_2}^2 \) given by
\[
\sigma_{T|\alpha,\theta_1,\theta_2}^2 = \left( \frac{\xi}{\alpha_{\theta_1,\theta_2}(\rho_0)} \right)^2 \mathbb{V} \ar \left( W_k^{(\alpha,1,\theta_2)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) W_k^{(\alpha,\theta_1,\theta_2)} \right)
\]

\( \sigma_{W|\alpha,\theta_1,\theta_2}^2 \) and \( \sigma_{W|\alpha,\theta_1,\theta_2} \) defined in (3.8) and (3.9), respectively.

iii) In \( \mathcal{R}_3 \), if we further assume that \( \lim_{n \to \infty} \sqrt{k}A_0(n/k) = \lambda_A \), \( \lim_{n \to \infty} \sqrt{k}A_0(n/k)B_0(n/k) = \lambda_B \) and \( \lim_{n \to \infty} \sqrt{k}A_0(n/k)/U_0(n/k) = \lambda_{AU} \), we get
\[
\tilde{\rho}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau,q)} - \rho_q \xrightarrow{d_{n \to \infty}} N' \left( \tilde{\mu}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau,q)}, \tilde{\sigma}_{\rho_0|T,\alpha,\theta_1,\theta_2,q}^2 \right),
\]
where, with \( \tilde{\lambda} = \lim_{n \to \infty} 1/(A_0(n/k)U_0(n/k)) \neq 0 \), \( \tilde{w}_{(\alpha,\theta_1,\theta_2)}(\xi,\rho,\lambda_A) = g_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) \), \( g_{\rho|T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) \) and \( \tilde{\sigma}_{T|\alpha,\theta_1,\theta_2,q}^2 \) defined in (3.11), (3.16) and (3.17), respectively,
\[
\tilde{\mu}_{\rho_0|T}^{(\alpha,\theta_1,\theta_2,\tau,q)} = \frac{u^{(\alpha,\theta_1,\theta_2,\tau)}(\rho_0)\lambda_A + \rho^{(\alpha,\theta_1,\theta_2)}(\rho_0)}{1 + \xi \lambda q},
\]
\[
\tilde{\sigma}_{\rho_0|T,\alpha,\theta_1,\theta_2,q}^2 = \frac{\sigma_{\rho_0|T,\alpha,\theta_1,\theta_2,q}^2}{(1 + \xi \lambda q)^2} = \frac{\sigma_{T|\alpha,\theta_1,\theta_2,q}^2}{(1 + \xi \lambda q)^2} t_{\alpha,\theta_1,\theta_2}(\rho_0) \right\}^2.
\]

We finally present the non-degenerate behaviour of the PORT-\( \rho \) estimators, in (1.18).

**Corollary 3.1.** Under the validity of the third-order condition in (2.3), with \( \rho = \rho_0, \rho' = \rho'_0 < 0 \), and for the particular case \( (\alpha, \theta_1, \theta_2) = (1, 2, 3) \), we have the validity of the following asymptotic distributional representation for the PORT-\( \rho \) estimator, \( \tilde{\rho}_{k}^{(\tau,q)} \), in (1.18).

i) In \( \mathcal{R}_1 \), and with the same notation as before for \( \mathcal{R}_{11}, \mathcal{R}_{12} \) and \( \mathcal{R}_{13} \),
\[
\tilde{\rho}_{k}^{(\tau,q)} = \rho_q + \tilde{\sigma}_{\rho_0,q}^{2} \tilde{W}_k^{R_1}
\]
\[
+ \left\{ \frac{\chi_q \gamma_{\rho_0|T}(\xi,\rho)}{U_0(n/k)} (1 + \alpha_q(1)), \frac{\chi_q \gamma_{\rho_0|T}(\xi,\rho)}{U_0(n/k)} + \frac{\chi_q \gamma_{\rho_0|T}(\xi,\rho)}{U_0(n/k)} (1 + \alpha_q(1)), \right\}
\]
\[
\text{in } \mathcal{R}_{11}, \text{ in } \mathcal{R}_{12} \text{ and } \mathcal{R}_{13},
\]

\[ \text{with } \alpha_q(1), \gamma_{\rho_0|T}(\xi,\rho) \text{ and } U_0(n/k) \text{ defined in (1.18).} \]
where $W_{k}^{R_{1}}$ is asymptotically standard normal,

$$y_{p_{0}|T}(\xi) = \frac{6\xi(-4+\xi(-13+2\xi(-3+2\xi(2+\xi)^{2}))) - \xi(3+\xi)(1+2\xi)^{3}(3+2\xi)^{2}}{12(1+\xi)^{2}(1+2\xi)^{3}},$$

$$z_{p_{0}|T}(\xi, \rho_{0}) = \frac{(1+\xi)^{3}\rho_{0}(\xi+\rho_{0})}{\xi^{2}(1-\rho_{0})^{3}},$$

and $\sigma_{p_{0},q}^{2} = (1+\xi)^{6}(2\xi^{2}+2\xi+1)/(\xi\chi_{q})^{2}$. 

