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ON THE MAXIMUM LIKELIHOOD ESTIMATOR FOR IRREGULARLY OBSERVED TIME SERIES DATA FROM COGARCH(1,1) MODELS

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Abstract:

- In this paper, we study the asymptotic properties of the maximum likelihood estimator (MLE) in COGARCH(1,1) models driven by Lévy processes as proposed by Maller *et al.* ([13]). We show that the MLE is consistent and asymptotically normal under some conditions relevant to the moments of the driving Lévy process and the sampling scheme.

Key-Words:

- *COGARCH(1,1) models; maximum likelihood estimation; consistency; asymptotic normality; sampling scheme; irregular time spaces.*

AMS Subject Classification:

- 62F12, 62M86.

1. INTRODUCTION

GARCH models are prominent stochastic models in finance, designed to capture the time-varying conditional volatilities and heavy tail phenomenon of financial time series. We refer to Bollerslev ([5]); Bougerol and Picard ([6]); Nelson ([16]); Basrak *et al.* ([1]); Berkes *et al.* ([3]), and for its estimation, to Hall and Yao ([9]); Berkes and Horváth ([2]); Francq and Zakoïan ([8]). In ordinary discrete time GARCH models, time series are assumed to be equally spaced. However, in some situations, time series are often observed irregularly. This phenomenon happens, for instance, in tick-by-tick data and daily data which is not observed on weekends and holidays. To accommodate the irregularity of time spaces, several authors have made efforts to extend the discrete time GARCH model to a continuous time counterpart. Nelson ([15]) demonstrated that the discrete time GARCH process with Gaussian innovations is a finite approximation of a bivariate diffusion process. Therein, the limiting diffusion process is driven by two independent Brownian motions, which unfortunately undermines the spirit of GARCH processes since they are originally designed to have a single innovation sequence. Later, Klüppelberg *et al.* ([12]) proposed a continuous time GARCH (COGARCH) process driven by a Lévy process, which can be seen as an analogue of discrete time GARCH process. Also, Maller *et al.* ([13]) demonstrated that the discrete time GARCH process embeds in COGARCH processes and further, the embedded GARCH process converges in a strong sense to the original COGARCH process that embeds it as the discrete grid used for obtaining the embedded process gets finer (cf. Theorem 2.1 of [13]). For more details, we refer to Kallsen and Vesenmayer ([11]).

Concerning the estimation of COGARCH parameters, Haug *et al.* ([10]) considered a method of moment estimator which is suitable for equally spaced time series and verified its consistency and asymptotic normality under some regularity conditions, which, however, is not directly applicable to irregularly spaced time series. On the other hand, Müller ([14]) proposed an MCMC-based estimation for COGARCH(1,1) models driven by a compound Poisson process, which is suitable for irregularly spaced time series, which, however, has a defect that computation is somewhat intensive. Maller *et al.* ([13]) proposed using a Gaussian maximum likelihood estimator (MLE) in COGARCH(1,1) models but its asymptotic properties such as consistency and asymptotic normality has not been thoroughly investigated yet in the literature. Motivated by this, we are led to study the asymptotic behavior of the MLE in COGARCH(1,1) models. Since some empirical study to evaluate finite sample performance has been already implemented by [13], here we focus on the rigorous verification of the asymptotic properties of the MLE.

The organization of this paper as follows. In Section 2.1, we give a brief review for COGARCH(1,1) processes. In Section 2.2, we present the main result of this paper. In Section 3, we provide the proof for the result presented in Section 2.2.

2. THE COGARCH(1,1) MODEL AND ESTIMATION

2.1. COGARCH(1,1) Processes

In this subsection, we summarize the COGARCH(1,1) process. Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t : t \geq 0\})$ be a filtered probability space satisfying the usual conditions:

- $\mathcal{F}_0$ has all the measurable sets of P-measure 0,
- each $\mathcal{F}_t$ is right continuous, i.e., $\mathcal{F}_t = \bigcap_{t<s} \mathcal{F}_s$.

Let $L := \{L_t, \mathcal{F}_t : t \geq 0\}$ be a càdlàg Lévy process with characteristic triplet (γ, ϕ, Π) satisfying $\int_{\mathbb{R}} \min\{1, x^2\}\Pi(dx) < \infty$. The characteristic function of L_t is given by

$$u \mapsto \mathrm{E}e^{\mathrm{i}uL_t} = \exp\left\{\mathrm{i}t\gamma u - \frac{t\phi^2 u^2}{2} + t\int_{\mathbb{R}} \{e^{\mathrm{i}ux} - 1 - \mathrm{i}ux1_{(|x|\leq 1)}\}\Pi(dx)\right\}.$$

which is called Lévy-Khintchine's representation (cf. Theorem 43 of Chapter I of Protter ([17])). In this paper, we assume $\phi = 0$.

Let $\eta^\circ > 0$, $\varphi^\circ > 0$, and $\beta^\circ > 0$ satisfying $\eta^\circ > \varphi^\circ$. Define $\Delta L_s := L_s - L_{s-}$ and

$$X_t := \eta^\circ t - \sum_{0<s\leq t} \log\left(1 + \varphi^\circ (\Delta L_s)^2\right),$$

which is a càdlàg process. Let σ_0^2 be an integrable random variable which is independent of $\{L_t\}$. Define

$$\sigma_t^2 := \left(\beta^\circ \int_0^t e^{X_s} ds + \sigma_0^2\right) e^{-X_{t-}},$$

which is a càglàd process. According to Proposition 3.2 of Klüppelberg *et al.* ([12]), the process $\{\sigma_t^2\}$ satisfies the stochastic integral equation

$$\sigma_t^2 - \sigma_0^2 = \int_0^t (\beta^\circ - \eta^\circ \sigma_s^2) ds + \varphi^\circ \sum_{0<s<t} \sigma_s^2 (\Delta L_s)^2. \tag{2.1}$$

Note that due to $\phi = 0$, L is a quadratic pure jump, i.e., $[L, L]_t - [L, L]_0 = \sum_{0<s\leq t} (\Delta L_s)^2$ (cf. p. 71 of [17]) and (2.1) is rewritten as

$$\sigma_t^2 - \sigma_0^2 = \int_0^t (\beta^\circ - \eta^\circ \sigma_s^2) ds + \varphi^\circ \int_{(0,t)} \sigma_s^2 d[L, L]_s, \tag{2.2}$$

i.e., $\{\sigma_{t+}^2\}$ is the almost surely unique and càdlàg solution of the stochastic differential equation

$$d\sigma_{t+}^2 = (\beta^\circ - \eta^\circ \sigma_t^2)dt + \varphi^\circ \sigma_t^2 d[L, L]_t.$$

Later, we take σ_0^2 so that the solution is strictly stationary (see (3.1)). Finally, we define the integrated COGARCH(1,1) process as

$$G_t := \int_{(0,t]} \sigma_s dL_s, \quad t \geq 0.$$

2.2. Gaussian ML Estimation

In this subsection, we consider the maximum likelihood estimation method as proposed by Maller *et al.* ([13]) and study its asymptotic properties. Particularly, we consider the situation in which $\{G_t : t \geq 0\}$ is observed discretely with irregular time spaces. For each $n \in \mathbb{N}$, we set $N = N_n \in \mathbb{N}$,

$$0 = t_0 < t_1 < \cdots < t_N < \infty, \quad \Delta t_k := t_k - t_{k-1},$$

and

$$Y_{nk} := G_{t_k} - G_{t_{k-1}},$$

where $\{\Delta t_k\}$ are allowed to be nonidentical. By putting $\Delta := \Delta_n := \max\{\Delta t_1, ..., \Delta t_N\}$, we assume that $\Delta \to 0$ and $t_N \to \infty$ as $n \to \infty$.

Let $\theta^\circ = (\beta^\circ, \varphi^\circ, \eta^\circ)'$ be the vector of (unknown) true parameters. Let $\theta = (\beta, \eta, \varphi)'$ and

$$\Theta := \{\theta = (\beta, \eta, \varphi) : \beta_* \leq \beta \leq \beta^*, \eta_* \leq \eta \leq \eta^*, \varphi_* \leq \varphi \leq \varphi^*, \eta - \varphi \geq c_*\},$$

where $0 < \beta_* < \beta^* < \infty$, $0 < \eta_* < \eta^* < \infty$, $0 < \varphi_* < \varphi^* < \infty$, and $0 < c_* < \infty$. We assume that $\theta^\circ \in \Theta$.

Following [13], we set $\tilde{\sigma}_{n,k}^2(\theta)$ $(k = 0, 1, 2, ..., N)$ to be the solution of the recursion formula:

$$\begin{aligned}
\tilde{\sigma}_{n0}^2(\theta) &:= \frac{\beta}{\eta - \varphi}, \\
\tilde{\sigma}_{nk}^2(\theta) &:= \beta \Delta t_k + e^{-\eta \Delta t_k} \tilde{\sigma}_{n,k-1}^2(\theta) + \varphi e^{-\eta \Delta t_k} Y_{nk}^2 \quad \text{for } k = 1, 2, ..., N.
\end{aligned}$$

More precisely,

$$\tilde{\sigma}_{nk}^2(\theta) = \beta \sum_{i=0}^{k-1} \Delta t_{k-i} e^{-\eta(t_k - t_{k-i})} + e^{-\eta t_k} \tilde{\sigma}_{n0}^2(\theta) + \varphi e^{-\eta t_k} \sum_{i=1}^{k} e^{\eta t_{k-i}} Y_{n,k-i+1}^2,$$

which can be viewed as an estimate of $\sigma_{t_k}^2$ when $\theta = \theta^\circ$. By observing the argument:

$$\mathrm{E}\left\{(G_{t+h} - G_t)^2 | \mathcal{F}_t\right\} = \left(\sigma_t^2 - \frac{\beta^\circ}{\eta^\circ - \varphi^\circ}\right)\left(\frac{\exp\{(\eta^\circ - \varphi^\circ)h\} - 1}{\eta^\circ - \varphi^\circ}\right) + \frac{\beta^\circ h}{\eta^\circ - \varphi^\circ},$$

provided that $\mathrm{E}\{L_1^2\} = 1$ and $\{\sigma_t^2\}$ is strictly stationary (see the proof of Proposition 5.1 of [12]), we use the terms:

$$\tilde{\rho}_{nk}^2(\theta) := \left(\tilde{\sigma}_{n,k-1}^2(\theta) - \frac{\beta}{\eta - \varphi}\right)\left(\frac{\exp\{(\eta - \varphi)\Delta t_k\} - 1}{\eta - \varphi}\right) + \frac{\beta \Delta t_k}{\eta - \varphi}$$

as estimates of conditional variances of Y_{nk} when $\theta = \theta^\circ$.

Let $m = m_n$ be a positive integer. Then we define a Gaussian log-likelihood function of $\theta = (\beta, \varphi, \eta)$ as

$$\mathcal{L}_N(\theta) := \sum_{k=m}^{N} \mathrm{l}_{nk}(\theta)\Delta t_k, \quad \mathrm{l}_{nk}(\theta) = -\left(\frac{Y_{nk}^2}{\tilde{\rho}_{nk}^2(\theta)} + \log \frac{\tilde{\rho}_{nk}^2(\theta)}{\Delta t_k}\right),$$

which is slightly different from that of [13] in which Δt_k does not appear. Below, we show that $\hat{\theta}_n$, a measurable maximum point of $\mathcal{L}_N$, i.e.,

$$\mathcal{L}_N(\hat{\theta}_n) = \max_{\theta \in \Theta} \mathcal{L}_N(\theta)$$

is consistent and asymptotically normal under some regularity conditions such as

- **C1**: $\theta^\circ \in \Theta$. $\Delta \to 0$ and $t_N \to \infty$. $t_m = o(t_N)$ and $e^{-\eta_* t_m} = O(\Delta^{1/2})$.
- **C2**: $\phi = 0$, i.e., $\{L_t : t \geq 0\}$ is a quadratic pure jump.
- **C3**: $\mathrm{E}\{L_1\} = 0$, $\mathrm{E}\{L_1^2\} = 1$, and $\mathrm{E}\{L_1^4\} < \infty$; $\Psi(2) < 0$, where $\Psi(z) := \log \mathrm{E}e^{-zX_1}$.
- **C4**: θ° is an interior point of Θ; $t_N\Delta \to 0$; $\mathrm{E}|G_h|^{4+\delta} = O(h)$ for some $\delta > 0$; $\int_{\mathbb{R}} x^3 d\Pi(x) = 0$.

The following is the main result of this paper, the proof of which is presented in the next section.

Theorem 2.1. *Under* **C1-C3**,

$$\hat{\theta}_n \xrightarrow{P} \theta^\circ. \tag{2.3}$$

Suppose that **C4** *also holds. Then,*

$$\sqrt{t_N}(\hat{\theta}_n - \theta^\circ) \Rightarrow N(\mathbf{0}, \tau\Sigma^{-1}), \tag{2.4}$$

where

$$\tau := \int_{\mathbb{R}} x^4 \Pi(dx) = \lim_{h \downarrow 0} \frac{h\mathrm{E}\left\{(G_h - G_0)^4 | \mathcal{F}_0\right\}}{\left\{\mathrm{E}\left\{(G_h - G_0)^2 | \mathcal{F}_0\right\}\right\}^2}$$

and Σ *is a positive definite matrix presented in Proposition 3.3.*

3. PROOFS

In what follows, K denotes a generic constant. We begin with the existence of a strictly stationary solution of (2.2). Let $\{L_t^*\}$ be an independent copy of $\{L_t : 0 \le t < \infty\}$. We extend the time domain of $\{L_t\}$ and $\{X_t\}$ to $\mathbb{R}$ by letting

$$L_t := -L^*_{(-t)-} \quad \text{for } -\infty < t < 0$$

and

$$X_t := \eta^\circ t + \sum_{t<s\le 0} \log\left(1 + \varphi^\circ (\Delta L_s)^2\right) \qquad \text{for } -\infty < t < 0.$$

Note that $\{L_t : t \in \mathbb{R}\}$ and $\{X_t : t \in \mathbb{R}\}$ are càdlàg processes and still have independent and strictly stationary increments. We define

$$\sigma_u^2 := \beta^\circ \int_{-\infty}^u e^{X_v - X_{u-}} dv \qquad \text{for } u \le 0. \tag{3.1}$$

Lemma 3.1. *Suppose that* **C3** *holds. Then, σ_u^2 is square integrable.*

Proof: Note that

$$\mathrm{E}\left\{\int_{-\infty}^u e^{X_v - X_{u-}} dv\right\}^2 = \lim_{h\to\infty} \mathrm{E}\left\{\int_{u-h}^u e^{X_v - X_{u-}} dv\right\}^2 < \infty,$$

(cf. the proof of Proposition 4.1 of [12]). This completes the proof. □

It can be easily checked that $\{\sigma_u^2\}$ with $\sigma_0^2 = \int_{-\infty}^0 e^{X_v - X_{0-}} dv$ is the almost surely unique strictly stationary solution of (2.2).

