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THE EXACT JOINT DISTRIBUTION OF CONCOMITANTS OF ORDER STATISTICS AND THEIR ORDER STATISTICS UNDER NORMALITY

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Abstract:

- In this work we derive the exact joint distribution of a linear combination of concomitants of order statistics and linear combinations of their order statistics in a multivariate normal distribution. We also investigate a special case of related joint distributions discussed by He and Nagaraja (2009).

Key-Words:

- *unified skew-normal; L-statistic; multivariate normal distribution; order statistic; concomitant; orthant probability.*

AMS Subject Classification:

- 62G30, 62H10.

1. INTRODUCTION

Suppose that the joint distribution of two n-dimensional random vectors $\mathbf{X}$ and $\mathbf{Y}$ follows a $2n$ dimensional multivariate normal vector with positive definite covariance matrix, i.e.

$$\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} \sim N_{2n}\left(\boldsymbol{\mu}=\begin{pmatrix}\boldsymbol{\mu}_{\mathbf{x}}\\ \boldsymbol{\mu}_{\mathbf{y}}\end{pmatrix},\ \sum=\begin{pmatrix}\sum_{\mathbf{xx}} & \sum_{\mathbf{xy}}\\ \sum_{\mathbf{xy}}^T & \sum_{\mathbf{yy}}\end{pmatrix}\right) \tag{1.1}$$

where $\boldsymbol{\mu}_{\mathbf{x}}$, $\boldsymbol{\mu}_{\mathbf{y}}$ are respectively the mean vectors and $\sum_{\mathbf{xx}}$, $\sum_{\mathbf{yy}}$ are the positive definite variance matrices of $\mathbf{X}$ and $\mathbf{Y}$, while $\sum_{\mathbf{xy}}$ is their covariance matrix. Let $\mathbf{X}_{(n)}=(X_{1:n},\ X_{2:n},\ldots,\ X_{n:n})^T$ be the vector of order statistics obtained from $\mathbf{X}$ and $\mathbf{Y}_{(n)}=(Y_{1:n},\ Y_{2:n},\ldots,\ Y_{n:n})^T$ be the vector of order statistics obtained from $\mathbf{Y}$. Further, let $\mathbf{Y}_{[n]}=(Y_{[1:n]},\ Y_{[2:n]},\ldots,\ Y_{[n:n]})^T$ be the vector of Y-variates paired with the order statistics of $\mathbf{X}$. The elements of $\mathbf{Y}_{[n]}$ are called the concomitants of the order statistics of $\mathbf{X}$.

Nagaraja (1982) has obtained the distribution of a linear combination of order statistics from a bivariate normal random vector where the variables are exchangeable. Loperfido (2008a) has extended the results of Nagaraja (1982) to elliptical distributions. Arellano-Valle and Genton (2007) have expressed the exact distribution of linear combinations of order statistics from dependent random variables. Sheikhi and Jamalizadeh (2011) have showed that for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$, the distribution of $(X,\ \mathbf{a}^T\mathbf{Y}_{(2)},\ \mathbf{b}^T\mathbf{Y}_{(2)})^T$ is a singular skew-normal and carried out a regression analysis. Yang (1981) has studied the linear combination of concomitants of order statistics. Tsukibayashi (1998) has obtained the joint distribution of $(Y_{i:n}, Y_{[i:n]})$, while He and Nagaraja (2009) have obtained the joint distribution of $(Y_{i:n}, Y_{[j:n]})$ for all $i,\ j=1,\ 2,\ldots,\ n$. Goel and Hall (1994) have discussed the difference between concomitants and order statistics using the sum $\sum_{i=1}^n h(Y_{i:n}-Y_{[i:n]})$ for some smooth function h. Recently much attention has been focused on the connection between order statistics and skew-normal distributions (see e.g. Loperfido 2008a and 2008b and Sheikhi and Jamalizadeh 2011). In this article we shall obtain the joint distribution of $\mathbf{a}^T\mathbf{Y}_{(n)}$ and $\mathbf{b}^T\mathbf{Y}_{[n]}$, where $\mathbf{a}=(a_1,\ a_2,\ldots,\ a_n)^T$ and $\mathbf{b}=(b_1,\ b_2,\ldots,\ b_n)^T$ are arbitrary vectors in $\mathbb{R}^n$. Since we do not assume independence, our results are more general than those of He and Nagaraja (2009). On the other hand, He and Nagaraja (2009) have not assumed normality.

The concept of the skew-normal distribution was proposed independently by Roberts (1966), Ainger *et al.* (1977), Andel *et al.* (1984) and Azzalini (1985). The univariate random variable Y has a skew-normal distribution if its distribution can be written as

$$f_Y(y)=2\varphi\left(y;\ \mu,\ \sigma^2\right)\Phi\left(\lambda\frac{y-\mu}{\sigma}\right) \qquad y\in\mathbb{R} \tag{1.2}$$

where $\varphi\left(.;\ \mu,\ \sigma^2\right)$ is the normal density with mean μ and variance σ^2 and $\Phi(.)$ denotes the standard normal distribution function.

Following Arellano-Valle and Azzalini (2006), a d-dimensional random vector $\mathbf{Y}$ is said to have a unified multivariate skew-normal distribution ($\mathbf{Y} \sim SUN_{d,m}\left(\boldsymbol{\xi},\ \boldsymbol{\delta},\ \boldsymbol{\Omega},\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda}\right)$), if it has a density function of the form

(1.3)

$$f_{\mathbf{Y}}\left(\mathbf{y}\right) = \varphi_d\left(\mathbf{y};\, \boldsymbol{\xi},\ \boldsymbol{\Omega}\right) \frac{\Phi_m\left(\boldsymbol{\delta} + \boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\left(\mathbf{y}-\boldsymbol{\xi}\right);\ \ \boldsymbol{\Gamma} - \boldsymbol{\Lambda}^T\boldsymbol{\Omega}^{-1}\boldsymbol{\Lambda}\right)}{\Phi_m\left(\boldsymbol{\delta};\, \boldsymbol{\Gamma}\right)} \qquad \mathbf{y} \in \mathbb{R}^d$$

where $\varphi_d(.,\ \boldsymbol{\xi},\ \boldsymbol{\Omega})$ is the density function of a multivariate normal and $\Phi_m(.;\ \Sigma)$ is the multivariate normal cumulative function with the covariance matrix Σ.