ii) In $\mathcal{R}_{2}$, and again with the same notation as before for $\mathcal{R}_{21}, \mathcal{R}_{22}$ and $\mathcal{R}_{23}$,

$$\hat{\rho}_{k}^{(\tau_{q},q)} \overset{d}{=} \rho_{q} + \frac{\sigma_{p_{0},q}}{\sqrt{\hat{A}_{0}(n/k)}} W_{k}^{R_{2}} + \left\{ \begin{array}{ll}
\left( \frac{\chi_{q}f_{p_{0}|T}(\xi, \rho_{0})}{A_{0}(n/k)U_{0}(n/k)} \right) (1 + o_{p}(1)), & \text{in } \mathcal{R}_{21} \\
\left( m_{p_{0},\rho_{0}'|T} + \chi_{q}f_{p_{0}|T}(\xi, \rho_{0}) \right) (1 + o_{p}(1)), & \text{in } \mathcal{R}_{22} \\
\left( m_{p_{0},\rho_{0}'|T} + 1 + o_{p}(1) \right), & \text{in } \mathcal{R}_{23},
\end{array} \right.$$ 

where $m_{p_{0},\rho_{0}'|T} = u_{p|T}(\rho)A_{0}(n/k) + v_{p|T}(\rho, \rho')B_{0}(n/k)$, with $u_{p|T}(\rho) \equiv u_{p}(\tau = \tau_{q})$ and $v_{p|T}(\rho, \rho') \equiv v_{p,\rho'}$, given by

(3.18) $u_{p} \equiv u_{p}(\tau) = \frac{\rho(\rho(42-45\tau)+\rho^{3}(96-44\tau)+8\rho^{4}(\tau-3)+9\tau+2\rho^{2}(37\tau-60))}{12(1-3\rho^{2}+2\rho')^{2}}$

and

(3.19) $v_{p,\rho'} = (1-\rho)^{3}\rho' (\rho+\rho')/(\rho (1-\rho-\rho')^{2})$

respectively. Moreover, $W_{k}^{R_{2}}$ is asymptotically standard normal,

$$\sigma_{p_{0},q}^{2} \equiv \sigma_{p_{0}}^{2} = \xi^{2}(1-\rho_{0})^{6}(2\rho_{0}^{2} - 2\rho_{0} + 1)/\rho_{0}^{2},$$

$$f_{p_{0}|T}(\xi, \rho_{0}) \equiv \frac{\xi^{2}(1-\rho_{0})^{3}(\xi+\rho_{0})}{(1+\xi)^{3}\rho_{0}}.$$ 

iii) In $\mathcal{R}_{3}$, and with $\tilde{\lambda} = \lim_{n \to \infty} 1/(A_{0}(n/k)U_{0}(n/k)) = (\xi\beta_{0}C)^{-1} \neq 0$, with $C$ given in (2.4),

$$\hat{\rho}_{k}^{(\tau_{q},q)} \overset{d}{=} \rho_{q} + \frac{\tilde{\sigma}_{p_{0},q}}{\sqrt{\tilde{A}_{0}(n/k)}} W_{k}^{R_{3}} + \left( \tilde{u}_{p_{0}|T}A_{0}(n/k) + \tilde{v}_{p_{0},\rho_{0}'|T}B_{0}(n/k) + \xi\chi_{q}\frac{\tilde{g}_{p,\rho_{0}'|T}}{U_{0}(n/k)} \right) (1 + o_{p}(1)),$$

where $W_{k}^{R_{3}}$ is an asymptotically standard normal r.v., $u_{p|T} \equiv u_{p}(\tau = \tau_{q})$ and $v_{p,\rho_{0}'|T} \equiv v_{p,\rho_{0}'}$, defined in (3.18) and (3.19), respectively, $\tilde{u}_{p|T} = u_{p|T}/(1 + \xi\chi_{q})$, $\tilde{v}_{p,\rho_{0}'|T} = v_{p,\rho_{0}'|T}/(1 + \xi\chi_{q})$, and $\xi_{p,\rho_{0}|T} = \xi_{p,\rho|T}/(1 + \xi\chi_{q})$, with $\bullet = g, y$, with

$$g_{\xi,\rho|T} = g_{\rho_{0},\rho_{0}|T} = g_{p|T}$$

$$= -\frac{6(4+\rho_{0}(-13+2\rho_{0}(3+2\rho_{0}(2-\rho_{0}^{2})))) + (3-\rho_{0})(3-2\rho_{0})(1-2\rho_{0})^{3}\tau}{6(1-\rho_{0})^{2}(1-2\rho_{0})^{3}}.$$
\[ \gamma_{\xi, \rho_{0}}|T = y_{-\rho_{0}, \rho_{0}}|T \equiv y_{\rho_{0}}|T = \frac{(3-\rho_{0})(1-3\rho_{0})^{2}}{2\rho_{0}} b(\rho_{0}, \tau), \]

\[
\begin{align*}
\gamma_{\rho, \tau} &= -\frac{(\rho-2)^{2}(4\rho-2)}{4(1-\rho)^{4}} + \frac{\tau^{2}}{(1-\rho)^{3}} - \frac{1}{2(1-\rho)^{2}} + 2 \frac{1-\rho}{(1-2\rho)^{2}} - \frac{1}{1-3\rho}\left\{(1-\rho)\left[-(\rho+3)(5\rho(\rho+3)+12)(2\rho+1)\tau - 6(6+\rho(3+2\rho)(4\rho^{5}+24\rho^{4}+42\rho^{3}+31\rho^{2}+14\rho+9))\right]\right\} \\
&= \frac{(1-\rho)}{12(3-\rho)(1+\rho)(1+2\rho)^{3} + \tilde{\sigma}_{\rho_{0}, q}^{2} = (1-\rho_{0})^{6}(2\rho_{0}^{2} - 2\rho_{0} + 1)\left(1 - \tilde{\lambda}\rho_{0}\right)^{2}.}
\end{align*}
\]

### 3.3. A few comments and conclusions

- We consider that the class of PORT-\(\rho\) estimators introduced and studied in this article is, from a theoretical point of view, a nice alternative to the classical \(\rho\)-estimators whenever, in a real data analysis, we are led to a bad performance of the classical estimators. Such a bad performance is usually due to the existence of a location \(s \neq 0\) in the available data, associated with unshifted models with \(\xi + \rho_{0} < 0\), a quite common situation in practical applications.