3.1. The Proof of Consistency

In this subsection, we assume that **C1-C3** hold. Note that

$$\sigma_0^2(\theta) := \beta/\eta + \varphi \int_{(-\infty,0)} e^{\eta u} \sigma_u^2 d[L,L]_u$$

is integrable, since

$$\mathrm{E}\int_{(-\infty,0]} e^{\eta u}\sigma_u^2 d[L,L]_u \;=\; \lim_{h\to\infty} \mathrm{E}\int_{(-h,0]} e^{\eta u}\sigma_u^2 d[L,L]_u = \mathrm{E}\sigma_0^2 \int_0^\infty e^{-\eta u} du < \infty.$$

We set

$$\sigma_t^2(\theta) := \beta/\eta + (\sigma_0^2(\theta) - \beta/\eta)e^{-\eta t} + \varphi e^{-\eta t}\int_{(0,t)} e^{\eta s}\sigma_s^2 d[L,L]_s, \quad (t > 0)$$

which is a càglàd process.

Lemma 3.2. *$\{\sigma_t^2(\theta)\}$ is strictly stationary and satisfies the stochastic differential equation*

$$d\sigma_{t+}^2(\theta) = (\beta - \eta\sigma_t^2(\theta))dt + \varphi\sigma_t^2 d[L,L]_t. \tag{3.2}$$

Especially, $\sigma_t^2(\theta) \geq \beta/\eta$ and $\mathrm{E}\sigma_t^4(\theta) < \infty$.

Proof: Note that

$$\sigma_{t+}^2(\theta) - \sigma_{0+}^2(\theta) = \left(\sigma_0^2(\theta) - \beta/\eta\right)\left(e^{-\eta t} - 1\right) + \varphi e^{-\eta t}\int_{(0,t]} e^{\eta s}\sigma_s^2 d[L,L]_s.$$

By using Fubini's theorem, we can see that

$$\begin{aligned}
&\int_0^t (\beta - \eta\sigma_s^2(\theta))ds = \int_0^t (\beta - \eta\sigma_{s+}^2(\theta))ds \\
&= \int_0^t \left\{-\eta(\sigma_0^2(\theta) - \beta/\eta)e^{-\eta s} - \varphi\eta e^{-\eta s}\int_{(0,s]} e^{\eta u}\sigma_u^2 d[L,L]_u\right\} ds \\
&= \left(\sigma_0^2(\theta) - \beta/\eta\right)\left(e^{-\eta t} - 1\right) - \varphi\int_0^t \eta e^{-\eta s}\int_{(0,s]} e^{\eta u}\sigma_u^2 d[L,L]_u ds \\
&= \left(\sigma_0^2(\theta) - \beta/\eta\right)\left(e^{-\eta t} - 1\right) - \varphi\int_{(0,t]}\left\{\int_u^t \eta e^{-\eta s}ds\right\} e^{\eta u}\sigma_u^2 d[L,L]_u \\
&= \left(\sigma_0^2(\theta) - \beta/\eta\right)\left(e^{-\eta t} - 1\right) - \varphi\int_{(0,t]}\left(e^{-\eta u} - e^{-\eta t}\right) e^{\eta u}\sigma_u^2 d[L,L]_u \\
&= \left(\sigma_0^2(\theta) - \beta/\eta\right)\left(e^{-\eta t} - 1\right) - \varphi\int_{(0,t]}\sigma_u^2 d[L,L]_u + \varphi e^{-\eta t}\int_{(0,t]} e^{\eta u}\sigma_u^2 d[L,L]_u,
\end{aligned}$$

and which implies (3.2). Now that the strict stationarity can be easily checked, $\sigma_0^2(\theta) \geq \beta/\eta$ obviously implies $\sigma_t^2(\theta) \geq \beta/\eta$. Moreover,

$$\sigma_0^2(\theta) \leq \frac{\beta}{\eta} + \sum_{j=0}^{\infty} e^{-\eta j}\int_{(-j-1,-j]}\sigma_u^2 d[L,L]_u,$$

which indicates the square integrability since $\mathrm{E}\{\int_{(0,1]}\sigma_u^2 d[L,L]_u\}^2 < \infty$ due to **C3**. This gives the lemma. □

Lemma 3.3. $\sigma_0^2(\theta^\circ) = \sigma_0^2$ a.s. *Hence,* $\sigma_t^2(\theta^\circ) = \sigma_t^2$ a.s. *for every* $t \geq 0$ *and* $\sigma_t^2 \geq \beta^\circ/\eta^\circ$.

Proof: By using Fubini's theorem, we obtain

$$\begin{aligned}
\sigma_0^2(\theta^\circ) &= \beta^\circ/\eta^\circ + \varphi^\circ \int_{(-\infty,0)} e^{\eta^\circ u}\sigma_u^2 d[L,L]_u \\
&= \beta^\circ/\eta^\circ + \varphi^\circ \int_{(-\infty,0)} e^{\eta^\circ u}\left(\beta^\circ \int_{-\infty}^{u} e^{X_v - X_{u-}} dv\right) d[L,L]_u \\
&= \beta^\circ/\eta^\circ + \beta^\circ\varphi^\circ \int_{-\infty}^{0} \left(\int_{(v,0)} e^{\eta^\circ u - X_{u-}} d[L,L]_u\right) e^{X_v} dv \\
&= \beta^\circ \int_{-\infty}^{0} e^{\eta^\circ v} dv + \beta^\circ\varphi^\circ \int_{-\infty}^{0} \left(\int_{(v,0)} e^{\eta^\circ u - X_{u-}} d[L,L]_u\right) e^{X_v} dv \\
&= \beta^\circ \int_{-\infty}^{0} \left(e^{\eta^\circ v - X_v} + \varphi^\circ \int_{(v,0)} e^{\eta^\circ u - X_{u-}} d[L,L]_u\right) e^{X_v} dv.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
e^{\eta^\circ v - X_v} &= \exp\left\{-\sum_{v<s\le 0} \log(1+\varphi^\circ(\Delta L_s)^2)\right\} \\
&= \sum_{v<w\le 0}\left[\exp\left\{-\sum_{w\le s\le 0} \log(1+\varphi^\circ(\Delta L_s)^2)\right\}\right. \\
&\qquad \left. - \exp\left\{-\sum_{w<s\le 0} \log(1+\varphi^\circ(\Delta L_s)^2)\right\}\right] + 1 \\
&= -\varphi^\circ \sum_{v<w\le 0} \exp\left\{-\sum_{w\le s\le 0} \log(1+\varphi^\circ(\Delta L_s)^2)\right\} (\Delta L_w)^2 + 1 \\
&= -\varphi^\circ \int_{(v,0]} e^{\eta^\circ w - X_{w-}} d[L,L]_w + 1,
\end{aligned}$$

and thus,

$$\sigma_0^2(\theta^\circ) = \beta^\circ \int_{-\infty}^{0} \left\{1 - \varphi^\circ e^{-X_{0-}}(\Delta L_0)^2\right\} e^{X_v} dv.$$

Since $(\Delta L_0)^2 = 0 = X_{0-}$ a.s., we obtain

$$\sigma_0^2(\theta^\circ) = \beta^\circ \int_{-\infty}^{0} e^{X_v - X_{0-}} dv = \sigma_0^2 \quad \text{a.s.}$$

This verifies the uniqueness of the solution of (3.2) and completes the proof. □

The following proposition plays a key role in proving the consistency.

Proposition 3.1. *If $\sigma_0^2(\theta) = \sigma_0^2$ a.s., then $\theta = \theta^\circ$. Hence,*

$$\Upsilon(\theta) := -\mathrm{E}\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta)} + \log\sigma_0^2(\theta)\right\}, \quad \theta \in \Theta$$

has the unique maximum at $\theta = \theta^\circ$. Moreover, $\Upsilon(\theta)$ is uniformly continuous in $\theta \in \Theta$.

Proof: Suppose that $\sigma_0^2(\theta) = \sigma_0^2$, a.s. Then, we have

$$\int_{(-\infty,0)} \left\{\varphi e^{\eta u} - \varphi^\circ e^{\eta^\circ u}\right\} \sigma_u^2 d[L,L]_u = \beta^\circ/\eta^\circ - \beta/\eta, \quad \text{a.s.}$$

Also, by the strict stationarity, for every t,

$$\int_{(-\infty,t)} \left\{\varphi e^{\eta(t-u)} - \varphi^\circ e^{\eta^\circ(t-u)}\right\} \sigma_u^2 d[L,L]_u = \beta^\circ/\eta^\circ - \beta/\eta, \quad \text{a.s.}$$

which implies $\sigma_t^2(\theta) = \sigma_t^2$, a.s. Moreover, both the processes are càglàd processes and so are indistinguishable. Thus, we have

$$\int_0^t (\beta - \eta\sigma_s^2)ds + \varphi \int_{(0,t)} \sigma_s^2 d[L,L]_s = \int_0^t (\beta^\circ - \eta^\circ\sigma_s^2)ds + \varphi^\circ \int_{(0,t)} \sigma_s^2 d[L,L]_s.$$

Suppose that $\varphi^\circ \neq \varphi$. Then,

$$\int_{(0,t)} \sigma_s^2 d[L,L]_s = \frac{1}{\varphi - \varphi^\circ} \int_0^t \left\{\beta^\circ - \beta + (\eta - \eta^\circ)\sigma_s^2\right\} ds,$$

which implies that there exist constants α, γ such that

$$\sigma_t^2 = \sigma_0^2 + \int_0^t (\alpha + \gamma\sigma_s^2)ds.$$

If $\gamma \neq 0$, $\sigma_t^2 = \gamma^{-1}\{(\alpha + \gamma\sigma_0^2)e^{\gamma t} - \alpha\}$, which contradicts the strictly stationarity of $\{\sigma_t^2\}$. On the other hand, if $\gamma = 0$, $\sigma_t^2 = \sigma_0^2 + \alpha t$. In this case, $\alpha \neq 0$ contradicts the strictly stationarity of $\{\sigma_t^2\}$ as well. Thus, $\alpha = \gamma = 0$, which in turn produces $\sigma_t^2 = \sigma_0^2$ a.s. for every $t > 0$. Then, we should have

$$0 = \beta^\circ - \eta^\circ\sigma_0^2 + \varphi^\circ\sigma_0^2\{[L,L]_1 - [L,L]_0\} \quad \text{a.s.}$$

However, the above is also false since $[L,L]_1 - [L,L]_0$ is independent of σ_0^2. Therefore, $\varphi = \varphi^\circ$. If $\eta \neq \eta^\circ$, then $\sigma_t^2 = c$ for some constant c. Thus from the same reasoning, we conclude that $\eta = \eta^\circ$, and $\beta^\circ = \beta$.

Now, we have that for $h > 0$, η_1, η_2 satisfying $\eta_* \leq \eta_2 < \eta_1 \leq \eta^*$,

$$\begin{aligned}
&\left\| \int_{(-\infty,0)} |e^{\eta_1 u} - e^{\eta_2 u}|\sigma_u^2 d[L,L]_u \right\|_2 = \left\| \int_{(-\infty,0)} e^{\eta_2 u}|e^{(\eta_1-\eta_2)u} - 1|\sigma_u^2 d[L,L]_u \right\|_2 \\
\leq\ & \Bigg\| \sup_{-h<u<0} |e^{(\eta_1-\eta_2)u} - 1| \int_{(-\infty,0)} e^{\eta_2 u}\sigma_u^2 d[L,L]_u \\
&+ e^{-\eta_2 h} \int_{(-\infty,-h]} e^{\eta_2(u+h)}\sigma_u^2 d[L,L]_u \Bigg\|_2 \\
\leq\ & \left(\sup_{-h<u<0} |e^{(\eta_1-\eta_2)u} - 1| + e^{-\eta_2 h} \right) \left\| \int_{(-\infty,0]} e^{\eta_2 u}\sigma_u^2 d[L,L]_u \right\|_2,
\end{aligned}$$

which implies

$$\lim_{\delta\to 0}\sup_{|\eta_1-\eta_2|<\delta}\left\|\int_{(-\infty,0)} e^{\eta_1 u}\sigma_u^2 d[L,L]_u - \int_{(-\infty,0)} e^{\eta_2 u}\sigma_u^2 d[L,L]_u\right\|_2 = 0.$$

This in turn implies that Υ is continuous. So the proposition is established. □

The proof of the consistency is based on the uniform convergence of the likelihood function, which can be obtained from the ergodic theorem and smoothness condition on the likelihood function.