If $\Sigma^* = \begin{pmatrix} \boldsymbol{\Gamma} & \boldsymbol{\Lambda}^T \\ \boldsymbol{\Lambda} & \boldsymbol{\Omega} \end{pmatrix}$ is a singular matrix we say that the distribution of $\mathbf{X}$ is singular unified skew-normal and write $SSUN_{d,m}(\boldsymbol{\xi},\ \boldsymbol{\delta}, \boldsymbol{\Omega},\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda})$. For more details see Arellano-Valle and Azzalini (2006) and Sheikhi and Jamalizadeh (2011).

In Section 2, we show that for two vectors $\mathbf{a}$ and $\mathbf{b}$, the joint distribution of $\mathbf{a}^T\mathbf{Y}_{(n)}$ and $\mathbf{b}^T\mathbf{Y}_{[n]}$ belongs to the unified multivariate skew-normal family. We also discuss special cases of these distributions under the setting of independent normal random variables. Finally, in section 3 we present a numerical application of our results.

2. MAIN RESULTS

Define $S(\mathbf{X})$ as the class of all permutation of components of the random vector $\mathbf{X}$, i.e. $S(\mathbf{X}) = \{\mathbf{X}^{(i)} = \mathbf{P}_i\mathbf{X};\ i = 1,\ 2, ...,\ n!\}$, where $\mathbf{P}_i$ is an $n \times n$ permutation matrix. Also, suppose $\boldsymbol{\Delta}$ is the difference matrix of dimension $(n-1) \times n$ such that the ith row of $\boldsymbol{\Delta}$ is $\mathbf{e}^T_{n,\,i+1} - \ \mathbf{e}^T_{n,\,i}$, $i = 1, ..., n-1$, where $\mathbf{e}_1,\ \mathbf{e}_2, ...,\ \ \mathbf{e}_n$ are n-dimensional unit basis vectors. Then $\boldsymbol{\Delta}\mathbf{X} = (X_2 - X_1,\ X_3 - X_2, ...,\ X_n - X_{n-1})^T$. (See e.g. Crocetta and Loperfido 2005).

Further, let $\mathbf{X}^{(i)}$ and $\mathbf{Y}^{(i)}$ be the ith permutation of the random vectors $\mathbf{X}$ and $\mathbf{Y}$ respectively. We write $G_{ij}(\mathbf{t},\ \boldsymbol{\xi},\ \sum) = P\left(\boldsymbol{\Delta}\mathbf{X}^{(i)} \geq \mathbf{0},\ \ \boldsymbol{\Delta}\mathbf{Y}^{(j)} \geq \mathbf{0}\right)$.

Theorem 2.1. *Suppose the matrix* $\begin{pmatrix} \boldsymbol{\Delta} \\ \mathbf{a}^T \\ \mathbf{b}^T \end{pmatrix}$ *is of full rank. Then under the assumption of model (1.1) the cdf of the random vector* $\left(\mathbf{a}^T\mathbf{Y}_{(n)},\ \mathbf{b}^T\mathbf{Y}_{[n]}\right)^T$ *is the mixture*

$$F_{\mathbf{a}^T\mathbf{Y}_{(n)},\ \mathbf{b}^T\mathbf{Y}_{[n]}}(\,y_1, y_2) = \sum_{i=1}^{n!}\sum_{j=1}^{n!} F_{SUN}(\,y_1, y_2\,;\ \boldsymbol{\xi}_{ij}, \boldsymbol{\delta}_{ij}, \boldsymbol{\Gamma}_{ij}, \boldsymbol{\Omega}_{ij}, \boldsymbol{\Lambda}_{ij})\, G_{ij}(t, \boldsymbol{\xi}, \sum)$$

where $F_{SUN}(.\,;\boldsymbol{\xi}_{ij},\ \boldsymbol{\delta}_{ij},\boldsymbol{\Gamma}_{ij},\boldsymbol{\Omega}_{ij},\boldsymbol{\Lambda}_{ij})$ *is the cdf of unified multivariate skew-normal with*

$$\boldsymbol{\xi}_{ij}=\begin{pmatrix}\mathbf{a}^T\boldsymbol{\mu}_{\mathbf{y}}^{(i)}\\ \mathbf{b}^T\boldsymbol{\mu}_{\mathbf{y}}^{(i)}\end{pmatrix},\ \boldsymbol{\delta}_{ij}=\begin{pmatrix}\boldsymbol{\Delta}\boldsymbol{\mu}_{\mathbf{x}}^{(i)}\\ \boldsymbol{\Delta}\boldsymbol{\mu}_{\mathbf{y}}^{(j)}\end{pmatrix},\ \boldsymbol{\Gamma}_{ij}=\begin{pmatrix}\boldsymbol{\Delta}\sum_{\mathbf{xx}}^{(ii)}\boldsymbol{\Delta}^T & \boldsymbol{\Delta}\sum_{\mathbf{xy}}^{(ij)}\boldsymbol{\Delta}^T\\ & \boldsymbol{\Delta}\sum_{\mathbf{yy}}^{(jj)}\boldsymbol{\Delta}^T\end{pmatrix},$$

$$\boldsymbol{\Omega}_{ij}=\begin{pmatrix}\mathbf{a}^T\sum_{\mathbf{yy}}^{(ii)}\mathbf{a} & \mathbf{a}^T\sum_{\mathbf{yy}}^{(ij)}\mathbf{b}\\ & \mathbf{b}^T\sum_{\mathbf{yy}}^{(jj)}\mathbf{b}\end{pmatrix}\qquad \textit{and}\qquad \boldsymbol{\Lambda}_{ij}=\begin{pmatrix}\boldsymbol{\Delta}\sum_{\mathbf{xy}}^{(ii)}\mathbf{a} & \boldsymbol{\Delta}\sum_{\mathbf{xy}}^{(ij)}\mathbf{b}\\ \boldsymbol{\Delta}\sum_{\mathbf{yy}}^{(ij)}\mathbf{a} & \boldsymbol{\Delta}\sum_{\mathbf{yy}}^{(jj)}\mathbf{b}\end{pmatrix}^T$$

where $\boldsymbol{\mu}_{\mathbf{x}}^{(i)}$ *and* $\boldsymbol{\mu}_{\mathbf{y}}^{(j)}$ *are respectively the mean vectors of the* i*th permutation of the random vector* $\mathbf{X}$ *and the* j*th permutation of the random vector* $\mathbf{Y}$ *and* $\sum_{\mathbf{xx}}^{(ii)}=Var(\mathbf{X}^{(i)})$, $\sum_{\mathbf{yy}}^{(jj)}=Var(\mathbf{Y}^{(j)})$ *and* $\sum_{\mathbf{xy}}^{(ij)}=Cov(\mathbf{X}^{(i)},\ \mathbf{Y}^{(j)})$.