- Concomitantly, the development and the theoretical study of a new class of PORT-estimators of the functional \(A\), in (1.6), can lead us to SORB EVI-estimators, invariant for changes in location and MVRB for an adequate choice of \(q\), i.e. EVI-estimators of the type of the ones in Caeiro et al. (2005), Gomes et al. (2007) and Gomes et al. (2008c), but invariant for changes in location, the so-called PORT-MVRB EVI-estimators. Note that these PORT-MVRB EVI-estimators have already been studied for finite samples in Gomes et al. (2011, 2012), and exhibit a quite interesting performance.

### 4. A SMALL-SCALE MONTE-CARLO SIMULATION

We next present in Figures 1 and 2, respectively the mean values (E) and the root mean squared errors (RMSE), of the classical estimator \(\hat{\rho}_{k}^{(0)}\) and the PORT-\(\rho\) estimators \(\left\{\hat{\rho}_{k}^{(0,q)}\right\}_{q=0,0.1,0.25}\) as defined in Eq. (1.18), as a function of the sample fraction \(k/n\), for sample sizes \(n = 5000\) and \(n = 10000\). The results are associated with the output of a small-scale simulation, of size 5000, related to underlying Fréchet parents \(F_{0}(x) = \exp(-x^{-1/\xi}), x > 0\), with \(\xi = 0.25\), and the shifted model \(F_{s}(x) = \exp\left(-(x-s)^{-1/\xi}\right), x > s\), with \(s = 1\).
There is indeed only a light improvement in all estimators as the sample size increases, and a high volatility of the classical $\rho$-estimators for shifted models, as can be seen, in either Figure 1 or in Figure 2, where the RMSE of such estimator is above 2, even for $n = 10000$. For smaller values of $n$, the sample paths of all estimators are even more volatile, particularly for small sample fractions $k/n$. But if we consider a much larger sample size, $n = 100000$, there is a clear improvement only in the classical $\rho$-estimators for shifted models, as can be seen, in Figure 3.
We now would like to emphasise the following points:

• The stability of the classical $\rho$-estimators around the ‘target’ for large $k$ can be fictitious or even non-existent, unless the model is an unshifted model. As can be seen in Figures 1 and 3, left, the classical $\rho$-estimator associated with the unshifted model, $\hat{\rho}_k(0)|s=0$ is close to $-1$ for large values of $k$, as expected, but the $\rho$-estimator associated with the shifted model, $\hat{\rho}_k(0)|s=1$, that should converge to $-0.25$, exhibits no stability in the sample paths.

• We are in the region $\xi + \rho_0 < 0$ ($\xi = 0.25$, $\rho_0 = -1$). Consequently, the PORT-$\rho$ estimator should converge to $-\xi = -0.25$ for $\chi_q \neq 0$ and to $\rho_0 = -1$ for $\chi_q = 0$. Unfortunately, the pattern of the PORT-$\rho$ estimators does not depend strongly on $\chi_q$. If we decide for a large value of $k$, we obtain a value close to $-1$ if $\chi_q = 0$, but a value not a long way from $-1$ when $\chi_q \neq 0$. But if we look at the region of $k/n$ close to 0.2, the PORT-$\rho$ estimators associated with $\chi_q \neq 0$ are reasonably close to $-\xi = -0.25$, with a not too large RMSE. We shall thus be again confronted with an adequate choice of the threshold $k$.

• This means that for shifted models or PORT-$\rho$ estimators associated with $\chi_q \neq 0$, the optimal level is clearly attained for not very large $k$, as can be seen in Figures 2 and 3, right, when we look at the minimal RMSE.

• For $\chi_q = 0$, the PORT-$\rho$ estimator is able to beat the classical one regarding minimum RMSE, even for very large sample sizes.

• Similar comments apply to other simulated underlying models.
• The choice of the tuning parameters τ and τ_q is also crucial. We have here used τ_q = τ = 0. The choice τ = 0 has been heuristically suggested and used before for the ρ-estimation and the region |ρ| ≤ 1, but it is possibly not the most adequate choice for the PORT-ρ estimation. This is another interesting topic out of the scope of this paper.

5. PROOFS

Proof: [Lemma 2.1]. We begin by writing

\[ \ln U_q(tx) - \ln U_q(t) = \ln \frac{U_q(tx)}{U_q(t)} - \frac{x_q}{t_q} = \ln \left( \frac{U_q(tx)}{U_q(t)} \right) - \frac{x_q}{t_q} \]

= \xi \ln x + \ln \left( x^{\xi} \frac{U_q(tx)}{U_q(t)} \right) + \ln \left( 1 - \frac{x_q}{U_q(tx)} \right) - \ln \left( 1 - \frac{x_q}{U_q(t)} \right). \]

Using Taylor’s expansion of \ln(1 + x), as x → 0, we obtain

\[ \ln U_q(tx) - \ln U_q(t) = \xi \ln x + \ln \left( x^{\xi} \frac{U_q(tx)}{U_q(t)} \right) - \frac{x_q}{U_q(t)} + o\left( \frac{1}{U_q(t)} \right), \]