Lemma 3.4. *Let* $\sigma_{n,k-1}^2(\theta) := \sigma_{t_{k-1}}^2(\theta)$*. Then,*

$$\frac{1}{t_N}\sum_{k=m}^{N}\left\{\frac{\sigma_{n,k-1}^2}{\sigma_{n,k-1}^2(\theta)} + \log\sigma_{n,k-1}^2(\theta)\right\}\Delta t_k \xrightarrow{P} \mathrm{E}\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta)} + \log\sigma_0^2(\theta)\right\}.$$

Proof: Since

$$\begin{aligned}&\sup_{t_{k-1}<u\le t_k}|\sigma_{n,k-1}^2(\theta)-\sigma_u^2(\theta)|\\ &\le \varphi e^{\eta\Delta t_k}\int_{t_{k-1}}^{t_k}\sigma_t^2 d[L,L]_t + \left(1-e^{-\eta\Delta t_k}\right)\varphi\int_{-\infty}^{t_{k-1}} e^{-\eta(t_{k-1}-t)}\sigma_t^2 d[L,L]_t\end{aligned}$$

and

$$\sup_{t_{k-1}<s\le t_k}\sigma_s^2(\theta)\le\sigma_{t_{k-1}}^2(\theta)+\varphi\int_{t_{k-1}}^{t_k}\sigma_t^2 d[L,L]_t,$$

we have that $\mathrm{E}\sup_{t_{k-1}<s\le t_k}\sigma_s^4(\theta)<\infty$ and

$$\max_{m\le k\le N}\left\|\sup_{t_{k-1}<u\le t_k}|\sigma_{n,k-1}^2(\theta)-\sigma_u^2(\theta)|\right\|_2 = o(1).$$

Thus, we have

$$\begin{aligned}&\mathrm{E}\left|\frac{1}{t_N}\sum_{k=m}^{N}\left\{\frac{\sigma_{n,k-1}^2}{\sigma_{n,k-1}^2(\theta)}+\log\sigma_{n,k-1}^2(\theta)\right\}\Delta t_k - \frac{1}{t_N}\int_{t_m}^{t_N}\left\{\frac{\sigma_s^2}{\sigma_s^2(\theta)}+\log\sigma_s^2(\theta)\right\}ds\right|\\ &\le\frac{1}{t_N}\sum_{k=m}^{N}\mathrm{E}\sup_{t_{k-1}<s\le t_k}\left\{\left|\frac{\sigma_{n,k-1}^2}{\sigma_{n,k-1}^2(\theta)}-\frac{\sigma_s^2}{\sigma_s^2(\theta)}\right|+\left|\log\sigma_{n,k-1}^2(\theta)-\log\sigma_s^2(\theta)\right|\right\}\Delta t_k\to 0.\end{aligned}$$

On the other hand, by the ergodic theorem,

$$\frac{1}{t_N}\int_{t_m}^{t_N}\left\{\frac{\sigma_s^2}{\sigma_s^2(\theta)}+\log\sigma_s^2(\theta)\right\}ds\xrightarrow{P}\mathrm{E}\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta)}+\log\sigma_0^2(\theta)\right\},$$

(cf. Lemma A.1). Hence, the lemma is validated. □

Lemma 3.5. *There exists a constant $c > 0$ such that for all large n,*

$$\min_{m \le k \le N} \inf_{\theta \in \Theta} \tilde{\sigma}^2_{nk}(\theta) \wedge \min_{m \le k \le N} \inf_{\theta \in \Theta} \frac{\tilde{\rho}^2_{nk}(\theta)}{\Delta t_k} > c \quad \text{a.s.}$$

Proof: Since

$$\tilde{\sigma}^2_{nk}(\theta) \ge \beta \sum_{i=0}^{k-1} \Delta t_{k-i} e^{-\eta(t_k - t_{k-i})} \quad \text{a.s.},$$

we have that for all large n,

$$\min_{m \le k \le N} \inf_{\theta \in \Theta} \tilde{\sigma}^2_{nk}(\theta) \ge \frac{\beta_*}{2\eta^*} > 0 \quad \text{a.s.}$$

and

$$\min_{m \le k \le N} \left\{ \inf_{\theta \in \Theta} \tilde{\sigma}^2_{n,k-1}(\theta) - \sup_{\theta \in \Theta} \frac{\beta}{\eta - \varphi} \left\{ \frac{e^{(\eta-\varphi)\Delta t_k} - 1}{(\eta - \varphi)\Delta t_k} - 1 \right\} \right\} \ge \frac{\beta_*}{4\eta^*} > 0 \quad \text{a.s.}$$

This completes the proof. □

Now, we prove that $\{\tilde{\sigma}^2_{nk}(\theta)\}$ approximates to $\{\sigma^2_{nk}(\theta)\}$ well.

Lemma 3.6.

$$\mathrm{E}\left(\int_{(0,h]} G_{s-}\sigma_s dL_s \right)^2 = O(h^2) \quad \textit{as } h \to 0.$$

Proof: From Corollary 4.1 of [12], we obtain

$$\mathrm{E}|\sigma_t^2 - \sigma_0^2|^2 = 2\{\mathrm{Var}(\sigma_0^2) - \mathrm{Cov}(\sigma_t^2, \sigma_0^2)\} = 2\,\mathrm{Var}(\sigma_0^2)\{1 - e^{t\Psi(1)}\},$$

i.e., $\mathrm{E}|\sigma_t^2 - \sigma_0^2|^2 = O(t)$ as $t \to 0$. Further, for $h > 0$,

$$\begin{aligned} \mathrm{E}\left(\int_{(0,h]} G_{s-}\sigma_s dL_s \right)^2 &= \int_{(0,h]} \mathrm{E}\{G^2_{s-}\sigma^2_s\} ds \\ &= \int_{(0,h]} \mathrm{E}G^2_{s-}\{\sigma^2_s - \sigma^2_0\} ds + \int_{(0,h]} \mathrm{E}G^2_{s-}\sigma^2_0 ds. \end{aligned}$$

Since

$$\left| \mathrm{E}G^2_{s-}\{\sigma^2_s - \sigma^2_0\} \right| \le \mathrm{E}^{1/2} G^4_{s-} \mathrm{E}^{1/2}(\sigma^2_s - \sigma^2_0)^2 = O(s) \quad \text{as } s \to 0$$

and

$$\mathrm{E}\{G^2_{s-}\sigma^2_0\} = \mathrm{E}\left\{ \mathrm{E}\left(G^2_{s-} | \mathcal{F}_0 \right) \sigma^2_0 \right\} = O(s) \quad \text{as } s \to 0,$$

the lemma is established. □

Lemma 3.7. *Suppose that $e^{-\eta_* t_m} = O(\Delta^{1/2})$. Then,*

$$\max_{m\le k\le N} \|\sigma^2_{nk}(\theta) - \tilde\sigma^2_{nk}(\theta)\|_2 = O(\Delta^{1/2}).$$

Proof: Since

$$\tilde\sigma^2_{nk}(\theta) = \beta \sum_{i=0}^{k} \Delta t_{k-i} e^{-\eta(t_k - t_{k-i})} + e^{-\eta t_k}\tilde\sigma^2_{n0}(\theta) + \varphi e^{-\eta t_k} \sum_{i=1}^{k} e^{\eta t_{k-i}} Y^2_{n,k-i+1}$$

and

$$\begin{aligned}
&\sum_{i=1}^{k} e^{-\eta(t_k-t_{k-i})} Y^2_{n,k-i+1} \\
&= \sum_{i=1}^{k} e^{-\eta(t_k-t_{k-i})} \left\{ [G,G]_{t_{k-i+1}} - [G,G]_{t_{k-i}} + 2\int_{(t_{k-i},t_{k-i+1}]} (G_{u-} - G_{t_{k-i}}) dG_u \right\} \\
&= \sum_{i=1}^{k} e^{-\eta(t_k-t_{k-i})} \left\{ \int_{(t_{k-i},t_{k-i+1}]} \sigma^2_u d[L,L]_u + 2\int_{(t_{k-i},t_{k-i+1}]} (G_{u-} - G_{t_{k-i}}) \sigma_u dL_u \right\},
\end{aligned}$$

we only have to deal with

$$(3.3) \qquad e^{-\eta t_k} \sum_{0<s\le t_k} e^{\eta s}\sigma^2_s d[L,L]_s - e^{-\eta t_k} \sum_{i=1}^{k} e^{\eta t_{k-i}} \int_{(t_{k-i},t_{k-i+1}]} \sigma^2_u d[L,L]_u$$

and

$$(3.4) \qquad \sum_{i=1}^{k} e^{-\eta(t_k-t_{k-i})} \int_{(t_{k-i},t_{k-i+1}]} (G_{u-} - G_{t_{k-i}}) \sigma_u dL_u.$$

Note that (3.3) is bounded by

$$\begin{aligned}
&\sum_{i=1}^{k} (e^{\eta\Delta t_{k-i+1}} - 1) e^{-\eta(t_k-t_{k-i})} \int_{(t_{k-i},t_{k-i+1}]} \sigma^2_u d[L,L]_u \\
&\quad = \sum_{i=1}^{k} (e^{\eta\Delta t_{k-i+1}} - 1) e^{-\eta(t_k-t_{k-i})} \int_{(t_{k-i},t_{k-i+1}]} \sigma^2_u du \\
&\qquad + \sum_{i=1}^{k} (e^{\eta\Delta t_{k-i+1}} - 1) e^{-\eta(t_k-t_{k-i})} \int_{(t_{k-i},t_{k-i+1}]} \sigma^2_u d\left\{[L,L]_u - u\right\},
\end{aligned}$$

where the second term is a sum of martingale differences. Thus, the L^2-norm of (3.3) is $O(\Delta^{1/2})$ uniformly in $m \le k \le N$, since

$$\begin{aligned}
\mathrm{E}\left(\int_{(s,t]} \sigma^2_u d\left\{[L,L]_u - u\right\} \right)^2 &= \mathrm{E}\left(\int_{(0,t-s]} \sigma^2_u d\left\{[L,L]_u - u\right\} \right)^2 \\
&= \mathrm{E}\sigma^4_0 \cdot \mathrm{E}[L,L]^2_1 \cdot (t-s).
\end{aligned}$$

Moreover, since (3.4) is also a sum of martingale differences, the L^2-norm of (3.4) is $O(\Delta^{1/2})$ due to Lemma 3.6. Hence, the proof is completed. □

For vector $\mathbf{x} = (x_1, x_2, x_3)'$, we denote $|\mathbf{x}| := \sqrt{\mathbf{x}'\mathbf{x}}$.

Lemma 3.8.

$$\max_{m\le k\le N}\left\|\sup_{\theta\in\Theta}\frac{1}{\tilde\rho^2_{nk}(\theta)}\left|\frac{\partial}{\partial\theta}\tilde\rho^2_{nk}(\theta)\right|\right\|_2 < \infty.$$

Proof: Due to Lemma 3.5, we have

$$\begin{aligned}\sup_{\theta\in\Theta}\frac{1}{\tilde\rho^2_{nk}(\theta)}\left|\frac{\partial}{\partial\theta}\tilde\rho^2_{nk}(\theta)\right| &\le K\sup_{\theta\in\Theta}\left\{\left|\frac{\partial}{\partial\theta}\tilde\sigma^2_{n,k-1}(\theta)\right| + O(\Delta t_k)\tilde\sigma^2_{n,k-1}(\theta)\right\}\\ &\le K\left\{1+\sum_{i=1}^{k-1}e^{-(\eta_*/2)(t_{k-1}-t_{k-i-1})}Y^2_{n,k-i-1}\right\}.\end{aligned}$$

Further, according to the proof of Lemma 3.7,

$$\max_{m\le k\le N}\left\|\sum_{i=1}^{k-1}e^{-(\eta_*/2)(t_{k-1}-t_{k-i-1})}Y^2_{n,k-i-1} - \int_{(0,t_{k-1}]}e^{-(\eta_*/2)(t_{k-1}-s)}\sigma^2_s d[L,L]_s\right\|_2 \to 0.$$

Since

$$\max_{m\le k\le N}\left\|\int_{(0,t_{k-1}]}e^{-(\eta_*/2)(t_{k-1}-s)}\sigma^2_s d[L,L]_s\right\|_2 \le \left\|\int_{(-\infty,0]}e^{-(\eta_*/2)s}\sigma^2_s d[L,L]_s\right\|_2 < \infty,$$

the lemma is validated. □

In fact, Lemma 3.10 below shows a more general result. However, Lemma 3.8 is sufficient to verify the consistency. Finally, we verify the uniform convergence of the likelihood function. In what follows, we denote

$$\rho^2_{nk}(\theta) := \left(\sigma^2_{n,k-1}(\theta) - \frac{\beta}{\eta-\varphi}\right)\left(\frac{\exp\{(\eta-\varphi)\Delta t_k\}-1}{\eta-\varphi}\right) + \frac{\beta\Delta t_k}{\eta-\varphi}$$

and $\rho^2_{nk} := \rho^2_{nk}(\theta^\circ)$.

Proposition 3.2.

$$\sup_{\theta\in\Theta}\left|\frac{1}{t_N}\mathcal{L}_N(\theta) - \Upsilon(\theta)\right| = o_P(1).$$

Proof: We have

$$\begin{aligned}&\frac{1}{t_N}\sum_{k=m}^{N}\mathrm{l}_{nk}(\theta)\Delta t_k\\ &= \frac{1}{t_N}\sum_{k=m}^{N}\{\mathrm{l}_{nk}(\theta) - \mathrm{E}(\mathrm{l}_{nk}(\theta)|\mathcal{F}_{n,k-1})\}\Delta t_k + \frac{1}{t_N}\sum_{k=m}^{N}\mathrm{E}(\mathrm{l}_{nk}(\theta)|\mathcal{F}_{n,k-1})\Delta t_k\\ &= \frac{1}{t_N}\sum_{k=m}^{N}\{\mathrm{l}_{nk}(\theta) - \mathrm{E}(\mathrm{l}_{nk}(\theta)|\mathcal{F}_{n,k-1})\}\Delta t_k - \frac{1}{t_N}\sum_{k=m}^{N}\left\{\frac{\rho^2_{nk}}{\tilde\rho^2_{nk}(\theta)} + \log\frac{\tilde\rho^2_{nk}(\theta)}{\Delta t_k}\right\}\Delta t_k,\end{aligned}$$

where the first term is a sum of martingale differences, which converges to 0 in probability. Then, we obtain from Lemmas 3.5 and 3.7 that

$$\max_{m\le k\le N}\left\|\frac{\rho_{nk}^2}{\tilde\rho_{nk}^2(\theta)}-\frac{\rho_{nk}^2}{\rho_{nk}^2(\theta)}+\log\tilde\rho_{nk}^2(\theta)-\log\rho_{nk}^2(\theta)\right\|_1 \le K\max_{m\le k\le N}\left(\left\|\sigma_{n,k-1}^2\right\|_2+1\right)\left\|\tilde\sigma_{n,k-1}^2(\theta)-\sigma_{n,k-1}^2(\theta)\right\|_2\longrightarrow 0$$

and

$$\max_{m\le k\le N}\sup_{\theta\in\Theta}\left|\frac{\rho_{nk}^2(\theta)}{\Delta t_k\sigma_{n,k-1}^2(\theta)}-1\right|=O(\Delta),$$

which implies

$$\frac{1}{t_N}\sum_{k=m}^N\left\{\frac{\rho_{nk}^2}{\tilde\rho_{nk}^2(\theta)}+\log\frac{\tilde\rho_{nk}^2(\theta)}{\Delta t_k}\right\}\Delta t_k-\frac{1}{t_N}\sum_{k=m}^N\left\{\frac{\sigma_{n,k-1}^2}{\sigma_{n,k-1}^2(\theta)}+\log\sigma_{n,k-1}^2(\theta)\right\}\Delta t_k = o_P(1).$$