Proof: We have

$$\begin{aligned}F_{\mathbf{a}^T\mathbf{Y}_{(n)},\,\mathbf{b}^T\mathbf{Y}_{[n]}}(y_1,\,y_2) &= P\left(\mathbf{a}^T\mathbf{Y}_{(n)}\le y_1,\ \mathbf{b}^T\mathbf{Y}_{[n]}\le y_2\right)\\ &=\sum_{i=1}^{n!}\sum_{j=1}^{n!}P\left(\mathbf{a}^T\mathbf{Y}^{(i)}\le y_1,\ \mathbf{b}^T\mathbf{Y}^{(j)}\le y_2|\mathbf{A}^{(ij)}\right)P\left(\mathbf{A}^{(ij)}\right)\end{aligned}$$

where $\mathbf{A}^{(ij)}=\{\boldsymbol{\Delta}\mathbf{X}^{(i)}\ge\mathbf{0},\ \boldsymbol{\Delta}\mathbf{Y}^{(j)}\ge\mathbf{0}\}$. Since

$$\begin{pmatrix}\boldsymbol{\Delta}\mathbf{X}^{(i)}\\ \boldsymbol{\Delta}\mathbf{Y}^{(j)}\\ \mathbf{a}^T\mathbf{Y}^{(i)}\\ \mathbf{b}^T\mathbf{Y}^{(j)}\end{pmatrix}_{2n\times1}=\begin{pmatrix}\boldsymbol{\Delta} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \boldsymbol{\Delta} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{a}^T\\ \mathbf{0} & \mathbf{b}^T & \mathbf{0}\end{pmatrix}_{2n\times3n}\begin{pmatrix}\mathbf{X}^{(i)}\\ \mathbf{Y}^{(j)}\\ \mathbf{Y}^{(i)}\end{pmatrix}_{3n\times1},\tag{2.1}$$

the full rank assumption implies nonsingularity of the matrix on the right hand side of (2.1). Furthermore,

$$\begin{pmatrix}\boldsymbol{\Delta}\mathbf{X}^{(i)}\\ \boldsymbol{\Delta}\mathbf{Y}^{(j)}\\ \mathbf{a}^T\mathbf{Y}^{(i)}\\ \mathbf{b}^T\mathbf{Y}^{(j)}\end{pmatrix}$$

$$\sim N_{2n}\left(\begin{pmatrix}\boldsymbol{\Delta}\boldsymbol{\mu}_{\mathbf{x}}^{(i)}\\ \boldsymbol{\Delta}\boldsymbol{\mu}_{\mathbf{y}}^{(j)}\\ \mathbf{a}^T\boldsymbol{\mu}_{\mathbf{y}}^{(i)}\\ \mathbf{b}^T\boldsymbol{\mu}_{\mathbf{y}}^{(j)}\end{pmatrix},\begin{pmatrix}\boldsymbol{\Delta}\sum_{\mathbf{xx}}^{(ii)}\boldsymbol{\Delta}^T & \boldsymbol{\Delta}\sum_{\mathbf{xy}}^{(ij)}\boldsymbol{\Delta}^T & \boldsymbol{\Delta}\sum_{\mathbf{xy}}^{(ii)}\mathbf{a} & \boldsymbol{\Delta}\sum_{\mathbf{xy}}^{(ij)}\mathbf{b}\\ & \boldsymbol{\Delta}\sum_{\mathbf{yy}}^{(jj)}\boldsymbol{\Delta}^T & \boldsymbol{\Delta}\sum_{\mathbf{yy}}^{(ij)}\mathbf{a} & \boldsymbol{\Delta}\sum_{\mathbf{yy}}^{(jj)}\mathbf{b}\\ & & \mathbf{a}^T\sum_{\mathbf{yy}}^{(ii)}\mathbf{a} & \mathbf{a}^T\sum_{\mathbf{yy}}^{(ij)}\mathbf{b}\\ & & & \mathbf{b}^T\sum_{\mathbf{yy}}^{(jj)}\mathbf{b}\end{pmatrix}\right).$$

Now, similar to Sheikhi and Jamalizadeh (2011), we immediately conclude that

$$\left(\mathbf{a}^T\mathbf{Y}^{(i)},\,\mathbf{b}^T\mathbf{Y}^{(j)}\right)^T|\,\boldsymbol{\Delta}\mathbf{X}^{(i)}\ge\mathbf{0},\,\boldsymbol{\Delta}\mathbf{Y}^{(j)}\ge\mathbf{0}\sim SUN_{2,\,2(n-1)}(\boldsymbol{\xi}_{ij},\boldsymbol{\delta}_{ij},\boldsymbol{\Gamma}_{ij},\boldsymbol{\Omega}_{ij},\boldsymbol{\Lambda}_{ij})\,.$$

This proves the Theorem. $\square$

Remark 2.1. If the rank of the matrix $\left(\boldsymbol{\Delta},\ \mathbf{a}^T, \mathbf{b}^T\right)^T$ is at most $n-1$, the joint distribution of $\left(\mathbf{a}^T\mathbf{Y}_{(n)},\ \mathbf{b}^T\mathbf{Y}_{[n]}\right)^T$ is a mixture of a unified skew-normals and a singular unified skew-normals. In this section we assume that the matrix $\left(\boldsymbol{\Delta},\ \mathbf{a}^T, \mathbf{b}^T\right)^T$ is of full rank. A special case will be investigated later in the paper.