= \xi \ln x + \ln \left( x^{\xi} \frac{U_q(tx)}{U_q(t)} \right) + \frac{x_q}{U_q(t)} \left( 1 - \frac{U_q(t)}{U_q(tx)} \right) + o\left( \frac{1}{U_q(t)} \right), \]

as t → ∞. Since \( U_0(tx) \sim x^\xi U_0(t) \), t → ∞, we thus have that

\[ \ln U_q(tx) - \ln U_q(t) - \xi \ln x \]

= \ln \left( x^{\xi} \frac{U_q(tx)}{U_q(t)} \right) + \frac{x_q}{U_q(t)} \left( 1 - x^{-\xi} \right) - \frac{x_q}{U_q(t)} \left( \frac{U_q(t)}{U_q(tx)} - x^{-\xi} \right) + o\left( \frac{1}{U_q(t)} \right). \]

Now, condition (1.6) with U, A and ρ replaced with U_0, A_0 and ρ_0, respectively, ascertains

\[ \ln U_q(tx) - \ln U_q(t) - \xi \ln x = A_0(t) \frac{x_q}{U_0(t)} \left( 1 - x^{-\xi} \right) - \frac{x_q}{U_0(t)} \left( \frac{U_0(t)}{U_0(tx)} - x^{-\xi} \right) + o\left( \frac{1}{U_0(t)} \right). \]

The precise result thus follows by noting that 1/U_0 ∈ RV_{-ξ} (hence \( x_q/U_0 \) is also in RV_{-ξ}) and that \( x^\xi U_0(t)/U_0(tx) - 1 \) divided by \( A_0(t) \) has the same limit as in (1.6), with the same second order parameter ρ_0 (cf. Proposition 6 and Corollary 7 of Neves, 2009). This result confirms a similar one for the rate of convergence of \( U_q(tx)/U_q(t) \) to \( x^\xi \) as obtained in Araújo Santos et al. (2006, Lemma 2.1).

Proof: [Proposition 2.2]. Using the same arguments as in Fraga Alves et al. (2009), bearing in mind the unshifted model (s = 0), we can write the PORT log-excesses of the observations over the random quantile \( X_{n_q; n} \), i.e. \( X_{n-i+n; n} - X_{n_q; n} \), for \( i = 1, \ldots, k \), in terms of the POT log-excesses, \( X_{n-i+n; n} - \chi_q \), over \( \chi_q := F^{-1}_0(q) = U_0(1/(1 - q)) \), as follows:

\[ \ln \left( X_{n-i+n; n} - X_{n_q; n} \right) = \ln \left( X_{n-i+n; n} - \chi_q \right) + \ln \left( 1 - \frac{X_{n-i+n; n} - \chi_q}{X_{n-i+n; n} - x_q} \right). \]
Now for the second term holds the inequality
\[ \ln \left( 1 - \frac{X_{n+1:n} - \chi_q}{X_{n+1:n} - \chi_q} \right) \leq \ln \left( 1 - \frac{X_{n+1:n} - \chi_q}{X_{n+1:n} - \chi_q} \right). \]
Since we are assuming \( \xi > 0 \) we have that \( X_{n:n} - \chi_q \xrightarrow{p} \infty \), which in conjunction with the asymptotical normality of the empirical quantile \( \sqrt{n} (X_{n:n} - \chi_q) = O_p(1) \) ascertains
\[
\sqrt{k} \ln \left( 1 - \frac{X_{n+1:n} - \chi_q}{X_{n+1:n} - \chi_q} \right) = \sqrt{k} \ln \left( \frac{X_{n+1:n} - \chi_q}{X_{n+1:n} - \chi_q} \right)(1 + o_p(1)) = \sqrt{k/n} o_p \left( \sqrt{n}(X_{n:n} - \chi_q) \right) = o_p \left( \sqrt{k/n} \right) \xrightarrow{p} 0.
\]
Then it is easily seen that, for any \( \alpha > 0 \), the PORT-moment statistics \( M_{n,k}^{(\alpha,q)} \) provided in (1.14) are asymptotically identically distributed to their POT-moment counterparts
\[
\tilde{M}_{n,k}^{(\alpha,q)} = \frac{1}{k} \sum_{i=1}^{k} \left( \ln \frac{X_{n+1:n} - \chi_q}{X_{n+1:n} - \chi_q} \right)^\alpha.
\]
In fact, \( \tilde{M}_{n,k}^{(\alpha,q)} \) differs from \( M_{n,k}^{(\alpha)} \) by a deterministic shift \( -\chi_q = -U_0(1/(1-q)) \) in the observations \( X_i, 1 \leq i \leq n \). Then the asymptotic results for \( \tilde{M}_{n,k}^{(\alpha,q)} \) can be obtained in view of the shifted observations from \( \tilde{X} := X_q = X_0 - \chi_q \), with associated \( U_q(t) = U_0(t) - \chi_q \).

