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NONCENTRAL GENERALIZED MULTIVARIATE BETA TYPE II DISTRIBUTION

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Abstract:

- The distribution of the variables that originates from monitoring the variance when the mean encountered a sustained shift is considered — specifically for the case when measurements from each sample are independent and identically distributed normal random variables. It is shown that the solution to this problem involves a sequence of dependent random variables that are constructed from independent noncentral chi-squared random variables. This sequence of dependent random variables are the key to understanding the performance of the process used to monitor the variance and are the focus of this article. For simplicity, the marginal (i.e. the univariate and bivariate) distributions and the joint (i.e. the trivariate) distribution of only the first three random variables following a change in the variance is considered. A multivariate generalization is proposed which can be used to calculate the entire run-length (i.e. the waiting time until the first signal) distribution.

Key-Words:

- *confluent hypergeometric functions; hypergeometric functions; multivariate beta distribution; noncentral chi-squared; shift in process mean and variance.*

AMS Subject Classification:

- 62E15, 62H10.

1. INTRODUCTION

We propose a noncentral generalized multivariate beta type II distribution constructed from independent noncentral chi-squared random variables using the variables in common technique. This is a new contribution to the existing beta type II distributions considered in the literature. Tang (1938) studied the distribution of the ratios of noncentral chi-squared random variables defined on the positive domain. He considered the ratio, consisting of independent variates, where the numerator was a noncentral chi-squared random variable while the denominator was a central chi-squared random variable, as well as the ratio where both the numerator and denominator were noncentral chi-squared random variables — this was applied to study the properties of analysis of variance tests under nonstandard conditions. Patnaik (1949) coined the phrase noncentral F for the first ratio mentioned above. The second ratio is referred to as the doubly noncentral F distribution. An overview of these distributions is given by Johnson, Kotz & Balakrishnan (1995). More recently Pe and Drygas (2006) proposed an alternative presentation for the doubly noncentral F by using the results on the product of two hypergeometric functions. In a bivariate context Gupta *et al.* (2009) derived a noncentral bivariate beta type I distribution, using a ratio of noncentral gamma random variables, that is defined on the unit square; applying the appropriate transformation will yield a noncentral beta type II distribution defined on the positive domain. The noncentral Dirichlet type II distribution was derived by Troskie (1967) as the joint distribution of $V_i = \frac{Y_i}{Y_{r+1}}$, $i=1,2,...,r$ where Y_i is chi-squared distributed and Y_{r+1} has a noncentral chi-squared distribution. Sánchez and Nagar (2003) derived the version where both Y_i and Y_{r+1} are noncentral gamma random variables.

Section 2 provides an overview of the practical problem which is the genesis of the random variables $U_0 = \frac{\lambda W_0}{X}$ and $U_j = \frac{\lambda W_j}{X+\lambda\sum_{k=0}^{j-1} W_k}$, $j=1,2,...,p$ with $\lambda > 0$ where X and W_i, $i=0,1,...,p$ are noncentral chi-squared distributed. In Section 3 the distribution of the first three random variables, i.e. U_0, U_1, U_2 is derived. Bivariate densities and univariate densities of (U_0, U_1, U_2) also receive attention. Section 4 proposes a multivariate extension, followed by shape analysis, an example and probability calculations in Sections 5 and 6, respectively.

2. PROBLEM STATEMENT

Adamski *et al.* (2012) proposed a generalized multivariate beta distribution; the dependence structure and construction of the random variables originate in a practical setting where the process mean is monitored, using a control chart (see e.g. Montgomery, 2009), when the measurements are independent and identically distributed having been collected from an Exp(θ) distribution, where θ was assumed to be unknown.

Monitoring the unknown process variance assuming that the observations from each independent sample are independent identically distributed (i.i.d.) normal random variables with the mean known was introduced by Quesenberry (1991). To gain insight into the performance of such a control chart, in other words, to determine the probability of detecting a shift immediately or after a number of samples, the joint distribution of the plotting statistics is needed. Exact expressions for the joint distribution of the plotting statistics for the chart proposed by Quesenberry (1991) can be obtained from the distribution derived by Adamski *et al.* (2012), the key difference is the fact that it is only the degrees of freedom of the chi-squared random variables that changes.