Note that due to Lemma 3.4, the pointwise convergence holds:

$$\frac{1}{t_N}\sum_{k=m}^N \mathrm{l}_{nk}(\theta)\Delta t_k\xrightarrow{P}\Upsilon(\theta),\quad \text{for each }\theta\in\Theta. \tag{3.5}$$

Below, we verify the uniform convergence. Letting $\dot{\mathrm{l}}_{nk}(\theta):=\frac{\partial}{\partial\theta}\mathrm{l}_{nk}(\theta)$, we have

$$\frac{1}{t_N}\sup_{|\theta_1-\theta_2|<h}|\mathcal{L}_N(\theta_1)-\mathcal{L}_N(\theta_2)|\le\frac{1}{t_N}\sum_{k=m}^N\sup_{\theta\in\Theta}\left|\dot{\mathrm{l}}_{nk}(\theta)\right|\Delta t_k h$$

since $\theta_1+\lambda(\theta_2-\theta_1)\in\Theta$ for any $\lambda\in(0,1)$. Moreover, due to Lemmas 3.1, 3.5, and 3.8,

$$\begin{aligned}\max_{m\le k\le N}\mathrm{E}\sup_{\theta\in\Theta}\left|\dot{\mathrm{l}}_{nk}(\theta)\right| &\le \max_{m\le k\le N}\mathrm{E}\sup_{\theta\in\Theta}\left\{\frac{Y_{nk}^2}{\tilde\rho_{nk}^2(\theta)}+1\right\}\left|\frac{1}{\tilde\rho_{nk}^2(\theta)}\frac{\partial}{\partial\theta}\tilde\rho_{nk}^2(\theta)\right|\\
&\le K\max_{m\le k\le N}\mathrm{E}\left\{\frac{Y_{nk}^2}{\Delta t_k}+1\right\}\sup_{\theta\in\Theta}\left|\frac{1}{\tilde\rho_{nk}^2(\theta)}\frac{\partial}{\partial\theta}\tilde\rho_{nk}^2(\theta)\right|\\
&= \max_{m\le k\le N}\mathrm{E}\left\{\frac{\mathrm{E}(Y_{nk}^2|\mathcal{F}_{t_{k-1}})}{\Delta t_k}+1\right\}\sup_{\theta\in\Theta}\left|\frac{1}{\tilde\rho_{nk}^2(\theta)}\frac{\partial}{\partial\theta}\tilde\rho_{nk}^2(\theta)\right|\\
&\le K\max_{m\le k\le N}\mathrm{E}\left\{\sigma_{t_{k-1}}^2+1\right\}\sup_{\theta\in\Theta}\left|\frac{1}{\tilde\rho_{nk}^2(\theta)}\frac{\partial}{\partial\theta}\tilde\rho_{nk}^2(\theta)\right|\\
&\le K\max_{m\le k\le N}\left\|\sup_{\theta\in\Theta}\left|\frac{1}{\tilde\rho_{nk}^2(\theta)}\frac{\partial}{\partial\theta}\tilde\rho_{nk}^2(\theta)\right|\right\|_2<\infty.\end{aligned}$$

Therefore, we obtain

$$\lim_{h\to0}\limsup_{n\to\infty}\mathrm{E}\frac{1}{t_N}\sup_{|\theta_1-\theta_2|<h}|\mathcal{L}_N(\theta_1)-\mathcal{L}_N(\theta_2)|=0. \tag{3.6}$$

Now, for given $h > 0$, take finitely many open balls $B_h(\theta_i) := \{\theta \in \Theta : |\theta - \theta_i| < h\}$ with $\theta_i \in \Theta$ such that $\Theta \subset \bigcup_i B_h(\theta_i)$. Then,

$$\begin{aligned}
&\sup_{\theta\in\Theta}\left|\frac{1}{t_N}\mathcal{L}_N(\theta) - \Upsilon(\theta)\right| \\
&\le \max_i \sup_{\theta\in B_h(\theta_i)}\left|\frac{1}{t_N}\mathcal{L}_N(\theta) - \frac{1}{t_N}\mathcal{L}_N(\theta_i)\right| + \max_i\left|\frac{1}{t_N}\mathcal{L}_N(\theta_i) - \Upsilon(\theta_i)\right| \\
&\quad + \max_i \sup_{\theta\in B_h(\theta_i)} |\Upsilon(\theta_i) - \Upsilon(\theta)|.
\end{aligned}$$

Thus, we obtain from (3.5) that for every $\epsilon > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} P\left(\sup_{\theta\in\Theta}\left|\frac{1}{t_N}\mathcal{L}_N(\theta) - \Upsilon(\theta)\right| > \epsilon\right) \\
&\le \limsup_{n\to\infty} P\left(\sup_{|\theta_1-\theta_2|<h}\left|\frac{1}{t_N}\mathcal{L}_N(\theta_1) - \frac{1}{t_N}\mathcal{L}_N(\theta_2)\right| > \frac{\epsilon}{3}\right) \\
&\quad + P\left(\sup_{|\theta_1-\theta_2|<h} |\Upsilon(\theta_1) - \Upsilon(\theta_2)| > \frac{\epsilon}{3}\right),
\end{aligned}$$

so that the uniform convergence is achieved by letting $h \to 0$ thanks to (3.6) and Proposition 3.1. □

The Proof of Consistency. Let $\epsilon > 0$ and $B_\epsilon(\theta^\circ) := \{\theta \in \Theta : |\theta - \theta^\circ| < \epsilon\}$. Then, $\Theta - B_\epsilon(\theta^\circ)$ is compact, since Θ is taken as a compact subset in $\mathbb{R}^3$.

$$H_n = \left\{\theta \in \Theta : \Upsilon(\theta) < \Upsilon(\theta^\circ) - \frac{1}{n}\right\}, \quad n \in \mathbb{N}$$

constitute a collection of open subsets relative to Θ, which covers $\Theta - B_\epsilon(\theta^\circ)$ since $\Upsilon(\theta) < \Upsilon(\theta^\circ)$ for each $\theta \in \Theta - B_\epsilon(\theta^\circ)$ (cf. Proposition 3.1). By virtue of compactness, there is $n_0 \in \mathbb{N}$ such that $\Theta - B_\epsilon(\theta^\circ) \subset H_{n_0}$, i.e.,

$$\sup\{\Upsilon(\theta) : \theta \in \Theta - B_\epsilon(\theta^\circ)\} \le \Upsilon(\theta^\circ) - \frac{1}{n_0}.$$

Therefore, by Proposition 3.2, we have that with probability tending to 1,

$$\sup\left\{\frac{1}{t_N}\mathcal{L}_N(\theta) : \theta \in \Theta - B_\epsilon(\theta^\circ)\right\} \le \Upsilon(\theta^\circ) - \frac{1}{2n_0}.$$

On the other hand,

$$\frac{1}{t_N}\mathcal{L}_N(\theta^\circ) \xrightarrow{P} \Upsilon(\theta^\circ), \quad \frac{1}{t_N}\mathcal{L}_N(\theta^\circ) \le \frac{1}{t_N}\mathcal{L}_N(\hat{\theta}_n).$$

Hence, $\lim_{n\to\infty} P\left(\hat{\theta}_n \in B_\epsilon(\theta^\circ)\right) = 1$. □

3.2. The Proof of Asymptotic Normality

In this subsection, we assume that **C1-C4** hold. By Taylor's theorem, we have

$$(3.7)\quad \mathbf{0} = \frac{1}{\sqrt{t_N}}\sum_{k=m}^{N}\dot{\mathrm{l}}_{nk}(\hat{\theta}_n)\Delta t_k$$
$$= \frac{1}{\sqrt{t_N}}\sum_{k=m}^{N}\dot{\mathrm{l}}_{nk}(\theta^\circ)\Delta t_k + \left\{\frac{1}{t_N}\sum_{k=m}^{N}\ddot{\mathrm{l}}_{nk}(\theta_n^*)\Delta t_k\right\}\cdot\sqrt{t_N}(\hat{\theta}_n-\theta^\circ),$$

where

$$\dot{\mathrm{l}}_{nk}(\theta^\circ) = \frac{\partial}{\partial\theta}\mathrm{l}_{nk}(\theta^\circ) = \left(\frac{Y_{nk}^2}{\tilde{\rho}_{nk}^2(\theta^\circ)}-1\right)\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\tilde{\rho}_{nk}^2(\theta^\circ),$$

$$\ddot{\mathrm{l}}_{nk}(\theta) = \frac{\partial}{\partial\theta\partial\theta'}\mathrm{l}_{nk}(\theta) = \left(1-2\frac{Y_{nk}^2}{\tilde{\rho}_{nk}^2(\theta)}\right)\frac{1}{\tilde{\rho}_{nk}^2(\theta)}\frac{\partial}{\partial\theta}\tilde{\rho}_{nk}^2(\theta)\frac{1}{\tilde{\rho}_{nk}^2(\theta)}\frac{\partial}{\partial\theta'}\tilde{\rho}_{nk}^2(\theta)$$
$$+\left(\frac{Y_{nk}^2}{\tilde{\rho}_{nk}^2(\theta)}-1\right)\frac{1}{\tilde{\rho}_{nk}^2(\theta)}\frac{\partial^2}{\partial\theta\partial\theta'}\tilde{\rho}_{nk}^2(\theta).$$

More precisely,

$$\ddot{\mathrm{l}}_{nk}(\theta_n^*) = \begin{pmatrix} \frac{\partial}{\partial\theta'}\frac{\partial}{\partial\beta}\mathrm{l}_{nk}(\theta^\circ+\lambda_1(\hat{\theta}_n-\theta^\circ)) \\ \frac{\partial}{\partial\theta'}\frac{\partial}{\partial\eta}\mathrm{l}_{nk}(\theta^\circ+\lambda_2(\hat{\theta}_n-\theta^\circ)) \\ \frac{\partial}{\partial\theta'}\frac{\partial}{\partial\varphi}\mathrm{l}_{nk}(\theta^\circ+\lambda_3(\hat{\theta}_n-\theta^\circ)) \end{pmatrix}$$

for some $\lambda_1,\lambda_2,\lambda_3\in(0,1)$.

Let $\frac{\partial^q}{\partial\beta^i\partial\eta^j\partial\varphi^l}$ be a differential operator of order q, where q,i,j,l are non-negative integers with $i+j+l=q$. Observe that

$$(3.8)\quad \frac{1}{\Delta t_k}\frac{\partial^q}{\partial\beta^i\partial\eta^j\partial\varphi^l}\tilde{\rho}_{nk}^2(\theta)$$
$$= \frac{\partial^q}{\partial\beta^i\partial\eta^j\partial\varphi^l}\tilde{\sigma}_{n,k-1}^2(\theta) + O(\Delta t_k)\left\{\sum_{p\le q}\sum_{a,b,c}\frac{\partial^p}{\partial\beta^a\partial\eta^b\partial\varphi^c}\tilde{\sigma}_{n,k-1}^2(\theta)+1\right\}$$

uniformly in $\theta \in \Theta$, and

$$(3.9)\quad \frac{\partial^q}{\partial\beta^a\partial\eta^b\partial\varphi^c}\tilde{\sigma}^2_{nk}(\theta) = \frac{\partial^q}{\partial\beta^a\partial\eta^b\partial\varphi^c}\left\{\beta\sum_{i=0}^{k-1}\Delta t_{k-i}e^{-\eta(t_k-t_{k-i})}\right\} + (-1)^q 1(a=0,c=0)\varphi\sum_{i=1}^{k}(t_k-t_{k-i})^q e^{-\eta(t_k-t_{k-i})}Y^2_{n,k-i+1} + (-1)^{q-1}1(a=0,c=1)\sum_{i=1}^{k}(t_k-t_{k-i})^{q-1}e^{-\eta(t_k-t_{k-i})}Y^2_{n,k-i+1} + \frac{\partial^q}{\partial\beta^a\partial\eta^b\partial\varphi^c}\frac{\beta e^{-\eta t_k}}{\eta-\varphi}.$$

Define

$$\frac{\partial^q}{\partial\beta^a\partial\eta^b\partial\varphi^c}\sigma^2_t(\theta) := \frac{\partial^q}{\partial\beta^a\partial\eta^b\partial\varphi^c}\frac{\beta}{\eta} + (-1)^q 1(a=0,c=0)\varphi\int_{-\infty<s<t}(t-s)^q e^{-\eta(t-s)}\sigma^2_s d[L,L]_s + (-1)^{q-1}1(a=0,c=1)\int_{-\infty<s<t}(t-s)^{q-1}e^{-\eta(t-s)}\sigma^2_s d[L,L]_s.$$

Below, we show that a nice approximation to $\left\{\frac{\partial}{\partial\theta}\sigma^2_{nk}(\theta)\right\}$ is achievable similarly to Lemma 3.7. For a random vector $\mathbf{X} = (X_1, X_2, X_3)'$, we denote $\|\mathbf{X}\|_2 := \sqrt{\mathrm{E}\mathbf{X}'\mathbf{X}}$.