Let us begin with the first moment of the log-excesses. With \( \{Y_i\}_{i=1,...,n} \) i.i.d. unit Pareto r.v.’s, we have the equality in distribution
\[
\{\tilde{X}_{n+1:n}\}_{i=1}^{n} \equiv \{X_{n+1:n} - \chi_q\}_{i=1}^{n} \equiv \{U_q(Y_{n+1:n})\}_{i=1}^{n}.
\]
and we can write,
\[
\left( \frac{\tilde{X}_{n+1:n}}{\tilde{X}_{n+1:n}} \right)^n_i = \{X_{n+1:n} - \chi_q\}_{i=1}^{n} \equiv \{U_q(Y_{n+1:n})\}_{i=1}^{n},
\]
We note that
\[
\ln U_q(tx) - \ln U_q(t) - (\ln U_0(tx) - \ln U_0(t))
\]
\[
= \ln \left( \frac{U_0(tx)}{U_0(t)} \right)_{\chi_q} - \ln \left( U_0(t) \right)_{\chi_q} - (\ln U_0(tx) - \ln U_0(t))
\]
\[
= \ln \left( x^{-\xi} U_0(tx) - 1 \right)_{\chi_q} - x^{-\xi} \chi_q_{\chi_q} + 1 \right) - \ln \left( x^{-\xi} U_0(tx) - 1 \right) + 1 - \ln (1 - \chi_q_{\chi_q}).
\]
Next, we deal with the first two terms in the above. Towards this end, we define for each \( x > 0 \),
\[
y_1(t) := \left( x^{-\xi} \frac{U_0(tx)}{U_0(t)} - 1 \right) - x^{-\xi} \frac{\chi_q}{\chi_q},
\]
\[
y_2(t) := x^{-\xi} \frac{U_0(tx)}{U_0(t)} - 1,
\]
with \( y_1(t) \) and \( y_2(t) \) converging to zero as \( t \to \infty \) (see text in the end of the proof of lemma 2.1). MacLaurin’s expansion of the logarithm, i.e. \( \ln(1 + y) = y - y^2/2 + o(y^3) \), applied to both \( y_1(t) \) and \( y_2(t) \) now yields

\[
\ln U_q(tx) - \ln U_q(t) - \left( \ln U_0(tx) - \ln U_0(t) \right)
= -x^{-\xi} \frac{\chi_y}{U_0(t)} - \frac{1}{2} \left( x^{-\xi} \frac{\chi_y}{U_0(t)} \right)^2 (1 + o(1)) + (x^{-\xi} \frac{U_0(tx)}{U_0(t)} - 1)x^{-\xi} \frac{\chi_y}{U_0(t)} (1 + o(1))
- \ln(1 - \frac{\chi_y}{U_0(t)}).
\]

In order to have a grasp at the remainder \( o(1) \)-terms, we require the following uniform bounds, which arise in connection with the third-order framework in (2.3) and Remark B.3.12 of de Haan and Ferreira (2006): for any \( \varepsilon, \delta > 0 \), there exists a \( t_0 = t_0(\varepsilon, \delta) \) such that for \( t \geq t_0 \), \( x \geq 1 \),

\[
\left| x^{-\xi} \frac{U_0(tx)}{U_0(t)} - 1 \right| \leq \frac{x^\varepsilon}{\ln(1 + x^\delta)}.
\]

Furthermore, since \( 0 < -\ln(1 - v) - v - v^2/2 < v^3/(3(1 - v)) \), \( v \in (0, 1) \), we can set \( v = \chi_y/U_0 \) in order to establish the upper bound

\[
\ln U_q(tx) - \ln U_q(t) - \left( \ln U_0(tx) - \ln U_0(t) \right)
- \xi \left( \frac{x^{-\xi}}{-\xi} \right) \frac{\chi_y}{U_0(t)} - \xi \left( \frac{x^{-2\xi}}{-2\xi} \right) \left( \frac{\chi_y}{U_0(t)} \right)^2 - x^{-\xi} \left( \frac{\chi_y}{U_0(t)} \right)^2 A_0(\chi_y) B_0(t)
\leq \frac{x^3}{3} \left( U_0^3(t)(1 - \frac{\chi_y}{U_0(t)}) \right)^{-1} + x^{-\xi} \frac{x^\rho + \rho_0 - 1}{\rho_0 + \rho_0} \chi_y \frac{A_0(\chi_y)}{U_0(t)} B_0(t) + \varepsilon \left| \frac{A_0(\chi_y)}{U_0(t)} \right| B_0(t) \left| x^{-\xi} + \rho_0 + \rho_0' + \delta \right|.
\]

We can also establish a similar lower bound. In this development, and with respect to the right hand side of (5.1), assuming \( k = k_n \) an intermediate sequence of positive integers, i.e. such that (1.13) holds, then taking average across \( i = 1, 2, ..., k \), for arbitrary \( \varepsilon, \delta > 0 \), the weak law of large numbers ensures that

\[
M_{n,k}^{(1,q)} - M_{n,k}^{(1)} = \frac{\chi_y}{U_0(n/k)} \left( \frac{\xi}{1 + \xi} + \frac{\xi}{1 + 2\xi} \frac{\chi_y}{U_0(n/k)} + \frac{\chi_y}{1 + \xi} \frac{A_0(n/k)}{U_0(t)} \right) (1 + o_p(1))
\]

We are then led to (2.16) for \( \alpha = 1 \) where

\[
\frac{\xi}{1 + \xi} = \xi \mu_1^{(2)}(-\xi), \quad \frac{1}{(1 + \xi)(1 + \xi - \rho_0)} = \mu_1^{(2)}(\xi, \rho_0) \quad \text{and} \quad \frac{\xi}{1 + 2\xi} = \xi \mu_1^{(2)}(-2\xi).
\]