Let (X_i, Y_i), $i = 1,\ 2, \ldots,\ n$ be a random sample of size n from a bivariate normal $N_2\left(\mu_x,\ \mu_y,\ \sigma_x^2,\ \sigma_y^2,\ \rho\right)$, then the model (1.1) reduces to the following:

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim N_{2n}\left(\boldsymbol{\mu} = \begin{pmatrix} \mu_x \mathbf{1}_n \\ \mu_y \mathbf{1}_n \end{pmatrix}, \sum = \begin{pmatrix} \sum_{\mathbf{xx}} & \sum_{\mathbf{xy}} \\ & \sum_{\mathbf{yy}} \end{pmatrix}\right) \tag{2.2}$$

where $\sum_{\mathbf{xx}} = \sigma_x^2 \mathbf{I}_n$, $\sum_{\mathbf{yy}} = \sigma_y^2 \mathbf{I}_n$ and $\sum_{\mathbf{xy}} = \rho\sigma_x\sigma_y \mathbf{1}_n \mathbf{1}_n^T$ where ρ is the correlation coefficient between X and Y.

The following corollary describes the joint distribution of a linear combination of concomitants of order statistics and a linear combination of their order statistics under the independence assumption.

Corollary 2.1. *Suppose the matrix $\left(\boldsymbol{\Delta},\ \mathbf{a}^T,\ \mathbf{b}^T\right)^T$ is of full rank. Then under the assumption of model (2.2) the distribution of the random vector $\left(\mathbf{a}^T\mathbf{Y}_{(n)},\ \mathbf{b}^T\mathbf{Y}_{[n]}\right)^T$ is $SUN_{2,\ 2(n-1)}\left(\boldsymbol{\xi},\ \mathbf{0}_{2n-2},\ \boldsymbol{\Omega},\ \boldsymbol{\Gamma},\ \boldsymbol{\Lambda}\right)$ where*

$$\boldsymbol{\xi} = \begin{pmatrix} \mu_x \mathbf{a}^T \mathbf{1}_n \\ \mu_y \mathbf{b}^T \mathbf{1}_n \end{pmatrix},\ \ \boldsymbol{\Omega} = \sigma_y^2 \begin{pmatrix} \mathbf{a}^T\mathbf{a} & \mathbf{a}^T\mathbf{b} \\ & \mathbf{b}^T\mathbf{b} \end{pmatrix},\ \ \boldsymbol{\Gamma} = \begin{pmatrix} \sigma_x^2 \boldsymbol{\Delta}\boldsymbol{\Delta}^T & \rho\sigma_x\sigma_y \boldsymbol{\Delta}\boldsymbol{\Delta}^T \\ & \sigma_y^2 \boldsymbol{\Delta}\boldsymbol{\Delta}^T \end{pmatrix},$$

$$\boldsymbol{\Lambda} = \begin{pmatrix} \rho\sigma_x\sigma_y \boldsymbol{\Delta}\mathbf{a} & \rho\sigma_x\sigma_y \boldsymbol{\Delta}\mathbf{b} \\ \sigma_y^2 \boldsymbol{\Delta}\mathbf{a} & \sigma_y^2 \boldsymbol{\Delta}\mathbf{b} \end{pmatrix}.$$

Proof: We have

$$\begin{aligned} F_{\mathbf{a}^T\mathbf{Y}_{(n)},\ \mathbf{b}^T\mathbf{Y}_{[n]}}(y_1,\ y_2) &= P\left(\mathbf{a}^T\mathbf{Y}_{(n)} \le y_1,\ \mathbf{b}^T\mathbf{Y}_{[n]} \le\ y_2\right) \\ &= \sum_{i=1}^{n!}\sum_{j=1}^{n!} P\left(\mathbf{a}^T\mathbf{Y}^{(i)} \le y_1, \mathbf{b}^T\mathbf{Y}^{(j)} \le y_2 | \mathbf{A}^{(ij)}\right)\ P\left(\mathbf{A}^{(ij)}\right). \end{aligned}$$

Since $P\left(\boldsymbol{\Delta}\mathbf{X}^{(i)} \ge \mathbf{0}, \boldsymbol{\Delta}\mathbf{Y}^{(j)} \ge \mathbf{0}\right) = \left(\frac{1}{n!}\right)^2$, $i, j = 1, \ldots, n!$, by independence we have

$$F_{\mathbf{a}^T\mathbf{Y}_{(n)},\ \mathbf{b}^T\mathbf{Y}_{[n]}}(y_1,\ y_2) = P\left(\mathbf{a}^T\mathbf{Y} \le y_1,\ \mathbf{b}^T\mathbf{Y} \le y_2 | \boldsymbol{\Delta}\mathbf{X} \ge \mathbf{0},\ \boldsymbol{\Delta}\mathbf{Y} \ge \mathbf{0}\right).$$

Moreover, $\left(\boldsymbol{\Delta}\mathbf{X},\ \boldsymbol{\Delta}\mathbf{Y},\ \mathbf{a}^T\mathbf{Y},\ \mathbf{b}^T\mathbf{Y}\right)^T$ follows an $2n$ dimensional multivariate normal distribution with $\boldsymbol{\mu} = \left(\mathbf{0}_{n-1},\ \mathbf{0}_{n-1},\ \mu_y \mathbf{a}^{\mathbf{T}}\mathbf{1}_n,\ \mu_y \mathbf{b}^{\mathbf{T}}\mathbf{1}_n\right)^T$ and

$$\sum = \begin{pmatrix} \sigma_x^2 \boldsymbol{\Delta}\boldsymbol{\Delta}^T & \rho\sigma_x\sigma_y \boldsymbol{\Delta}\boldsymbol{\Delta}^T & \rho\sigma_x\sigma_y \boldsymbol{\Delta}\mathbf{a} & \rho\sigma_x\sigma_y \boldsymbol{\Delta}\mathbf{b} \\ & \sigma_y^2 \boldsymbol{\Delta}\boldsymbol{\Delta}^T & \sigma_y^2 \boldsymbol{\Delta}\mathbf{a} & \sigma_y^2 \boldsymbol{\Delta}\mathbf{b} \\ & & \sigma_y^2 \mathbf{a}^T\mathbf{a} & \sigma_y^2 \mathbf{a}^T\mathbf{b} \\ & & & \sigma_y^2 \mathbf{b}^T\mathbf{b} \end{pmatrix}.$$