Lemma 3.9.

$$\max_{m\le k\le N}\left\|\frac{\partial}{\partial\theta}\tilde{\sigma}^2_{nk}(\theta) - \frac{\partial}{\partial\theta}\sigma^2_{nk}(\theta)\right\|_2 = O(\Delta^{1/2}).$$

Proof: We can express

$$\sum_{i=1}^{k}(t_k-t_{k-i})\,e^{-\eta(t_k-t_{k-i})}Y^2_{n,k-i+1} = \sum_{i=1}^{k}(t_k-t_{k-i})\,e^{-\eta(t_k-t_{k-i})}\left\{\int_{(t_{k-i},t_{k-i+1}]}\sigma^2_u\,d[L,L]_u + 2\int_{(t_{k-i},t_{k-i+1}]}(G_{u-}-G_{t_{k-i}})\sigma_u\,dL_u\right\}$$

and

$$\left|\sum_{i=1}^{k}(t_k-t_{k-i})\,e^{-\eta(t_k-t_{k-i})}\int_{(t_{k-i},t_{k-i+1}]}\sigma^2_u d[L,L]_u - \int_0^{t_k}(t_k-u)e^{-\eta(t_k-u)}\sigma^2_u d[L,L]_u\right| \le \sum_{i=1}^{k}\sup_{u\in(t_{k-i},t_{k-i+1}]}\left|(t_k-t_{k-i})\,e^{-\eta(t_k-t_{k-i})} - (t_k-u)e^{-\eta(t_k-u)}\right|\int_{(t_{k-i},t_{k-i+1}]}\sigma^2_u d[L,L]_u \le K\sum_{i=1}^{k}\Delta t_{k-i+1}e^{-\eta(t_k-t_{k-i})}(t_k-t_{k-i}+1)\int_{(t_{k-i},t_{k-i+1}]}\sigma^2_u d[L,L]_u.$$

By virtue of the above facts, the lemma can be proven in the same fashion to prove Lemma 3.7. □

Lemma 3.10. *For any $p > 0$ and any nonnegative integer q,*

$$\max_{m\le k\le N} \mathrm{E}\sup_\theta \left| \frac{1}{\tilde\sigma^2_{n,k-1}(\theta)} \frac{\partial^q \tilde\sigma^2_{n,k-1}(\theta)}{\partial\beta^i\partial\eta^j\partial\varphi^l} \right|^p \vee \mathrm{E}\sup_\theta \left| \frac{1}{\sigma^2_0(\theta)} \frac{\partial^q \sigma^2_0(\theta)}{\partial\beta^i\partial\eta^j\partial\varphi^l} \right|^p < \infty \tag{3.10}$$

and

$$\max_{m\le k\le N} \mathrm{E}\sup_\theta \left| \frac{1}{\tilde\rho^2_{nk}(\theta)} \frac{\partial^q \tilde\rho^2_{nk}(\theta)}{\partial\beta^i\partial\eta^j\partial\varphi^l} \right|^p \vee \mathrm{E}\sup_\theta \left| \frac{1}{\rho^2_{nk}(\theta)} \frac{\partial^q \rho^2_{nk}(\theta)}{\partial\beta^i\partial\eta^j\partial\varphi^l} \right|^p < \infty.$$

Proof: Assume that $p > 1$. In view of (3.9), we have

$$\left| \frac{1}{\tilde\sigma^2_{n,k-1}(\theta)} \frac{\partial^q \tilde\sigma^2_{n,k-1}(\theta)}{\partial\beta^a\partial\eta^b\partial\varphi^c} \right| \le K \frac{1 + \sum_{l=0}^1 \sum_{i=1}^k (t_k - t_{k-i})^{q-l}\, e^{-\eta(t_k - t_{k-i})} Y^2_{n,k-i+1}}{\beta/\eta + \varphi \sum_{i=1}^k e^{-\eta(t_k-t_{k-i})} Y^2_{n,k-i+1}}.$$

Since $\frac{x}{c+x} \le x^{1/p}$ holds for every $x > 0$ and $c > 0$, letting $B_j := \{i : j \le t_i < j+1,\ i \le k\}$, we have

$$\begin{aligned}
&\mathrm{E}\sup_\theta \left| \frac{\sum_{i=1}^k (t_k - t_{k-i})^q\, e^{-\eta(t_k-t_{k-i})} Y^2_{n,k-i+1}}{\beta/\eta + \varphi\sum_{i=1}^k e^{-\eta(t_k-t_{k-i})} Y^2_{n,k-i+1}} \right|^p \\
&\le \mathrm{E}\sup_\theta \left| \sum_{j=0}^{[t_k]} ([t_k] - j + 1)^q \frac{\sum_{i\in B_j} e^{-\eta(t_k-t_i)} Y^2_{n,i+1}}{\beta/\eta + \varphi\sum_{i\in B_j} e^{-\eta(t_k-t_i)} Y^2_{n,i+1}} \right|^p \\
&\le \mathrm{E}\sup_\theta \left| \frac{1}{\varphi} \sum_{j=0}^{[t_k]} ([t_k] - j + 1)^q \left(\sum_{i\in B_j} e^{-\eta(t_k-t_i)} Y^2_{n,i+1} \right)^{1/p} \right|^p \\
&\le K\mathrm{E} \left| \sum_{j=0}^{[t_k]} ([t_k] - j + 1)^q\, e^{-\eta_*/p([t_k]-j)} \left(\sum_{i\in B_j} Y^2_{n,i+1} \right)^{1/p} \right|^p \\
&\le K \left| \sum_{j=0}^{[t_k]} ([t_k] - j + 1)^q\, e^{-\eta_*/p([t_k]-j)} \left(\mathrm{E}\sum_{i\in B_j} Y^2_{n,i+1} \right)^{1/p} \right|^p < \infty
\end{aligned}$$

uniformly in $m \le k \le N$. Similarly, we also have

$$\frac{1}{\sigma^2_0(\theta)} \frac{\partial^q \sigma^2_0(\theta)}{\partial\beta^i\partial\eta^j\partial\varphi^l} \le K \frac{1 + \sum_{l=0}^1 \int_{-\infty}^0 (-u)^{q-l}\, e^{\eta u} \sigma^2_u d[L,L]_u}{\beta/\eta + \varphi\int_{-\infty}^0 e^{\eta u}\sigma^2_u d[L,L]_u}$$

and

$$
\begin{aligned}
&\mathrm{E}\sup_\theta \left| \frac{\int_{-\infty}^0 (-u)^q\, e^{\eta u}\sigma_u^2 d[L,L]_u}{\beta/\eta + \varphi \int_{-\infty}^0 e^{\eta u}\sigma_u^2 d[L,L]_u} \right|^p \\
&\le\ \mathrm{E}\sup_\theta \left| \sum_{j=0}^\infty (j+1)^q\, \frac{\int_{-j-1}^{-j} e^{\eta u}\sigma_u^2 d[L,L]_u}{\beta/\eta + \varphi \int_{-j-1}^{-j} e^{\eta u}\sigma_u^2 d[L,L]_u} \right|^p \\
&\le\ \mathrm{E}\sup_\theta \left| \frac{1}{\varphi} \sum_{j=0}^\infty (j+1)^q \left(\int_{-j-1}^{-j} e^{\eta u}\sigma_u^2 d[L,L]_u \right)^{1/p} \right|^p \\
&\le\ K\mathrm{E} \left| \sum_{j=0}^\infty (j+1)^q\, e^{-(\eta_*/p)j} \left(\int_{-j-1}^{-j} \sigma_u^2 d[L,L]_u \right)^{1/p} \right|^p \\
&\le\ K \left| \sum_{j=0}^\infty (j+1)^q\, e^{-(\eta_*/p)j} \left(\mathrm{E} \int_{-j-1}^{-j} \sigma_u^2 d[L,L]_u \right)^{1/p} \right|^p < \infty.
\end{aligned}
$$

Therefore, we obtain (3.10).

Now, due to (3.8),

$$
\begin{aligned}
\left| \frac{1}{\tilde\rho_{nk}^2(\theta)} \frac{\partial^q \tilde\rho_{nk}^2(\theta)}{\partial\beta^i\partial\eta^j\partial\varphi^l} \right| \ \le\ K &\left| \frac{1}{\sigma_{n,k-1}^2(\theta)} \frac{\partial^q \tilde\sigma_{n,k-1}^2(\theta)}{\partial\beta^i\partial\eta^j\partial\varphi^l} \right| \\
&+ O(\Delta t_k) \left\{ \sum_{p\le q} \sum_{a,b,c} \left| \frac{1}{\sigma_{n,k-1}^2(\theta)} \frac{\partial^p \tilde\sigma_{n,k-1}^2(\theta)}{\partial\beta^a\partial\eta^b\partial\varphi^c} \right| + 1 \right\}
\end{aligned}
$$

uniformly in $\theta \in \Theta$. Further, a similar argument can be applied to $\sup_\theta \left| \frac{1}{\rho_{nk}^2(\theta)} \frac{\partial^q \rho_{nk}^2(\theta)}{\partial\beta^i\partial\eta^j\partial\varphi^l} \right|$. Hence, the lemmas is proved. □

Below, we establish a central limit theorem for the asymptotic normality. To this end, we show that the score function can be approximated by a sum of square integrable martingale differences. For vector $\mathbf{x} = (x_1, x_2, x_3)'$, we denote $|\mathbf{x}| := \sqrt{\mathbf{x}'\mathbf{x}}$. And for random vector $\mathbf{X} = (X_1, X_2, X_3)'$, we denote $\|\mathbf{X}\|_1 := \mathrm{E}|\mathbf{X}|$.

Lemma 3.11. *Suppose that $t_N\Delta \to 0$. Then,*

$$
\begin{aligned}
&\frac{1}{\sqrt{t_N}} \sum_{k=m}^N \left(\frac{Y_{nk}^2}{\tilde\rho_{nk}^2(\theta^\circ)} - 1 \right) \frac{1}{\tilde\rho_{nk}^2(\theta^\circ)} \frac{\partial}{\partial\theta} \tilde\rho_{nk}^2(\theta^\circ)\Delta t_k \\
&= \frac{1}{\sqrt{t_N}} \sum_{k=m}^N \left(\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)} - 1 \right) \frac{1}{\rho_{nk}^2(\theta^\circ)} \frac{\partial}{\partial\theta} \rho_{nk}^2(\theta^\circ)\Delta t_k + o_P(1) \qquad (3.11) \\
&= \frac{1}{\sqrt{t_N}} \sum_{k=m}^N \left(\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)} - 1 \right) \frac{1}{\sigma_{n,k-1}^2(\theta^\circ)} \frac{\partial}{\partial\theta} \sigma_{n,k-1}^2(\theta^\circ)\Delta t_k + o_P(1). \qquad (3.12)
\end{aligned}
$$

Proof: Due to Lemmas 3.5, 3.7, 3.9, and 3.10, we have

$$\begin{aligned}
&\left\|\left(\frac{Y_{nk}^2}{\tilde\rho_{nk}^2(\theta^\circ)}-\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)}\right)\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right\|_1\\
&= \mathrm{E}\left\{Y_{nk}^2\left|\left(\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}-\frac{1}{\rho_{nk}^2(\theta^\circ)}\right)\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right|\right\}\\
&= \mathrm{E}\left\{\mathrm{E}\left\{Y_{nk}^2|\mathcal{F}_{t_{k-1}}\right\}\left|\left(\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}-\frac{1}{\rho_{nk}^2(\theta^\circ)}\right)\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right|\right\}\\
&= \mathrm{E}\left|\left(\frac{\rho_{nk}^2(\theta^\circ)}{\tilde\rho_{nk}^2(\theta^\circ)}-1\right)\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right|\\
&\le \left\|\frac{\rho_{nk}^2(\theta^\circ)-\tilde\rho_{nk}^2(\theta^\circ)}{\tilde\rho_{nk}^2(\theta^\circ)}\right\|_2\left\|\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right\|_2\\
&\le K\left\|\sigma_{n,k-1}^2(\theta^\circ)-\tilde\sigma_{n,k-1}^2(\theta^\circ)\right\|_2\left\|\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right\|_2 = O(\Delta^{1/2})
\end{aligned}$$

uniformly in $m \le k \le N$, and

$$\begin{aligned}
&\left\|\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)}\left\{\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}-\frac{1}{\rho_{nk}^2(\theta^\circ)}\frac{\partial\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right\}\right\|_1\\
&= \mathrm{E}\left\{\mathrm{E}(Y_{nk}^2|\mathcal{F}_{t_{k-1}})\frac{1}{\rho_{nk}^2(\theta^\circ)}\left|\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}-\frac{1}{\rho_{nk}^2(\theta^\circ)}\frac{\partial\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right|\right\}\\
&= \left\|\frac{1}{\tilde\rho_{nk}^2(\theta^\circ)}\frac{\partial\tilde\rho_{nk}^2(\theta^\circ)}{\partial\theta}-\frac{1}{\rho_{nk}^2(\theta^\circ)}\frac{\partial\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right\|_1\\
&\le K\left\{\left\|\frac{\partial\tilde\sigma_{n,k-1}^2(\theta^\circ)}{\partial\theta}-\frac{\partial\sigma_{n,k-1}^2(\theta^\circ)}{\partial\theta}\right\|_1\right.\\
&\qquad\left.+\left\|\tilde\sigma_{n,k-1}^2(\theta^\circ)-\sigma_{n,k-1}^2(\theta^\circ)\right\|_2\left\|\frac{1}{\rho_{n,k}^2(\theta^\circ)}\frac{\partial\rho_{nk}^2(\theta^\circ)}{\partial\theta}\right\|_2\right\}\\
&= O(\Delta^{1/2}),
\end{aligned}$$

uniformly in $m \le k \le N$. Thus, (3.11) follows.

Further, note that

$$\begin{aligned}
\frac{1}{\rho_{nk}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\rho_{nk}^2(\theta^\circ) &= \frac{1}{\sigma_{n,k-1}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_{n,k-1}^2(\theta^\circ)\\
&\quad+O(\Delta)\left\{\left|\frac{1}{\sigma_{n,k-1}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_{n,k-1}^2(\theta^\circ)\right|+1\right\}
\end{aligned}$$

uniformly in $m \le k \le N$ and

$$\begin{aligned}
\mathrm{E}\left| \frac{1}{\sqrt{t_N}} \sum_{k=m}^{N} \left(\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)} - 1 \right) \frac{1}{\sigma_{n,k-1}^2(\theta^\circ)} \frac{\partial}{\partial\theta} \sigma_{n,k-1}^2(\theta^\circ) \Delta t_k \right| \\
\le \frac{1}{\sqrt{t_N}} \sum_{k=m}^{N} \mathrm{E}\left(\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)} + 1 \right) \left| \frac{1}{\sigma_{n,k-1}^2(\theta^\circ)} \frac{\partial}{\partial\theta} \sigma_{n,k-1}^2(\theta^\circ) \right| \Delta t_k \\
\le \frac{1}{\sqrt{t_N}} \sum_{k=m}^{N} 2\mathrm{E} \left| \frac{1}{\sigma_{n,k-1}^2(\theta^\circ)} \frac{\partial}{\partial\theta} \sigma_{n,k-1}^2(\theta^\circ) \right| \Delta t_k = O(t_N^{1/2}),
\end{aligned}$$

so that (3.12) holds. This completes the proof. □

The following proposition and lemma are concerned with the stability of the sum of conditional variances of the score function.