Let us next consider a general \( \alpha \). Similarly as before, we can write

\[
\left( \ln U_q(tx) - \ln U_q(t) \right)^\alpha - \left( \ln U_0(tx) - \ln U_0(t) \right)^\alpha = \frac{\alpha(\ln x)^\alpha \chi_y}{U_0(t)} \left( \frac{1}{\ln x} \left( x^{-\xi} - 1 \right) \right)
+ \frac{1}{\ln x} \left( x^{-\xi} - 1 \right) \left( \frac{\chi_y}{U_0(t)} \right)^2 \left( x^{-\xi} - 1 \right) \left( \frac{x^{-2\xi}}{-2\xi} \right) A_0(t)
+ \frac{1}{\ln x} \left( \frac{x^{-\xi}}{-\xi} \right) \left( \frac{\chi_y}{U_0(t)} \right)^2 \left( x^{-\xi} - 1 \right) \left( \frac{\chi_y}{U_0(t)} \right)^2 \left( x^{-\xi} - 1 \right) \right) + o(1/U_0^2(t)).
\]
Considering again \( k = k_n \) as an intermediate sequence of integers, i.e. (1.13) holds, the same type of arguments of the previous case (\( \alpha = 1 \)), and the weak law of large numbers enable us to write (2.16) for any \( \alpha > 0 \).

\[ \text{Proof: [Theorem 3.2].} \] (i) In the region \( \mathcal{R}_1 \), \( A_0(t) = o(1/U_0(t)) \), as \( t \to \infty \), the third and last term of the right-hand side of (3.6) is the dominant one, and the r.v.'s \( D^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n,k}((T)/U_0(n/k)) \) converge in probability to \( \alpha \tau_q \xi_{\alpha \tau_q} \chi_{q} d_{\alpha, \theta_1, \theta_2}(-\xi) \) provided that (3.7) holds, i.e. if \( \sqrt{k}/U_0(n/k) \to \infty \), as \( n \to \infty \) (see Remark 3.1). Moreover, we get

\[
\frac{D^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n,k}}{1/U_0(n/k)} \xrightarrow{d} \xi_{\alpha \tau_q} \left( \alpha \tau_q \chi_{q} d_{\alpha, \theta_1, \theta_2}(-\xi) + \frac{\tau_q W_k^{(\alpha, \theta_1, \theta_2)}}{\sqrt{k}} U_0(n/k) \right) + \alpha \tau_q \left\{ \frac{d_{\alpha, \theta_1, \theta_2} \left( \omega_0(n/k)/U_0(n/k) \right)}{\xi} \right\}.
\]

For levels \( k \) such that (1.13) holds, with \( W_k^{(\alpha, \theta_1, \theta_2)} \) given in (1.2), and with \( T^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n,k} \) defined in (1.15), we can say that if (3.7) holds,

\[
T^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n,k} \xrightarrow{d} \frac{t_{\alpha, \theta_1, \theta_2}(-\xi) + \left( \frac{d_{\alpha, \theta_1, \theta_2}(-\xi)}{\alpha \chi_{q}} \right) - 1 \left( W_k^{(\alpha, \theta_1, \theta_2)} - t_{\alpha, \theta_1, \theta_2}(-\xi) W_k^{(\alpha, \theta_1, \theta_2)} \right)}{\alpha \chi_{q} \sqrt{k}/U_0(n/k)} + \frac{\chi_{q} y_{\tau q}^{(\alpha, \theta_1, \theta_2, \tau_q)}(\xi, \alpha \tau_q, \xi)}{U_0(n/k)}.
\]

For sequences of positive intermediate integers \( k = k_n \) such that \( k_n = o(n) \), \( \sqrt{k}/U_0(n/k) \to \infty \), \( \sqrt{k}/U_0(n/k) \to \lambda \) and \( \sqrt{k}/U_0^2(n/k) \to \lambda' \), as \( n \to \infty \), let us consider the following cases:

- If \( \xi + \rho_0 < -\xi \) and \( \chi_{q} \neq 0 \), then

\[
T^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n,k} \xrightarrow{d} \frac{t_{\alpha, \theta_1, \theta_2}(-\xi)}{\xi} + \frac{\left( \frac{d_{\alpha, \theta_1, \theta_2}(-\xi)}{\alpha \chi_{q}} \right) - 1 \left( W_k^{(\alpha, \theta_1, \theta_2)} - t_{\alpha, \theta_1, \theta_2}(-\xi) W_k^{(\alpha, \theta_1, \theta_2)} \right)}{\alpha \chi_{q} \sqrt{k}/U_0(n/k)} + \frac{\chi_{q} y_{\tau q}^{(\alpha, \theta_1, \theta_2, \tau_q)}(\xi, \alpha \tau_q, \xi)}{U_0(n/k)},
\]

and

\[
\frac{\sqrt{k}}{U_0(n/k)} \left( T^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n,k} - t_{\alpha, \theta_1, \theta_2}(-\xi) \right) \xrightarrow{d} \frac{\tilde{N}(\mu_{T_{\alpha, \theta_1, \theta_2, \tau_q, q}}, \sigma_{T_{\alpha, \theta_1, \theta_2, \tau_q, q}})}{n \to \infty} \to \lambda_{U} \chi_{q} y_{\tau q}^{(\alpha, \theta_1, \theta_2, \tau_q)}(\xi, \alpha \tau_q, \xi),
\]

where \( \mu_{T_{\alpha, \theta_1, \theta_2, \tau_q, q}} = \lambda_{U} \chi_{q} y_{\tau q}^{(\alpha, \theta_1, \theta_2, \tau_q)}(\xi, \alpha \tau_q, \xi) \) defined in (3.11).