So, as in Theorem 2.1 the proof is completed. □

We easily find that $\mathbf{\Gamma} = [\gamma_{ij}]$, where

$$\gamma_{ij} = \begin{cases} 2\sigma_x^2 & |i-j| = 0 \\ -\sigma_x^2 & |i-j| = 1 \\ 0 & |i-j| = 2, ..., 2(n-1) \end{cases}$$

and $\mathbf{\Lambda} = \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}_2 \end{bmatrix}$ with $\mathbf{\Lambda}_1 = \left(\lambda_{11}, ..., \ \lambda_{(n-1)1}\right)^T$ where $\lambda_{k1} = \sigma_x^2 \left\{a_{k+1} - a_k\right\}$, $k = 1, ..., n-1$ and $\mathbf{\Lambda}_2 = \left(\lambda_{12}, ..., \ \lambda_{(n-1)2}\right)^T$ where $\lambda_{k2} = \sigma_y^2 \left\{b_{k+1} - b_k\right\}$, $k = 1, ..., n-1$.

Let the difference matrix $\mathbf{\Delta}_1$ of dimension $n-1 \times n$ be such that its first $i-1$ rows are $\mathbf{e}_{n,1}^T - \mathbf{e}_{n,k}^T$, $k = 2, 3, ..., i-1$ and the last $n-i$ rows are $\mathbf{e}_{n,k}^T - \mathbf{e}_{n,1}^T$, $k = i, \ ... \ n-1$. Also, let the matrix $\mathbf{\Delta}_2$ of dimension $n-1 \times n$ be such that its first $j-1$ rows are $\mathbf{e}_{n,1}^T - \mathbf{e}_{n,k}^T$, $k = 2, 3, ..., j-1$ and the last $n-j$ rows are $\mathbf{e}_{n,j}^T - \mathbf{e}_{n,1}^T$, $k = j, \ ... \ n-1$ and $\mathbf{1}_{n,i}$ be a $n-1$ dimensional vector with the first i elements equal to 1 and the rest -1. Further, let $\mathbf{X}_i$ be a permutation of the random vector $\mathbf{X}$, such that its ith element is located in the first place.

Theorem 2.2. *For a random sample of size n from a bivariate normal random vector (X, Y), the joint distribution of $Y_{i:n}, \ Y_{[j:n]}$ is*

$$\begin{aligned} F_{Y_{i:n}, \ Y_{[j:n]}}(y_1, \ y_2) \ = \ & k_1 F_{SUN}(\min(y_1, \ y_2)\,; \ \mu_y, \ \mathbf{0}_{2(n-1)}, \ \sigma_y^2, \ \mathbf{\Gamma}, \ \mathbf{\Lambda}) \\ & + k_2 F_{SSUN}(y_1, \ y_2\,; \ \mu_y \mathbf{1}_2 \ , \mathbf{0}_{2(n-1)}, \ \sigma_y^2 \mathbf{I}_2, \ \mathbf{\Gamma}, \ \mathbf{\Lambda}) \end{aligned}$$

where $F_{SUN}(.\,; \ \mu_y, \ \mathbf{0}_{2(n-1)}, \ \sigma_y^2, \ \mathbf{\Gamma}, \ \mathbf{\Lambda})$ is the cdf of a non-singular unified multivariate skew-normal distribution $SUN_{1, \ 2n-2}\left(\mu_y, \ \mathbf{0}, \ \sigma_y^2, \ \mathbf{\Gamma}, \ \mathbf{\Lambda}\right)$ with

$$\mathbf{\Gamma} = \begin{pmatrix} \sigma_x^2 \mathbf{\Delta}_1 \mathbf{\Delta}_1^T & \rho\sigma_x\sigma_y \mathbf{\Delta}_1 \mathbf{\Delta}_1^T \\ & \sigma_y^2 \mathbf{\Delta}_1 \mathbf{\Delta}_1^T \end{pmatrix}, \ \mathbf{\Lambda} = \begin{pmatrix} \rho\sigma_x\sigma_y \mathbf{1}_{n,i} \\ \sigma_y^2 \mathbf{1}_{n,i} \end{pmatrix}$$

and $F_{SSUN}(.\,; \ \mu_y \mathbf{1}_2, \ \mathbf{0}_{2(n-1)}, \ \sigma_y^2 \mathbf{I}_2, \ \mathbf{\Gamma}, \ \mathbf{\Lambda})$ is the cdf of a singular unified multivariate skew-normal distribution $SSUN_{2, \ 2n-2}(\mu_y \mathbf{1}_2 \ , \mathbf{0}_{2(n-1)}, \ \sigma_y^2 \mathbf{I}_2, \ \mathbf{\Gamma}, \ \mathbf{\Lambda})$ where $\mathbf{I}_2$ is an identity matrix of dimension 2 and

$$\mathbf{\Gamma} = \begin{pmatrix} \sigma_x^2 \mathbf{\Delta}_2 \mathbf{\Delta}_2^T & \rho\sigma_x\sigma_y \mathbf{\Delta}_2 \mathbf{\Delta}_1^T \\ & \sigma_y^2 \mathbf{\Delta}_1 \mathbf{\Delta}_1^T \end{pmatrix}, \ \mathbf{\Lambda} = \begin{pmatrix} -\rho\sigma_x\sigma_y \mathbf{J}_{n-1} & \rho\sigma_x\sigma_y \mathbf{1}_{n,j} \\ \sigma_y^2 \mathbf{1}_{n,i} & -\sigma_y^2 \mathbf{J}_{n-1} \end{pmatrix},$$

$k_1 = n!(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(-2\rho))^n$ *and* $k_2 = n(n-1)((n-1)!)^2(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(-2\rho))^n$.