Proposition 3.3. *Suppose that* **C2-C3** *hold and* $\int_{\mathbb{R}} x^3 d\Pi(x) = 0$. *Then, as* $h \downarrow 0$,

$$\mathrm{E}\left\{ (G_{t+h} - G_t)^4 | \mathcal{F}_t \right\} = h \left(\int_{\mathbb{R}} x^4 \Pi(dx) + o(1) \right) \sigma_t^4, \tag{3.13}$$

$$\mathrm{E}\left\{ (G_{t+h} - G_t)^2 | \mathcal{F}_t \right\} = h(1 + o(1)) \sigma_t^2 \tag{3.14}$$

uniformly in $t \ge 0$, *and therefore,*

$$\tau := \int_{\mathbb{R}} x^4 \Pi(dx) = \lim_{h\downarrow 0} \frac{h\mathrm{E}\left\{ (G_{t+h} - G_t)^4 | \mathcal{F}_t \right\}}{\left\{ \mathrm{E}\left\{ (G_{t+h} - G_t)^2 | \mathcal{F}_t \right\} \right\}^2} \quad \textit{for every } t \ge 0.$$

Further,

$$\Sigma := \mathrm{E} \frac{1}{\sigma_0^4(\theta^\circ)} \frac{\partial}{\partial\theta} \sigma_0^2(\theta^\circ) \frac{\partial}{\partial\theta'} \sigma_0^2(\theta^\circ)$$

is positive definite.

Proof: We defer the proof of (3.13) and (3.14) to Lemma A.2. Since $\mathrm{E}\frac{1}{\sigma_0^4(\theta^\circ)} \left| \frac{\partial}{\partial\theta} \sigma_0^2(\theta^\circ) \right|^2 < \infty$ (cf. Lemma 3.10), Σ is well defined and symmetric. Moreover, since we have that for $\lambda \in \mathbb{R}^3$, $\lambda' \frac{\partial}{\partial\theta} \sigma_0^2(\theta^\circ) = 0$ a.s. if and only if $\lambda = \mathbf{0}$, Σ is positive definite. □

For a matrix $A = (a_{ij})_{i,j=1,2,3}$, we denote $|A| := \left(\sum_{ij} |a_{ij}|^2 \right)^{1/2}$.

Lemma 3.12.

$$\frac{1}{t_N} \sum_{k=m}^{N} \frac{1}{\sigma_{n,k-1}^4(\theta^\circ)} \frac{\partial}{\partial\theta} \sigma_{n,k-1}^2(\theta^\circ) \frac{\partial}{\partial\theta'} \sigma_{n,k-1}^2(\theta^\circ) \Delta t_k \xrightarrow{P} \Sigma.$$

Proof: Notice

$$\begin{aligned}
&\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma^4_{n,k-1}(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma^2_{n,k-1}(\theta^\circ)\frac{\partial}{\partial\theta'}\sigma^2_{n,k-1}(\theta^\circ)-\frac{1}{\sigma^4_u(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma^2_u(\theta^\circ)\frac{\partial}{\partial\theta'}\sigma^2_u(\theta^\circ)\right|\\
&\le\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma^2_{n,k-1}(\theta^\circ)}-\frac{1}{\sigma^2_u(\theta^\circ)}\right|\left|\frac{\partial}{\partial\theta}\sigma^2_{n,k-1}(\theta^\circ)\frac{1}{\sigma^2_{n,k-1}(\theta^\circ)}\frac{\partial}{\partial\theta'}\sigma^2_{n,k-1}(\theta^\circ)\right|\\
&\quad+\sup_{u\in(t_{k-1},t_k]}\frac{1}{\sigma^2_u(\theta^\circ)}\left|\left(\frac{\partial}{\partial\theta}\sigma^2_{n,k-1}(\theta^\circ)-\frac{\partial}{\partial\theta}\sigma^2_u(\theta^\circ)\right)\frac{1}{\sigma^2_{n,k-1}(\theta^\circ)}\frac{\partial}{\partial\theta'}\sigma^2_{n,k-1}(\theta^\circ)\right|\\
&\quad+\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma^2_u(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma^2_u(\theta^\circ)\frac{\partial}{\partial\theta'}\sigma^2_{n,k-1}(\theta^\circ)\right|\left|\frac{1}{\sigma^2_{n,k-1}(\theta^\circ)}-\frac{1}{\sigma^2_u(\theta^\circ)}\right|\\
&\quad+\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma^2_u(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma^2_u(\theta^\circ)\frac{1}{\sigma^2_u(\theta^\circ)}\left(\frac{\partial}{\partial\theta'}\sigma^2_{n,k-1}(\theta^\circ)-\frac{\partial}{\partial\theta'}\sigma^2_u(\theta^\circ)\right)\right|.
\end{aligned}$$

We concentrate on the third term since the other terms can be treated similarly.

Note that

$$\begin{aligned}
&\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma^2_{n,k-1}(\theta^\circ)}-\frac{1}{\sigma^2_u(\theta^\circ)}\right|\left|\frac{\partial}{\partial\theta}\sigma^2_{n,k-1}(\theta^\circ)\frac{1}{\sigma^2_{n,k-1}(\theta^\circ)}\frac{\partial}{\partial\theta'}\sigma^2_{n,k-1}(\theta^\circ)\right|\\
&\le K\sup_{u\in(t_{k-1},t_k]}|\sigma^2_{n,k-1}(\theta^\circ)-\sigma^2_u(\theta^\circ)|\left|\frac{1}{\sigma^2_{n,k-1}(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma^2_{n,k-1}(\theta^\circ)\frac{1}{\sigma^2_u(\theta^\circ)}\frac{\partial}{\partial\theta'}\sigma^2_u(\theta^\circ)\right|.
\end{aligned}$$

Since

$$\begin{aligned}
&\sup_{u\in(t_{k-1},t_k]}|\sigma^2_{n,k-1}(\theta^\circ)-\sigma^2_u(\theta^\circ)|\\
&\le\varphi^\circ e^{\eta^\circ\Delta t_k}\int_{t_{k-1}}^{t_k}\sigma^2_t d[L,L]_t+\varphi^\circ\left(1-e^{-\eta^\circ\Delta t_k}\right)\int_{-\infty}^{t_{k-1}}e^{-\eta^\circ(t_{k-1}-t)}\sigma^2_t d[L,L]_t,
\end{aligned}$$

we can have

$$\max_{m\le k\le N}\left\|\sup_{u\in(t_{k-1},t_k]}|\sigma^2_{n,k-1}(\theta^\circ)-\sigma^2_u(\theta^\circ)|\right\|_2=o(1).$$

Similarly,

$$\max_{m\le k\le N}\left\|\sup_{u\in(t_{k-1},t_k]}\left|\frac{\partial}{\partial\theta}\sigma^2_{n,k-1}(\theta^\circ)-\frac{\partial}{\partial\theta}\sigma^2_u(\theta^\circ)\right|\right\|_2=o(1).$$

Moreover, we have that for $h>0$ and $p>1, q>0$,

$$\begin{aligned}
&\left|\frac{\int_{-\infty}^{h}(h-u)^{q}\, e^{-\eta(h-u)}\sigma_u^2 d[L,L]_u}{\beta/\eta+\varphi\int_{-\infty}^{h} e^{-\eta(h-u)}\sigma_u^2 d[L,L]_u}\right|^p \\
&\le \left|\frac{\int_{0}^{h} e^{-\eta(h-u)}\sigma_u^2 d[L,L]_u}{\beta/\eta+\varphi\int_{0}^{h} e^{-\eta(h-u)}\sigma_u^2 d[L,L]_u} + \sum_{j=0}^{\infty}(j+1)^q \frac{\int_{-j-1}^{-j} e^{-\eta(h-u)}\sigma_u^2 d[L,L]_u}{\beta/\eta+\varphi\int_{-j-1}^{-j} e^{-\eta(h-u)}\sigma_u^2 d[L,L]_u}\right|^p \\
&\le \left|\left(\int_0^h e^{-\eta(h-u)}\sigma_u^2 d[L,L]_u\right)^{1/p} + \frac{1}{\varphi}\sum_{j=0}^{\infty}(j+1)^q\left(\int_{-j-1}^{-j} e^{-\eta(h-u)}\sigma_u^2 d[L,L]_u\right)^{1/p}\right|^p \\
&\le \left|\left(\int_0^h \sigma_u^2 d[L,L]_u\right)^{1/p} + \frac{1}{\varphi}\sum_{j=0}^{\infty}(j+1)^q\, e^{-\eta(h+j)}\left(\int_{-j-1}^{-j} \sigma_u^2 d[L,L]_u\right)^{1/p}\right|^p .
\end{aligned}$$

Thus, it can be seen that

$$\left\|\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma_u^2(\theta^\circ)}\frac{\partial}{\partial\theta'}\sigma_u^2(\theta^\circ)\right|\right\|_4 < \infty$$

and

$$\begin{aligned}
&\left\|\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma_{n,k-1}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_{n,k-1}^2(\theta^\circ)\frac{1}{\sigma_u^2(\theta^\circ)}\frac{\partial}{\partial\theta'}\sigma_u^2(\theta^\circ)\right|\right\|_2 \\
&\le \left\|\left|\frac{1}{\sigma_{n,k-1}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_{n,k-1}^2(\theta^\circ)\right|\right\|_4 \left\|\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma_u^2(\theta^\circ)}\frac{\partial}{\partial\theta'}\sigma_u^2(\theta^\circ)\right|\right\|_4 < \infty.
\end{aligned}$$

Therefore,

$$\mathrm{E}\sup_{u\in(t_{k-1},t_k]}\left|\frac{1}{\sigma_{n,k-1}^4(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_{n,k-1}^2(\theta^\circ)\frac{\partial}{\partial\theta'}\sigma_{n,k-1}^2(\theta^\circ) - \frac{1}{\sigma_u^4(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_u^2(\theta^\circ)\frac{\partial}{\partial\theta'}\sigma_u^2(\theta^\circ)\right| \to 0$$

uniformly in $m \le k \le N$. Then the lemma is validated by the ergodic theorem (cf. Lemma A.1). □

Now, we establish the asymptotical normality of the score function.

Proposition 3.4. *Suppose that there exists $\delta>0$ such that $\mathrm{E}|G_h|^{4+\delta} = O(h)$. Then,*

$$\frac{1}{\sqrt{t_N}}\sum_{k=m}^{N} \dot{\mathrm{l}}_{nk}(\theta^\circ)\Delta t_k \Rightarrow N(0,\tau\Sigma).$$

Proof: Due to Lemma 3.11, it suffices to show that

$$\frac{1}{\sqrt{t_N}}\sum_{k=m}^{N}\left(\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)}-1\right)\frac{1}{\sigma_{n,k-1}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_{n,k-1}^2(\theta^\circ)\Delta t_k \Rightarrow N(0,\tau\Sigma).$$

Let λ be any vector in $\mathbb{R}^3$ and

$$\xi_{n,k-1} := \frac{1}{\sigma^4_{n,k-1}(\theta^\circ)}\left\{\lambda'\frac{\partial}{\partial\theta}\sigma^2_{n,k-1}(\theta^\circ)\right\}^2.$$

Note that

$$\frac{\Delta t_k}{\sqrt{t_N}}\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)\frac{1}{\sigma^2_{n,k-1}(\theta^\circ)}\lambda'\frac{\partial}{\partial\theta}\sigma^2_{n,k-1}(\theta^\circ), \quad k=m,...,N$$

are row-wise martingale differences with respect to $\{\mathcal{F}_{t_k}: k=m,...,N\}$. Since due to Proposition 3.3 and Lemma 3.12

$$\frac{1}{t_N}\sum_{k=m}^{N}\left\{\Delta t_k\left(\frac{\mathrm{E}\left\{Y^4_{nk}|\mathcal{F}_{t_{k-1}}\right\}}{\rho^4_{nk}(\theta^\circ)}-1\right)\right\}\xi_{n,k-1}\Delta t_k \xrightarrow{P} \tau\lambda'\Sigma\lambda,$$

it suffices to verify Lindeberg's condition for martingale differences (cf. Theorem 35.12 of Billingsley ([4])). For $\epsilon>0$ and $A>0$, we have

$$\begin{aligned}
&\sum_{k=m}^{N}\mathrm{E}\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2\xi_{n,k-1}\frac{\Delta t_k^2}{t_N}I\left\{\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2\xi_{n,k-1}\frac{\Delta t_k^2}{t_N}>\epsilon\right\}\\
&\quad\le \sum_{k=m}^{N}\mathrm{E}\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2 A\frac{\Delta t_k^2}{t_N}I\left\{\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2 A\frac{\Delta t_k^2}{t_N}>\epsilon\right\}\\
&\qquad+\sum_{k=m}^{N}\mathrm{E}\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2\xi_{n,k-1}\frac{\Delta t_k^2}{t_N}I\left\{\xi_{n,k-1}>A\right\},
\end{aligned}$$

and further, due to Lemma 3.5 and the fact $\mathrm{E}|G_h|^{4+\delta}=O(h)$,

$$\begin{aligned}
&\sum_{k=m}^{N}\mathrm{E}\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2 A\frac{\Delta t_k^2}{t_N}I\left\{\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2 A\frac{\Delta t_k^2}{t_N}>\epsilon\right\}\\
&\le \sum_{k=m}^{N}\mathrm{E}\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2 A\frac{\Delta t_k^2}{t_N}\left|\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right|^{\delta/2}\left(\frac{A}{\epsilon}\right)^{\delta/4}\frac{\Delta t_k^{\delta/2}}{t_N^{\delta/4}}\\
&\le K\sum_{k=m}^{N}\mathrm{E}\left|\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right|^{2+\delta/2}\frac{\Delta t_k^{2+\delta/2}}{t_N^{1+\delta/4}}\le K\sum_{k=m}^{N}\left\{\left(\frac{\mathrm{E}|Y_{nk}|^{4+\delta}}{\Delta t_k^{2+\delta/2}}+1\right)\right\}\frac{\Delta t_k^{2+\delta/2}}{t_N^{1+\delta/4}}\\
&\le K\sum_{k=m}^{N}\left\{\left(\frac{O(\Delta t_k)}{\Delta t_k^{2+\delta/2}}+1\right)\right\}\frac{\Delta t_k^{2+\delta/2}}{t_N^{1+\delta/4}}\to 0,
\end{aligned}$$

and due to Proposition 3.3,

$$\begin{aligned}
&\sum_{k=m}^{N}\mathrm{E}\left(\frac{Y^2_{nk}}{\rho^2_{nk}(\theta^\circ)}-1\right)^2\xi_{n,k-1}\frac{\Delta t_k^2}{t_N}I\left\{\xi_{n,k-1}>A\right\}\\
&\quad=\sum_{k=m}^{N}\mathrm{E}\left\{\Delta t_k\left(\frac{\mathrm{E}\{Y^4_{nk}|\mathcal{F}_{t_{k-1}}\}}{\rho^4_{nk}(\theta^\circ)}-1\right)\xi_{n,k-1}\frac{\Delta t_k}{t_N}I\left\{\xi_{n,k-1}>A\right\}\right\}\\
&\quad\le K\mathrm{E}\left\{\xi_0 I\left\{\xi_0>A\right\}\right\}.
\end{aligned}$$

Then by letting $A\to\infty$, we establish the proposition. □

Note that every component of $\frac{1}{t_N}\sum_{k=m}^{N}\ddot{\mathrm{l}}_{nk}(\theta_n^*)\Delta t_k$ is expressed as

$$\frac{1}{t_N}\sum_{k=m}^{N}\frac{\partial^2}{\partial\beta^i\partial\eta^j\partial\varphi^l}\mathrm{l}_{nk}(\theta_n^*)\Delta t_k = \frac{1}{t_N}\sum_{k=m}^{N}\frac{\partial^2}{\partial\beta^i\partial\eta^j\partial\varphi^l}\mathrm{l}_{nk}(\theta^\circ)\Delta t_k + \frac{1}{t_N}\sum_{k=m}^{N}\frac{\partial}{\partial\theta'}\frac{\partial^2}{\partial\beta^i\partial\eta^j\partial\varphi^l}\mathrm{l}_{nk}(\delta\theta_n^* + (1-\delta)\theta^\circ)(\theta_n^* - \theta^\circ)\Delta t_k$$

with $\delta \in (0,1)$. The rest of this subsection is devoted to verifying the convergence of $\frac{1}{t_N}\sum_{k=m}^{N}\ddot{\mathrm{l}}_{nk}(\theta_n^*)\Delta t_k$.