- If \( \xi + \rho_0 = -\xi \) and \( \chi_{q} \neq 0 \), then

\[
T^{(\alpha, \theta_1, \theta_2, \tau_q, q)}_{n,k} \xrightarrow{d} \frac{t_{\alpha, \theta_1, \theta_2}(-\xi)}{\xi} + \frac{\left( \frac{d_{\alpha, \theta_1, \theta_2}(-\xi)}{\alpha \chi_{q}} \right) - 1 \left( W_k^{(\alpha, \theta_1, \theta_2)} - t_{\alpha, \theta_1, \theta_2}(-\xi) W_k^{(\alpha, \theta_1, \theta_2)} \right)}{\alpha \chi_{q} \sqrt{k}/U_0(n/k)} + \frac{\chi_{q} y_{\tau q}^{(\alpha, \theta_1, \theta_2, \tau_q)}(\xi, \alpha \tau_q, \xi)}{U_0(n/k)},
\]
and
\[
\frac{\sqrt{k}}{U_0(n/k)} \left( T_{\alpha,\theta_1,\theta_2,\tau_\eta,q}^{(\alpha,\theta_1,\theta_2,\tau_\eta,q)} - t_{\alpha,\theta_1,\theta_2}(-\xi) \right) \xrightarrow{\text{d}} \frac{d}{n \to \infty} N(\mu_{T[\alpha,\theta_1,\theta_2,\tau_\eta,q]}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho), \sigma_{T[\alpha,\theta_1,\theta_2]}^{(\alpha,\theta_1,\theta_2)}),
\]
where \( \mu_{T[\alpha,\theta_1,\theta_2,\tau_\eta,q]}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) \) and \( \sigma_{T[\alpha,\theta_1,\theta_2]}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) \) defined in (3.11) and (3.12), respectively.

- if \( \xi + \rho_0 > -\xi \) and \( \chi_\eta \neq 0 \), then
\[
T_{\alpha,\theta_1,\theta_2,\tau_\eta,q}^{(\alpha,\theta_1,\theta_2,\tau_\eta,q)} = \frac{d}{n \to \infty} \frac{\lambda_{T[\alpha,\theta_1,\theta_2,\tau_\eta,q]}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) + \lambda_{U,\eta}^{(\alpha,\theta_1,\theta_2,\tau_\eta,q)}(\xi,\rho)}{\chi_\eta}.
\]

(ii) In the region \( \xi + \rho_0 > 0 \), where \( 1/U_0(t) = o(A_0(t)) \), as \( t \to \infty \), or more generally in the region \( \mathcal{R}_2 \), the second term of the right-hand side of (3.6) is the dominant one. In \( \mathcal{R}_2 \), \( A_2(t) = A_0(t) \), so condition (3.7) can be rewritten as \( \sqrt{k}A_0(n/k) \to \infty \), as \( n \to \infty \) and if we assume that this condition holds,
\[
D_{\alpha,\theta_1,\theta_2,\tau_\eta,q}^{(\alpha,\theta_1,\theta_2,\tau_\eta,q)}(\xi) \xrightarrow{\text{d}} \frac{\sqrt{k}}{A_0(n/k)} \left( \alpha\tau_\eta \frac{d_{\alpha,\theta_1,\theta_2}(\rho_0)}{\xi} + \frac{\tau_\eta}{A_0(n/k)} \right)
\]
\[
+ u_{\alpha,\theta_1,\theta_2,\tau_\eta}(\rho_0)A_0(n/k)(1 + o_p(1)) + \frac{\alpha\tau_\eta}{A_0(n/k)} d_{\alpha,\theta_1,\theta_2}(-\xi).
\]
If \( \xi > -\rho_0 \) or \( \chi_\eta = 0 \), and (3.7) holds,
\[
T_{\alpha,\theta_1,\theta_2,\tau_\eta,q}^{(\alpha,\theta_1,\theta_2,\tau_\eta,q)} = \frac{d}{n \to \infty} \frac{\lambda_{T[\alpha,\theta_1,\theta_2,\tau_\eta,q]}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho) + \lambda_{U,\eta}^{(\alpha,\theta_1,\theta_2,\tau_\eta,q)}(\xi,\rho)}{\chi_\eta}.
\]
For sequences of positive intermediate integers \( k = k_n \) such that \( k_n = o(n) \), \( \sqrt{k}A_0(n/k) \to \infty \), \( \sqrt{k}A_0^2(n/k) \to \lambda_A \), \( \sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B \) and \( \sqrt{k}/U_0(n/k) \)
as $n \to \infty$, let us consider the following cases:

- if $0 < \xi + \rho_0 < -\rho_0$ and $x_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} = t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}(W_{n,k}^{(\alpha,\theta_1,\theta_2)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)W_{n,k}^{(\alpha,\theta_1,\theta_2)})}{\alpha \sqrt{k} A_0(n/k)} + \chi_q f_{T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)$$

and

$$\sqrt{k} A_0(n/k) \left( T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) \right) \xrightarrow{n \to \infty} \mathcal{N}(\mu_{T^{(\alpha,\theta_1,\theta_2,\tau_q,q)}}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0), \sigma_{T^{(\alpha,\theta_1,\theta_2)}}^2),$$

where $\mu_{T^{(\alpha,\theta_1,\theta_2,\tau_q,q)}} = \chi_q f_{T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\lambda_A + \chi_q f_{T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\lambda_B + f_{T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\lambda_A + f_{T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)\lambda_B$.