Proof: Let B_{ij} denote the event that Y_i is the ith order statistic among $\{Y_1, \ Y_2, ..., \ Y_n\}$ and X_j is the jth order statistic among $\{X_1, \ X_2, ..., \ X_n\}$. So,

$B_{ij} = \{\mathbf{\Delta}_1\mathbf{Y}_i > \mathbf{0},\ \mathbf{\Delta}_2\mathbf{X}_j > \mathbf{0}\}$ and we have

$$
\begin{aligned}
& F_{Y_{i:n},\ Y_{[j:n]}}(u,\ v) \\
=\ & P\left(Y_{i:n} \le u,\ Y_{[j:n]} \le v\right) \\
=\ & \sum_{i=1}^{n}\sum_{j=1}^{n} P\left(Y_i \le u,\ Y_j \le v | B_{ij}\right) P\left(B_{ij}\right) \\
=\ & \sum_{i=1}^{n} P\left(Y_i \le u,\ Y_i \le v | B_{ii}\right) P\left(B_{ii}\right) + \sum_{\substack{i=1 \\ i\neq j}}^{n}\sum_{j=1}^{n} P\left(Y_i \le u,\ Y_j \le v | B_{ij}\right) P\left(B_{ij}\right) \\
=\ & n! P\left(Y_1 \le \min(u,v) | B_{11}\right) P\left(B_{11}\right) \\
& + (n^2-n)((n-1)!)^2 P\left(Y_1 \le u, Y_2 \le v | B_{12}\right) P\left(B_{12}\right).
\end{aligned}
$$

The last equality holds by the independence assumption. Since the distribution of $Y_1|B_{11}$ is identical to the distribution of $Y_1|\{\mathbf{\Delta}_1\mathbf{Y}_1 > \mathbf{0},\ \mathbf{\Delta}_1\mathbf{X}_1 > \mathbf{0}\}$, we have

$$
\begin{pmatrix} \mathbf{\Delta}_1\mathbf{X}_1 \\ \mathbf{\Delta}_1\mathbf{Y}_1 \\ Y_1 \end{pmatrix} \sim N_{2n-1}\left(\begin{pmatrix} \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1} \\ \mu_y \end{pmatrix},\ \begin{pmatrix} \sigma_x^2\mathbf{\Delta}_1\mathbf{\Delta}_1^T & \rho\sigma_x\sigma_y\mathbf{\Delta}_1\mathbf{\Delta}_1^T & \rho\sigma_x\sigma_y\mathbf{1}_{n,i} \\ & \sigma_y^2\mathbf{\Delta}_1\mathbf{\Delta}_1^T & \sigma_y^2\mathbf{1}_{n,i} \\ & & \sigma_y^2 \end{pmatrix}\right).
$$

So, $Y_1|B_{11} \sim SUN_{1,\ 2n-2}\left(\mu_y,\ \mathbf{0}_{2(n-1)},\ \sigma_y^2,\ \mathbf{\Gamma},\ \mathbf{\Lambda}\right)$ where

$$
\mathbf{\Gamma} = \begin{pmatrix} \sigma_x^2\mathbf{\Delta}_1\mathbf{\Delta}_1^T & \rho\sigma_x\sigma_y\mathbf{\Delta}_1\mathbf{\Delta}_1^T \\ & \sigma_y^2\mathbf{\Delta}_1\mathbf{\Delta}_1^T \end{pmatrix} \qquad \text{and} \qquad \mathbf{\Lambda} = \begin{pmatrix} \rho\sigma_x\sigma_y\mathbf{1}_{n,i} \\ \sigma_y^2\mathbf{1}_{n,i} \end{pmatrix}.
$$

Also, the conditional distribution of Y_1 and Y_2 given B_{12} is the same as the distribution of $(Y_1,\ Y_2)^T|\{\mathbf{\Delta}_2\mathbf{X}_2 > \mathbf{0},\ \mathbf{\Delta}_1\mathbf{Y}_1 > \mathbf{0}\}$. Moreover, $(\mathbf{\Delta}_2\mathbf{X}_2,\ \mathbf{\Delta}_1\mathbf{Y}_1,\ Y_1,\ Y_2)^T$ follows a $2n$ multivariate singular normal distribution with rank $2n-1$, $\boldsymbol{\mu} = \left(\mathbf{0}_{n-1},\ \mathbf{0}_{n-1},\ \mu_y\mathbf{1}_2\right)^T$ and

$$
\sum = \begin{pmatrix} \sigma_x^2\mathbf{\Delta}_2\mathbf{\Delta}_2^T & \rho\sigma_x\sigma_y\mathbf{\Delta}_2\mathbf{\Delta}_1^T & -\rho\sigma_x\sigma_y\mathbf{J}_{n-1} & \rho\sigma_x\sigma_y\mathbf{1}_{n,j} \\ & \sigma_y^2\mathbf{\Delta}_1\mathbf{\Delta}_1^T & \sigma_y^2\mathbf{1}_{n,i} & -\sigma_y^2\mathbf{J}_{n-1} \\ & & \sigma_y^2 & 0 \\ & & & \sigma_y^2 \end{pmatrix}
$$

where $\mathbf{J}_{n-1} = (1,\ \mathbf{0}_{n-2})^T$.

We note that the matrix $(\mathbf{\Delta}_2\mathbf{X}_2,\ \mathbf{\Delta}_1\mathbf{Y}_1)^T$ is of full rank but $(\mathbf{\Delta}_2\mathbf{X}_2,\ \mathbf{\Delta}_1\mathbf{Y}_1,\ Y_1,\ Y_2)^T$ is not. Hence, according to the case (3) of Arellano-Valle and Azzalini (2006) we conclude that $(Y_1,\ Y_2)^T|\{\mathbf{\Delta}_2\mathbf{X}_2,\ \mathbf{\Delta}_1\mathbf{Y}_1 > \mathbf{0}\} \sim SSUN_{2,2n-2}(\mu_y\mathbf{1}_2,\ \mathbf{0}_{2(n-1)},\ \sigma_y^2\mathbf{I}_2,\ \mathbf{\Gamma},\ \mathbf{\Lambda})$ where

$$
\mathbf{\Gamma} = \begin{pmatrix} \sigma_x^2\mathbf{\Delta}_2\mathbf{\Delta}_2^T & \rho\sigma_x\sigma_y\mathbf{\Delta}_2\mathbf{\Delta}_1^T \\ & \sigma_y^2\mathbf{\Delta}_1\mathbf{\Delta}_1^T \end{pmatrix} \qquad \text{and} \qquad \mathbf{\Lambda} = \begin{pmatrix} -\rho\sigma_x\sigma_y\mathbf{J}_{n-1} & \rho\sigma_x\sigma_y\mathbf{1}_{n,j} \\ \sigma_y^2\mathbf{1}_{n,i} & -\sigma_y^2\mathbf{J}_{n-1} \end{pmatrix}.
$$