Lemma 3.13.

$$\frac{1}{t_N}\sum_{k=m}^{N}\left(1 - 2\frac{Y_{nk}^2}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right)\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\tilde{\rho}_{nk}^2(\theta^\circ)\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial}{\partial\theta'}\tilde{\rho}_{nk}^2(\theta^\circ)\Delta t_k \xrightarrow{P} -\Sigma$$

and

$$\frac{1}{t_N}\sum_{k=m}^{N}\left(\frac{Y_{nk}^2}{\tilde{\rho}_{nk}^2(\theta^\circ)} - 1\right)\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial^2}{\partial\theta\partial\theta'}\tilde{\rho}_{nk}^2(\theta^\circ)\Delta t_k \xrightarrow{P} \mathbf{0}.$$

Hence,

$$\frac{1}{t_N}\sum_{k=m}^{N}\ddot{\mathrm{l}}_{nk}(\theta^\circ)\Delta t_k \xrightarrow{P} -\Sigma.$$

Proof: For convenience, we set $\partial_1 := \frac{\partial}{\partial\beta^i\partial\eta^j\partial\varphi^l}$ and $\partial_2 := \frac{\partial}{\partial\beta^a\partial\eta^b\partial\varphi^c}$ to denote any differential operators of the first order. Due to Lemmas 3.5 and 3.7, we have

$$\begin{aligned}
&\mathrm{E}\left\{Y_{nk}^2\left|\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)} - \frac{1}{\rho_{nk}^2(\theta^\circ)}\right|\left|\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right|\right\} \\
&= \mathrm{E}\left\{\rho_{nk}^2\left|\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)} - \frac{1}{\rho_{nk}^2(\theta^\circ)}\right|\left|\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right|\right\} \\
&= \mathrm{E}\left\{\left|\frac{\rho_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)} - 1\right|\left|\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right|\right\} \\
&\le K\left\|\tilde{\sigma}_{n,k-1}^2(\theta^\circ) - \sigma_{n,k-1}^2(\theta^\circ)\right\|_2\left\|\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right\|_2 \\
&\longrightarrow 0,
\end{aligned}$$

uniformly in $m \le k \le N$, since

$$\max_{m\le k\le N}\left\|\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right\|_2 < \infty$$

(cf. Lemma 3.10). Moreover, due to Proposition 3.3,

$$
\begin{aligned}
&\frac{1}{t_N^2}\sum_{k=m}^{N}\mathrm{E}\left\{\left(\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)}-1\right)^2\left(\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right)^2\right\}\Delta t_k^2\\
&= \frac{1}{t_N^2}\sum_{k=1}^{N}\mathrm{E}\left\{\Delta t_k\left(\frac{\mathrm{E}\left(Y_{nk}^4|\mathcal{F}_{t_{k-1}}\right)}{\rho_{nk}^4(\theta^\circ)}-1\right)\left(\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right)^2\right\}\Delta t_k\\
&= \frac{K}{t_N^2}\sum_{k=1}^{N}\mathrm{E}\left(\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right)^2\Delta t_k \longrightarrow 0,
\end{aligned}
$$

so that we get

$$
\frac{1}{t_N}\sum_{k=m}^{N}\left(\frac{Y_{nk}^2}{\rho_{nk}^2(\theta^\circ)}-1\right)\left(\frac{\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right)\Delta t_k = o_P(1),
$$

i.e.,

$$
\frac{1}{t_N}\sum_{k=m}^{N}\left(\frac{Y_{nk}^2}{\tilde{\rho}_{nk}^2(\theta^\circ)}-1\right)\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial}{\partial\theta}\tilde{\rho}_{nk}^2(\theta^\circ)\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial}{\partial\theta'}\tilde{\rho}_{nk}^2(\theta^\circ)\Delta t_k \xrightarrow{P} \mathbf{0}.
$$

Similarly, we can see that

$$
\frac{1}{t_N}\sum_{k=m}^{N}\left(\frac{Y_{nk}^2}{\tilde{\rho}_{nk}^2(\theta^\circ)}-1\right)\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}\frac{\partial^2}{\partial\theta\partial\theta'}\tilde{\rho}_{nk}^2(\theta^\circ)\Delta t_k \xrightarrow{P} \mathbf{0}.
$$

On the other hand, we have

$$
\begin{aligned}
&\mathrm{E}\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}\left|\partial_1\tilde{\rho}_{nk}^2(\theta^\circ)-\partial_1\rho_{nk}^2(\theta^\circ)\right|\left|\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right|\\
&\le K\left\|\partial_1\tilde{\sigma}_{n,k-1}^2(\theta^\circ)-\partial_1\sigma_{n,k-1}^2(\theta^\circ)\right\|_2\left\|\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right\|_2 \longrightarrow 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{E}\left|\frac{1}{\tilde{\rho}_{nk}^2(\theta^\circ)}-\frac{1}{\rho_{nk}^2(\theta^\circ)}\right|\left|\partial_1\rho_{nk}^2(\theta^\circ)\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right| &= \mathrm{E}\left|\frac{\rho_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}-1\right|\left|\frac{\partial_1\rho_{nk}^2(\theta^\circ)}{\rho_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right|\\
&\le K\left\|\tilde{\sigma}_{n,k-1}^2(\theta^\circ)-\sigma_{n,k-1}^2(\theta^\circ)\right\|_2\left\|\frac{\partial_1\rho_{nk}^2(\theta^\circ)}{\rho_{nk}^2(\theta^\circ)}\frac{\partial_2\tilde{\rho}_{nk}^2(\theta^\circ)}{\tilde{\rho}_{nk}^2(\theta^\circ)}\right\|_2 \longrightarrow 0
\end{aligned}
$$

uniformly in $m \le k \le N$. Therefore,

$$
\frac{1}{t_N}\sum_{k=m}^{N}\ddot{\mathrm{l}}_{nk}(\theta^\circ)\Delta t_k = -\frac{1}{t_N}\sum_{k=m}^{N}\frac{1}{\sigma_{n,k-1}^4(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_{n,k-1}^2(\theta^\circ)\frac{\partial}{\partial\theta'}\sigma_{n,k-1}^2(\theta^\circ)\Delta t_k + o_P(1).
$$

Henceforth, the lemma is validated by Lemma 3.12. □

Lemma 3.14. *We have*

$$\max_{m\le k\le N} \mathrm{E}\sup_{\theta\in\Theta}\left|\frac{\partial^3}{\partial\beta^i\partial\eta^j\partial\varphi^l}\mathrm{l}_{nk}(\theta)\right| < \infty.$$

Hence,

$$\frac{1}{t_N}\sum_{k=m}^{N}\frac{\partial}{\partial\theta'}\frac{\partial^2}{\partial\beta^i\partial\eta^j\partial\varphi^l}\mathrm{l}_{nk}(\delta\theta_n^* + (1-\delta)\theta^\circ)(\theta_n^* - \theta^\circ)\Delta t_k \xrightarrow{P} 0.$$

Proof: Observe that $\frac{\partial^3}{\partial\beta^i\partial\eta^j\partial\varphi^l}\mathrm{l}_{nk}(\theta)$ is a finite sum of the terms:

$$\frac{Y_{nk}^2}{\tilde\rho_{nk}^2(\theta)}\prod_{i=1}^{3}\frac{\partial_i\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)},\quad \frac{Y_{nk}^2}{\tilde\rho_{nk}^2(\theta)}\frac{1}{\tilde\rho_{nk}^2(\theta)}\frac{\partial}{\partial\beta^a\partial\eta^b\partial\varphi^c}\tilde\rho_{nk}^2(\theta)\frac{1}{\tilde\rho_{nk}^2(\theta)}\frac{\partial^2}{\partial\beta^i\partial\eta^j\partial\varphi^l}\tilde\rho_{nk}^2(\theta),$$
$$\frac{Y_{nk}^2}{\tilde\rho_{nk}^2(\theta)}\frac{1}{\tilde\rho_{nk}^2(\theta)}\frac{\partial^3}{\partial\beta^a\partial\eta^b\partial\varphi^c}\tilde\rho_{nk}^2(\theta),$$

where $\partial_i, (i = 1, 2, 3)$ are differential operators of the first order. Now, by Lemmas 3.5 and 3.10,

$$\mathrm{E}\sup_\theta\frac{Y_{nk}^2}{\tilde\rho_{nk}^2(\theta)}\prod_{i=1}^{3}\left|\frac{\partial_i\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)}\right| \le K\mathrm{E}\frac{Y_{nk}^2}{\Delta t_k}\prod_{i=1}^{3}\sup_\theta\left|\frac{\partial_i\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)}\right|$$
$$\le K\mathrm{E}\frac{\rho_{nk}^2}{\Delta t_k}\prod_{i=1}^{3}\sup_\theta\left|\frac{\partial_i\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)}\right| \le K\mathrm{E}\left(\sigma_{n,k-1}^2 + O(\Delta)\right)\prod_{i=1}^{3}\sup_\theta\left|\frac{\partial_i\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)}\right|$$
$$\le K\|\sigma_{n,k-1}^2 + O(\Delta)\|_2\left\|\prod_{i=1}^{3}\sup_\theta\left|\frac{\partial_i\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)}\right|\right\|_2$$
$$\le K\|\sigma_{n,k-1}^2 + O(\Delta)\|_2\left\|\sup_\theta\left|\frac{\partial_1\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)}\right|\right\|_4\left\|\sup_\theta\left|\frac{\partial_2\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)}\right|\right\|_8\left\|\sup_\theta\left|\frac{\partial_3\tilde\rho_{nk}^2(\theta)}{\tilde\rho_{nk}^2(\theta)}\right|\right\|_8 < \infty$$

uniformly in $m \le k \le N$. The other terms can be treated in essentially the same fashion. Hence, the lemmas is asserted. □

The following proposition is due to Lemmas 3.13-3.14:

Proposition 3.5.

$$\frac{1}{t_N}\sum_{k=m}^{N}\ddot{\mathrm{l}}_{nk}(\theta_n^*)\Delta t_k \xrightarrow{P} -\Sigma.$$

The Proof of Asymptotic Normality. (2.4) can be proven by using standard arguments (cf. the proof of Theorem 2.2 in Francq and Zakoïan ([8])) and the results in (3.7) and Propositions 3.3-3.5. □

APPENDIX

Lemma A.1. *Suppose that* **C3** *holds. Then,*

$$\text{(A.1)} \qquad \frac{1}{t_N}\int_{t_m}^{t_N}\left\{\frac{\sigma_s^2}{\sigma_s^2(\theta)}+\log\sigma_s^2(\theta)\right\}ds \xrightarrow{P} \mathrm{E}\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta)}+\log\sigma_0^2(\theta)\right\},$$

$$\text{(A.2)} \qquad \frac{1}{t_N}\int_{t_m}^{t_N}\frac{1}{\sigma_s^4(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_s^2(\theta^\circ)\frac{\partial}{\partial\theta'}\sigma_s^2(\theta^\circ)ds \xrightarrow{P} \mathrm{E}\frac{1}{\sigma_0^4(\theta^\circ)}\frac{\partial}{\partial\theta}\sigma_0^2(\theta^\circ)\frac{\partial}{\partial\theta'}\sigma_0^2(\theta^\circ).$$

Proof: We only verify (A.1) since (A.2) can be proved similarly. Let $h>0$ and

$$\sigma_s^2(\theta,h) := \beta/\eta+\varphi\int_{(s-h,s)}e^{-\eta(s-u)}\sigma_u^2 d[L,L]_u.$$

Then we have

$$\begin{aligned}\text{(A.3)}\quad &\mathrm{E}\left|\frac{1}{t_N}\int_{t_m}^{t_N}\left\{\frac{\sigma_s^2}{\sigma_s^2(\theta)}+\log\sigma_s^2(\theta)\right\}ds-\frac{1}{t_N}\int_{t_m}^{t_N}\left\{\frac{\sigma_s^2}{\sigma_s^2(\theta,h)}+\log\sigma_s^2(\theta,h)\right\}ds\right|\\ &\le \mathrm{E}\left|\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta)}+\log\sigma_0^2(\theta)\right\}-\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta,h)}+\log\sigma_0^2(\theta,h)\right\}\right|\\ &\le \mathrm{E}K\left\{\sigma_0^2|\sigma_0^2(\theta)-\sigma_0^2(\theta,h)|\right\}\le K\|\sigma_0^2\|_2\|\sigma_0^2(\theta)-\sigma_0^2(\theta,h)\|_2\le Ke^{-\eta h}.\end{aligned}$$