- if $\xi + \rho_0 = -\rho_0$ and $x_q \neq 0$, then

$$T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} = t_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0))^{-1}(W_{n,k}^{(\alpha,\theta_1,\theta_2)} - t_{\alpha,\theta_1,\theta_2}(\rho_0)W_{n,k}^{(\alpha,\theta_1,\theta_2)})}{\alpha \sqrt{k} A_0(n/k)} + \chi_q f_{T}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0)$$

and

$$\sqrt{k} A_0(n/k) \left( T_{n,k}^{(\alpha,\theta_1,\theta_2,\tau_q,q)} - t_{\alpha,\theta_1,\theta_2}(\rho_0) \right) \xrightarrow{n \to \infty} \mathcal{N}(\mu_{T^{(\alpha,\theta_1,\theta_2,\tau_q,q)}}^{(\alpha,\theta_1,\theta_2)}(\xi,\rho_0), \sigma_{T^{(\alpha,\theta_1,\theta_2)}}^2),$$

where $\mu_{T^{(\alpha,\theta_1,\theta_2,\tau_q,q)}} = \mu_{T^{(\alpha,\theta_1,\theta_2,\tau_q,q)}} = v_{T}^{(\alpha,\theta_1,\theta_2)}(\rho_0)\lambda_B$, with $v_{T}^{(\alpha,\theta_1,\theta_2)}(\rho,\rho')$ defined in (3.13) and (3.14), respectively, and $\sigma_{T^{(\alpha,\theta_1,\theta_2)}}^2$ is defined in (3.17).
(iii) In the region $\mathcal{R}_3$, $A_0(t)$ and $1/U_0(t)$ are of the same order, i.e. the dominant terms of the right-hand side of (3.6) are the second and the third. In $\mathcal{R}_3$, $A_q(t) = A_0(t) + \xi \chi_q/U_0(t)$, so condition (3.7) can be rewritten as $\sqrt{k}A_0(n/k) \to \infty$, as $n \to \infty$. If we assume that this condition holds with $\lambda = \lim_{n \to \infty} 1/(A_0(n/k)U_0(n/k)) \neq 0$, then

$$
\frac{D^{(\alpha,\theta_1,\theta_2,\tau,q)}_{n,k}(\xi)}{A_0(n/k)} = \frac{\xi}{\alpha} \left\{ a_{\alpha,\theta_1,\theta_2}(\rho_0) + \xi \bar{\lambda} \chi_q d_{\alpha,\theta_1,\theta_2}(-\xi) \right\} + \frac{\tau_q W^{(\alpha,\theta_1,\theta_2)}_{\lambda_{\bar{\lambda}}}}{\sqrt{k}A_0(n/k)} \\
+ \frac{\alpha \tau_q}{\xi} \left\{ u_{\alpha,\theta_1,\theta_2,\tau_q}(\rho_0)A_0(n/k)(1 + o_p(1)) + v_{\alpha,\theta_1,\theta_2}(\rho_0, \rho_0')B_0(n/k)(1 + o_p(1)) \right\}
$$

If $\xi + \rho_0 = 0$ and $\chi_q \neq 0$, if we consider levels $k$ such that $(1.13)$ and $(3.7)$ hold, then $T^{(\alpha,\theta_1,\theta_2,\tau,q)}_{n,k}(d) = d_{\alpha,\theta_1,\theta_2}(\rho_0) + \frac{\xi(d_{\alpha,\theta_1,\theta_2}(\rho_0) - 1(W^{(\alpha,\theta_1,\theta_2)}_{\lambda_{\bar{\lambda}}},t_{\alpha,\theta_1,\theta_2}(\rho_0)W^{(\alpha,\theta_1,\theta_2)}_{\lambda_{\bar{\lambda}}} - t_{\alpha,\theta_1,\theta_2}(\rho_0))}{\alpha(1 + \xi \chi_q)\sqrt{k}A_0(n/k)} \\
+ \frac{\frac{W^{(\alpha,\theta_1,\theta_2,\tau_q)}_{\lambda}(\rho_0),A_0(n/k) + W^{(\alpha,\theta_1,\theta_2,\tau_q)}_{\lambda}(\rho_0, \rho_0')B_0(n/k)(1 + o_p(1))}{\alpha(1 + \xi \chi_q)\sqrt{k}A_0(n/k)} \\
+ \left\{ \frac{\xi \chi_q g^{(\alpha,\theta_1,\theta_2,\tau_q)}_{\lambda}(\xi, \rho_0)}{(1 + \xi \chi_q)U_0(n/k)} + \frac{\xi \chi_q \bar{\lambda}^2 g^{(\alpha,\theta_1,\theta_2,\tau_q)}_{\lambda}(\xi, \rho_0)}{(1 + \xi \chi_q)U_0(n/k)} \right\}(1 + o_p(1)),
$$

with $g^{(\alpha,\theta_1,\theta_2,\tau_q)}_{\lambda}(\xi, \rho_0)$, $v^{(\alpha,\theta_1,\theta_2,\tau_q)}_{\lambda}(\xi, \rho_0)$, $u^{(\alpha,\theta_1,\theta_2,\tau_q)}_{\lambda}(\xi, \rho_0)$ and $g^{(\alpha,\theta_1,\theta_2,\tau_q)}_{\lambda}(\xi, \rho_0)$ defined in (3.11), (3.13), (3.14) and (3.16), respectively. The proof of the theorem follows for sequences of positive intermediate integers $k = k_n$ such that $k_n = o(n)$, $\sqrt{k}A_0(n/k) \to \infty$, $\sqrt{k}A^2_0(n/k) \to \lambda_A$, $\sqrt{k}A_0(n/k)B_0(n/k) \to \lambda_B$ and $\sqrt{k}A_0(n/k)/U_0(n/k) \to \lambda_{AU}$, as $n \to \infty$. 

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**REFERENCES**