On the other hand, using the orthant probabilities (e.g. Kotz *et al.* 2000) we easily obtain

$$\begin{aligned} P\left(B_{11}\right) &= P(X_2 > X_1,\ X_3 > X_1, ...,\ X_3 > X_1,\ Y_2 > Y_1,\ Y_3 > Y_1, ...,\ Y_n > Y_1) \\ &= (P(X_2 > X_1,\ Y_2 > Y_1))^n \\ &= (\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(-2\rho))^n\,. \end{aligned}$$

So, $k_1 = n!(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(-2\rho))^n$. Similarly, $k_2 = n(n-1)((n-1)!)^2(\frac{1}{4} + \frac{1}{2\pi}\sin^{-1}(-2\rho))^n$.

This completes the proof. $\square$

Remark 2.2. As a special case, we assume $n = 2$, $(X,\ Y)^T \sim BN(0,\ 0,\ 1,\ 1,\ \rho)$, $i = 1$ and $j = 2$. Then the joint pdf of $Y_{1:2}$ and $Y_{[2:2]}$ is obtained as

$$F_{Y_{1:2},\ Y_{[2:2]}}(\,y_1,\ y_2) = k_1 F_{SUN}(\,\min(y_1,\ y_2)\,) + k_2 F_{SSUN}(\,y_1,\ y_2\,)$$

where k_1 and k_2 are as in Theorem 2.2 with $n = 2$ and $F_{SUN}(.)$ and $F_{SSUN}(.,.)$ are the cdfs of

$$\varphi((\,\min(y_1,\ y_2))\frac{\Phi_2\left((\rho,\ -1)^T\min(y_1,\ y_2);\ \ \mathbf{M}_1\right)}{\Phi_2\left((0,0)^T;\ \mathbf{M}_2\right)}$$

and

$$\varphi(y_1)\varphi(\,y_2)\frac{\Phi_2\left((\rho,\ 1)^T(y_1 -\ y_2);\ \ \mathbf{M}_3\right)}{\Phi_2\left((0,0)^T;\ \mathbf{M}_4\right)}$$

respectively where

$$\mathbf{M}_1 = \begin{pmatrix} 2-\rho^2 & \rho \\ \rho & 1 \end{pmatrix}, \qquad \mathbf{M}_2 = 2\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix},$$

$$\mathbf{M}_3 = \begin{pmatrix} 2-\rho^2 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathbf{M}_4 = 2\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix},$$

and their joint pdf is
(2.3)

$$f_{Y_{1:2},\ Y_{[2:2]}}(\,y_1,\ y_2) = \begin{cases} \varphi(y)\frac{\Phi_2\left((\rho,\ -1)^T y; \mathbf{M}_1\right)}{\Phi_2\left((0,0)^T;\ \mathbf{M}_2\right)} & if\ \ y_1 = y_2 = y \\ \varphi(y_1)\varphi(y_2)\frac{\Phi_2(\rho(y_2-y_1),\ y_2-y_1;\ \ \mathbf{M}_3)}{\Phi_2\left((0,0)^T;\ \mathbf{M}_4\right)} & if\ \ y_1 <\ y_2\,. \end{cases}$$

Remark 2.3. When X and Y are independent, the joint density (2.3) becomes

$$f_{Y_{1:2},\, Y_{[2:2]}}(y_1,\, y_2) = \begin{cases} 2\varphi(y)(1-\Phi\,(y)) & if \;\; y_1 = y_2 = y \\ 2\varphi(y_1)\varphi(y_2) & if \;\; y_1 < y_2 \end{cases}$$

which is the same as the joint distribution (8) of He and Nagaraja (2009) under these assumptions (see e.g. He, 2007, p. 35).

Furthermore, He and Nagaraja (2009) discussed some relations between $Y_{i:n}$ and $Y_{[j:n]}$ in a bivariate setting. In particular, they showed that $Corr(Y_{i:n}, Y_{[j:n]}) = Corr(Y_{n-i+1:n},\; Y_{[n-j+1:n]})$. The following remark shows that, in addition, the joint distribution of $Y_{i:n},\; Y_{[j:n]}$ and $Y_{n-i+1:n},\; Y_{[n-j+1:n]}$ belong to a same family and differ only in one parameter. The relation (24) of He and Nagaraja (2009) is a direct consequence.

Remark 2.4. Let B'_{ij} denote the event that Y_i is the $(n-i+1)$th order statistic among $\{Y_1,\; Y_2, ...,\; Y_n\}$ and X_j is the $(n-j+1)$th order statistic among $\{X_1,\; X_2, ...,\; X_n\}$. Then $B'_{ij} = \{\mathbf{\Delta}_1\mathbf{Y}_i < \mathbf{0},\; \mathbf{\Delta}_2\mathbf{X}_j < \mathbf{0}\} = \{-\mathbf{\Delta}_1\mathbf{Y}_i > \mathbf{0},\; -\mathbf{\Delta}_2\mathbf{X}_j > \mathbf{0}\}$. Hence, the joint distribution of $Y_{n-i+1:n},\; Y_{[n-j+1:n]}$ is

$$\begin{aligned} F_{Y_{n-i+1:n},\, Y_{[n-j+1:n]}}(y_1,\, y_2) \;=\;& k_1 F_{SUN}(\min(y_1,\, y_2)\,;\mu_y,\; \mathbf{0},\; \sigma_y^2,\; \mathbf{\Gamma},\; \mathbf{\Lambda}') \\ &+ k_2 F_{SSUN}(y_1,\, y_2\,;\mu_y\mathbf{1}_2\;,\mathbf{0},\; \sigma_y^2\mathbf{I}_2,\; \mathbf{\Gamma},\; \mathbf{\Lambda}') \end{aligned}$$

where $\mathbf{\Lambda}' = -\mathbf{\Lambda}$ and the parameters as in Theorem 2.2.