Note that

$$\sigma_s^2=\beta^\circ\int_{s-h}^{s}e^{X_u-X_{s-}}du+\sigma_{s-h}^2e^{-X_{s-h}-X_{s-}}$$

and thus,

$$\frac{\sigma_s^2}{\sigma_s^2(\theta,h)}+\log\sigma_s^2(\theta,h)\in\mathcal{G}_{s-h}^s,$$

where $\mathcal{G}_s^t := \sigma\{\sigma_u, L_u-L_s : s<u<t\}$. Let

$$\alpha(v) := \sup_{0\le t<\infty}\sup\left\{P(A\cap B)-P(A)P(B) : A\in\mathcal{G}_{-\infty}^t, B\in\mathcal{G}_{t+v}^\infty\right\},$$

and define α^* in the same way with replacing $\mathcal{G}_s^t$ by $\sigma\{\sigma_u^2 : s<u<t\}$. According to the proof of Theorem 3.5 of Haug *et al.* ([10]), we can have

$$0\le\alpha(v)\le 6\alpha^*(v)\to 0 \quad \text{as } v\to\infty,$$

(cf. Fasen ([7])), which implies that $\left\{\frac{\sigma_s^2}{\sigma_s^2(\theta,h)}+\log\sigma_s^2(\theta,h) : s\ge 0\right\}$ is càglàd, strictly stationary and strong mixing. Thus,

$$Z_i := \int_{i-1}^{i}\left(\frac{\sigma_s^2}{\sigma_s^2(\theta,h)}+\log\sigma_s^2(\theta,h)\right)ds\in\mathcal{G}_{i-h-1}^i, \quad i=1,2,...$$

is strictly stationary and ergodic. Then, since by the ergodic theorem,

$$\begin{aligned}\frac{1}{t_N}\int_{t_m}^{t_N}\left\{\frac{\sigma_s^2}{\sigma_s^2(\theta,h)}+\log\sigma_s^2(\theta,h)\right\}ds &= \frac{1}{t_N}\sum_{i=[t_m]+1}^{[t_N]} Z_i + o_P(1)\\ &\xrightarrow{P} \mathrm{E}\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta,h)}+\log\sigma_0^2(\theta,h)\right\}\end{aligned}$$

owing to (A.3), by letting $h\to\infty$, we get

$$\frac{1}{t_N}\int_{t_m}^{t_N}\left\{\frac{\sigma_s^2}{\sigma_s^2(\theta)}+\log\sigma_s^2(\theta)\right\}ds \xrightarrow{P} \mathrm{E}\left\{\frac{\sigma_0^2}{\sigma_0^2(\theta)}+\log\sigma_0^2(\theta)\right\}.$$

This completes the proof. □

Lemma A.2. *Suppose that* **C2-C3** *hold and* $\int_{\mathbb{R}} x^3 d\Pi(x)=0$. *Then, as* $h\downarrow 0$,

$$\begin{aligned}\mathrm{E}\left\{(G_{t+h}-G_t)^4|\mathcal{F}_t\right\} &= h\left(\int_{\mathbb{R}} x^4\Pi(dx)+o(1)\right)\sigma_t^4,\\ \mathrm{E}\left\{(G_{t+h}-G_t)^2|\mathcal{F}_t\right\} &= h(1+o(1))\sigma_t^2\end{aligned}$$

uniformly in $t\geq 0$.

Proof: By the strict stationarity, it suffices to consider the case $t=0$. For $h>0$,

$$\begin{aligned}(G_h-G_0)^2 &= 2\int_{(0,h]} G_{u-}dG_u+[G,G]_h = 2\int_{(0,h]} G_{u-}\sigma_u dL_u+\int_{(0,h]}\sigma_u^2 d[L,L]_u,\\ (G_h-G_0)^4 &= 2\int_{(0,h]} G_{s-}^2 dG_s^2+[G^2,G^2]_h\\ &= 4\int_{(0,h]} G_{s-}^3\sigma_s dL_s+2\int_{(0,h]} G_{s-}^2\sigma_s^2 d[L,L]_s\\ &\quad +4\int_{(0,h]} G_{s-}^2\sigma_s^2 d[L,L]_s+\int_{(0,h]}\sigma_s^4 d[[L,L],[L,L]]_s\\ &\quad +4\int_{(0,h]} G_{s-}\sigma_s^3 d[[L,L],L]_s,\end{aligned}$$

where

$$\mathrm{E}\left\{\int_{(0,h]} G_{s-}^3\sigma_s dL_s\middle|\mathcal{F}_0\right\}=0.$$

Since $\int_{\mathbb{R}} x^3 d\Pi(x)=0$,

$$\mathrm{E}\left\{\int_{(0,h]} G_{s-}\sigma_s^3 d[[L,L],L]_s\middle|\mathcal{F}_0\right\}=0.$$

Thus, we have

$$\mathrm{E}\left\{(G_h - G_0)^4|\mathcal{F}_0\right\} = 6\int_{(0,h]} \mathrm{E}\left\{G_{s-}^2\sigma_s^2|\mathcal{F}_0\right\} ds + \int_{\mathbb{R}} x^4\Pi(dx)\int_{(0,h]} \mathrm{E}\left\{\sigma_s^4|\mathcal{F}_0\right\} ds.$$

Let $Z_s = \int_{(0,s]} G_{u-}\sigma_u dL_u$. By the integration by parts and associativity (cf. [17]), we can write

$$\begin{aligned} Z_s\sigma_{s+}^2 \;=\; & \beta^\circ\int_{(0,s]} Z_{u-}du \;-\; \eta^\circ\int_{(0,s]} Z_{u-}\sigma_u^2 du \;+\; \varphi^\circ\int_{(0,s]} Z_{s-}\sigma_u^2 d[L,L]_u + \int_{(0,s]}\sigma_u^3 G_{u-}dL_u \\ & + \left[\int_{(0,\cdot]}(\beta^\circ - \eta^\circ\sigma_u^2)du \;+\; \varphi^\circ\int_{(0,\cdot]}\sigma_u^2 d[L,L]_u\,,\; \int_{(0,\cdot]} G_{u-}\sigma_u dL_u\right]_s. \end{aligned}$$

Note that for $F\in\mathcal{F}_0$,

$$\begin{aligned} \mathrm{E}\left\{\int_{(0,s]} Z_{u-}du\cdot 1_F\right\} \;&=\; \int_{(0,s]}\mathrm{E}\left\{Z_{u-}1_F\right\}du = \int_{(0,s]}\mathrm{E}\left\{\mathrm{E}(Z_{u-}|\mathcal{F}_0)1_F\right\}du = 0,\\ \mathrm{E}\left\{\int_{(0,s]} Z_{u-}\sigma_u^2 du\cdot 1_F\right\} \;&=\; \int_{(0,s]}\mathrm{E}\left\{Z_u\sigma_u^2 1_F\right\}du. \end{aligned}$$

Since

$$\begin{aligned} &\left[\int_{(0,\cdot]}(\beta^\circ - \eta^\circ\sigma_u^2)du + \varphi^\circ\int_{(0,\cdot]}\sigma_u^2 d[L,L]_u, \int_{(0,\cdot]} G_{u-}\sigma_u dL_u\right]\\ &= \left[\int_{(0,\cdot]}(\beta^\circ - \eta^\circ\sigma_u^2)du, \int_{(0,\cdot]} G_{u-}\sigma_u dL_u\right] + \left[\varphi^\circ\int_{(0,\cdot]}\sigma_u^2 d[L,L]_u, \int_{(0,\cdot]} G_{u-}\sigma_u dL_u\right]\\ &= \varphi^\circ\int_{(0,\cdot]} G_{u-}\sigma_u^3 d[[L,L],L]_u, \end{aligned}$$

we have

$$\mathrm{E}\left\{\left[\int_{(0,\cdot]}(\beta^\circ - \eta^\circ\sigma_u^2)du + \varphi^\circ\int_{(0,\cdot]}\sigma_u^2 d[L,L]_u, \int_{(0,\cdot]} G_{u-}\sigma_u dL_u\right]1_F\right\} = 0$$

due to $\int_{\mathbb{R}} x^3 d\Pi(x) = 0$. Thus,

$$\mathrm{E}\left\{Z_s\sigma_s^2 1_F\right\} = (\varphi^\circ - \eta^\circ)\int_0^s \mathrm{E}\left\{Z_u\sigma_u^2 1_F\right\}du, \quad \mathrm{E}\left\{Z_0\sigma_0^2 1_F\right\} = 0,$$

which implies $\mathrm{E}\left\{Z_s\sigma_s^2 1_F\right\} = 0$ for each $F\in\mathcal{F}_0$. This in turn implies $\mathrm{E}\left\{Z_s\sigma_s^2|\mathcal{F}_0\right\} = 0$ and

$$\begin{aligned} \mathrm{E}\left\{G_{s-}^2\sigma_s^2|\mathcal{F}_0\right\} \;&=\; \mathrm{E}\left\{G_s^2\sigma_s^2|\mathcal{F}_0\right\} = \mathrm{E}\left\{\sigma_s^2\int_{(0,s]}\sigma_u^2 d[L,L]_u + 2\sigma_s^2\int_{(0,s]} G_{u-}\sigma_u dL_u\,\middle|\,\mathcal{F}_0\right\}\\ &=\; \varphi_\circ^{-1}\mathrm{E}\left\{\sigma_s^2\left\{\sigma_{s+}^2 - \sigma_{0+}^2 - \beta^\circ s + \eta^\circ\int_0^s\sigma_u^2 du\right\}\middle|\,\mathcal{F}_0\right\}\\ &=\; \varphi_\circ^{-1}\left\{\mathrm{E}\{\sigma_s^4|\mathcal{F}_0\} - (\sigma_0^2 + \beta^\circ s)\mathrm{E}\{\sigma_s^2|\mathcal{F}_0\} + \eta^\circ\int_0^s\mathrm{E}\{\sigma_s^2\sigma_u^2|\mathcal{F}_0\}du\right\}. \end{aligned}$$

Since $\sigma_s^2 = \beta^\circ \int_0^s e^{-(X_{s-}-X_u)}du + \sigma_0^2 e^{-X_{s-}}$, we can have

$$\begin{aligned}
\mathrm{E}\{\sigma_s^2|\mathcal{F}_0\} &= \beta^\circ \int_0^s \mathrm{E}e^{-X_{s-u}}du + \sigma_0^2\mathrm{E}e^{-X_s} = \beta^\circ \int_0^s \mathrm{E}e^{-X_u}du + \sigma_0^2\mathrm{E}e^{-X_s} \\
&= \beta^\circ \int_0^s \mathrm{E}e^{-X_u}du + \sigma_0^2\mathrm{E}e^{-X_s} \\
&= \frac{\beta^\circ(e^{s\Psi(1)}-1)}{\Psi(1)} + e^{s\Psi(1)}\sigma_0^2.
\end{aligned}$$

Then, observing

$$\sigma_s^4 = \beta_\circ^2 \left\{\int_0^s e^{-(X_{s-}-X_u)}du\right\}^2 + 2\sigma_0^2 e^{-X_{s-}}\beta^\circ \int_0^s e^{-(X_{s-}-X_u)}du + \sigma_0^4 e^{-2X_{s-}},$$

we obtain

$$\begin{aligned}
\mathrm{E}\left\{\sigma_s^4|\mathcal{F}_0\right\} &= \beta_\circ^2\mathrm{E}\left\{\int_0^s e^{-X_u}du\right\}^2 + 2\beta^\circ\sigma_0^2\mathrm{E}\left\{\int_0^s e^{X_u-2X_s}du\right\} + \sigma_0^4\mathrm{E}e^{-2X_s} \\
&= \beta_\circ^2\left\{\frac{2}{\Psi(1)\Psi(2)} + \frac{2}{\Psi(2)-\Psi(1)}\left(\frac{e^{s\Psi(2)}}{\Psi(2)} - \frac{e^{s\Psi(1)}}{\Psi(1)}\right)\right\} \\
&\quad + 2\beta^\circ\sigma_0^2\frac{e^{s\Psi(2)}-e^{s\Psi(1)}}{\Psi(2)-\Psi(1)} + \sigma_0^4 e^{s\Psi(2)}
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\left\{\sigma_s^2\sigma_u^2|\mathcal{F}_0\right\} &= \mathrm{E}\left\{\sigma_u^2\mathrm{E}\left\{\sigma_s^2|\mathcal{F}_u\right\}\middle|\,\mathcal{F}_0\right\} \\
&= \mathrm{E}\left\{\sigma_u^2\frac{\beta^\circ(e^{(s-u)\Psi(1)}-1)}{\Psi(1)} + e^{(s-u)\Psi(1)}\sigma_u^4\middle|\,\mathcal{F}_0\right\} \\
&= \mathrm{E}\{\sigma_u^2|\mathcal{F}_0\}\frac{\beta^\circ\{e^{(s-u)\Psi(1)}-1\}}{\Psi(1)} + e^{(s-u)\Psi(1)}\mathrm{E}\left\{\sigma_u^4|\mathcal{F}_0\right\} \\
&= \left\{\frac{\beta^\circ(e^{u\Psi(1)}-1)}{\Psi(1)} + e^{u\Psi(1)}\sigma_0^2\right\}\frac{\beta^\circ\{e^{(s-u)\Psi(1)}-1\}}{\Psi(1)} \\
&\quad + e^{(s-u)\Psi(1)}\beta_\circ^2\left\{\frac{2}{\Psi(1)\Psi(2)} + \frac{2}{\Psi(2)-\Psi(1)}\left(\frac{e^{u\Psi(2)}}{\Psi(2)} - \frac{e^{u\Psi(1)}}{\Psi(1)}\right)\right\} \\
&\quad + 2e^{(s-u)\Psi(1)}\beta^\circ\sigma_0^2\frac{e^{u\Psi(2)}-e^{u\Psi(1)}}{\Psi(2)-\Psi(1)} + e^{(s-u)\Psi(1)}\sigma_0^4 e^{u\Psi(2)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{E}\left\{(G_h-G_0)^4|\mathcal{F}_0\right\} &= 6\int_{(0,h]}\mathrm{E}\left\{G_{s-}^2\sigma_s^2|\mathcal{F}_0\right\}ds + \int_{\mathbb{R}} x^4\Pi(dx)\int_{(0,h]}\mathrm{E}\left\{\sigma_s^4|\mathcal{F}_0\right\}ds \\
&= h\left(\int_{\mathbb{R}} x^4\Pi(dx) + o(1)\right)\sigma_0^4, \\
\mathrm{E}\left\{(G_h-G_0)^2|\mathcal{F}_0\right\} &= h(1+o(1))\sigma_0^2.
\end{aligned}$$

This completes the proof. □

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