3. NUMERICAL EXAMPLE

Loperfido (2008b), with the assumption of exchangeability, have estimated the distribution of extreme values of vision of left eye (Y_1) and vision of right eye (Y_2) and the conditional distribution of age (X), given these extreme values as a skew-normal family. Johnson and Wichern (2002, p.24) provide data consisting of mineral content measurements of three bones (radius, humerus, ulna) in two arms (dominant and non dominant) for each of 25 old women. We consider the following variables:

X_1 : Dominant radius

X_2 : Non dominant radius

Y_1 : Dominant ulna

Y_2 : Non dominant ulna

The sample data is presented in Table 1. We apply model (1.1) to this data and obtain the unbiased estimates of the parameters of these models as

$$\hat{\boldsymbol{\mu}}_{\mathbf{x}} = \begin{bmatrix} 0.8438 \\ 0.8191 \end{bmatrix}, \ \hat{\boldsymbol{\mu}}_{\mathbf{y}} = \begin{bmatrix} 0.7044 \\ 0.6938 \end{bmatrix}, \ \hat{\boldsymbol{\Sigma}}^2_{\mathbf{x}} = \begin{bmatrix} 0.0130 & 0.0103 \\ 0.0103 & 0.0114 \end{bmatrix},$$
$$\hat{\boldsymbol{\Sigma}}^2_{\mathbf{y}} = \begin{bmatrix} 0.0115 & 0.0088 \\ 0.0088 & 0.0105 \end{bmatrix} \text{ and } \hat{\boldsymbol{\Sigma}}_{\mathbf{xy}} = \begin{bmatrix} 0.0091 & 0.0085 \\ 0.0085 & 0.0105 \end{bmatrix}.$$

Table 1: Data of measurements of two bones in 25 old women.

Dominant radius	Non dominant radius	Dominant ulna	Non dominant ulna
1.103	1.052	0.873	0.872
0.842	0.859	0.590	0.744
0.925	0.873	0.767	0.713
0.857	0.744	0.706	0.674
0.795	0.809	0.549	0.654
0.787	0.799	0.782	0.571
0.933	0.880	0.737	0.803
0.799	0.851	0.618	0.682
0.945	0.876	0.853	0.777
0.921	0.906	0.823	0.765
0.792	0.825	0.686	0.668
0.815	0.751	0.678	0.546
0.755	0.724	0.662	0.595
0.880	0.866	0.810	0.819
0.900	0.838	0.723	0.677
0.764	0.757	0.586	0.541
0.733	0.748	0.672	0.752
0.932	0.898	0.836	0.805
0.856	0.786	0.578	0.610
0.890	0.950	0.758	0.718
0.688	0.532	0.533	0.482
0.940	0.850	0.757	0.731
0.493	0.616	0.546	0.615
0.835	0.752	0.618	0.664
0.915	0.936	0.869	0.868

Yang (1981) has considered general linear functions of the form

$$L = \frac{1}{n}\sum_{i=1}^{n} J(\frac{i}{n})Y_{[i:n]}$$

where J is a smooth function. He has established that L is asymptotically normal and may be used to construct consistent estimator of various conditional quantities such as $E(Y|X = x)$, $P(Y \in A|X = x)$ and $Var(Y|X = x)$. We assume that J is a quadratic function and estimate the joint distribution of L and the *sample midrange* of $\mathbf{Y}$, i.e. $T = \frac{1}{2}\sum_{i=1}^{2} Y_{i:n}$. The joint distribution of T and L is as in Theorem 2.1 with $\boldsymbol{\Delta} = \begin{pmatrix} -1 & 1 \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}^T$ and $\mathbf{b} = \begin{pmatrix} 1/8 & 1/2 \end{pmatrix}^T$.

In particular,

$$\boldsymbol{\xi}_{11}=\boldsymbol{\xi}_{21}=\begin{pmatrix}0.6991\\0.4345\end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\xi}_{12}=\boldsymbol{\xi}_{22}=\begin{pmatrix}0.6991\\0.4389\end{pmatrix}.$$

Also, if

$$M_n = n^{-1}\sum_{i=1}^{n} h(n)^{-1}K(\frac{(i/n)-F_n(x)}{h(n)})Y_{[i:n]}$$

where $F_n(x)$ is the proportion of the X_i less than or equal to x, $K(x)$ is some pdf on real line and $h(n)\to 0$ as $n\to\infty$, then M_n is a mean square consistent estimator of the regression function $E(Y|X=x)$. We assume that $K(x)$ is the pdf of the normal distribution with mean 0.8314 and variance 0.0108, i.e. $K(x)$ is the pdf of the radius. Moreover, we set $h(n)=\frac{1}{n-1}$. At $x=0.8$, we obtain $M_2 = 0.012Y_{[1:2]} + 0.515Y_{[2:2]}$. Again, the joint distribution of T and M_2 is as in Theorem 2.1 with $\boldsymbol{\Delta}=\begin{pmatrix}-1 & 1\end{pmatrix}$, $\mathbf{a}=\begin{pmatrix}1/2 & 1/2\end{pmatrix}^T$ and $\mathbf{b}=\begin{pmatrix}0.012 & 0.515\end{pmatrix}^T$.

4. CONCLUSION

In this paper we model the joint distribution of a linear combination of concomitants of order statistics and linear combinations of their order statistics as a unified skew-normal family assuming a multivariate normal distribution. However, there are many interesting further work which may be carried out. Viana and Lee (2006) have studied the covariance structure of two random vectors $\mathbf{X}_{(n)}$ and $\mathbf{Y}_{[n]}$ in the presence of a random variable Z. We may generalize their work by extending our results in the presence of one or more covariates. The results of this paper may be extended to elliptical distributions or using exchangeability assumption. Other results such as the regression analysis of concomitants using their order statistics are also of interest.

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