Monitoring of the unknown process variance when the known location parameter sustained a permanent shift leads to a noncentral version of the generalized multivariate beta distribution proposed by Adamski *et al.* (2012). To derive this new noncentral generalized multivariate beta type II distribution we proceed in two steps. First we describe the practical setting which motivates the derivation of the distribution, and secondly we derive the distributions in sections 3 and 4. To this end, let $(X_{i1}, X_{i2}, ..., X_{in_i})$, $i = 1, 2, ...$ represent successive, independent samples of size $n_i \geq 1$ measurements made on a sequence of items produced in time. Assume that these values are independent and identically distributed having been collected from a $N(\mu_0, \sigma^2)$ distribution where the parameters μ_0 and σ^2 denotes the known process mean and unknown process variance, respectively. Take note that a sample can even consist of an individual observation because the process mean is assumed to be known and the variance of the sample can still be calculated as $S_i^2 = (X_{i1} - \mu_0)^2$. Suppose that from sample (time period) $\kappa > 1$ the unknown process variance parameter has changed from σ^2 to $\sigma_1^2 = \lambda\sigma^2$ (also unknown) where $\lambda \neq 1$ and $\lambda > 0$, but the known process mean also encountered an unknown sustained shift from sample (time period) $h > 1$ onwards, i.e. it changed from μ_0 to μ_1 where μ_1 is also known. To clarify, the mean of the process at start-up is assumed to be known and denoted μ_0 but the time and the size of the shift in the mean will be unknown in a practical situation. In order to incorporate and/or evaluate the influence of these changes in the parameters on the performance of the control chart for the variance, we assume fixed/deterministic values for these parameters — essentially this implies then that the mean is known following the shift, i.e. denoted by μ_1. Therefore, the main interest is monitoring the process variance when the process mean is known, although this mean can suffer at some time an unknown shift. In practice it is important to note that even though the mean and the variance of the normal distribution can change independently, the performance of a Shewhart type control chart for the mean depends on the process variance and vice versa.

This dependency is due to the plotting statistics and the control limits used. The proposed control chart could thus be useful in practice when the control chart for monitoring the mean fails to detect the shift in the mean. For example, in case a small shift in the mean occurs and a Shewhart-type chart for the mean is used (which is known for the inefficiency in detecting small shifts compared to the EWMA (exponentially weighted moving average) and CUSUM (cumulative sum) charts for the mean which are better in detecting small shifts (Montgomery, 2009)) the shift might go undetected.

Based on the time of the shift in the process mean, this problem can be viewed in three ways, as illustrated in Figure 1.

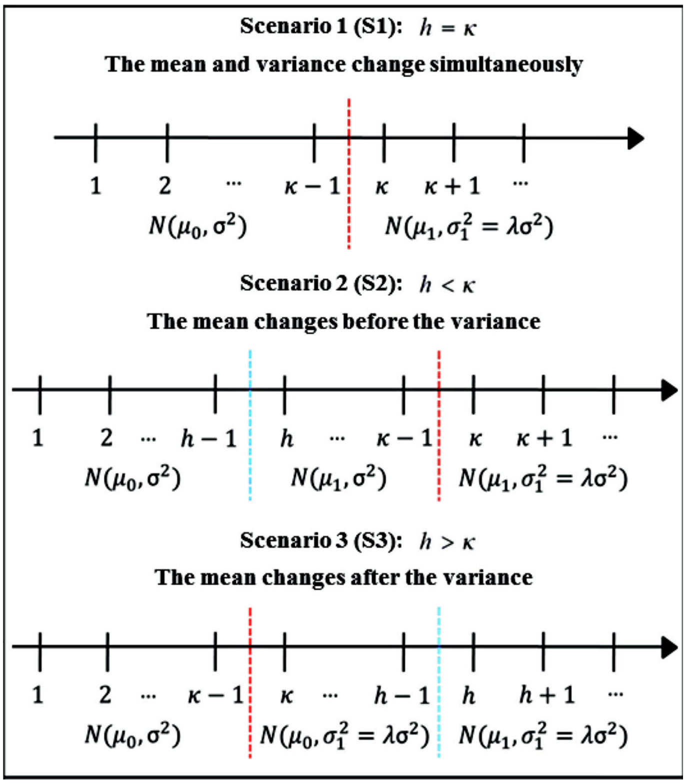


Figure 1: The different scenarios.

From Figure 1 we see the following:

Scenario 1: The mean and the variance change simultaneously from μ_0 to μ_1 and from σ^2 to σ_1^2, respectively. Note that, it is assumed that the shift in the process parameters occurs somewhere between samples $\kappa - 1$ and κ.

Scenario 2: The change in the mean from μ_0 to μ_1 occurs before the change in the variance from σ^2 to σ_1^2.

Scenario 3: The change in the variance from σ^2 to σ_1^2 occurs before the change in the mean from μ_0 to μ_1.

Because it is assumed that the process variance σ^2 is unknown, the first sample is used to obtain an initial estimate of σ^2. Thus, in the remainder of this article σ^2 is assumed to denote a point estimate of the unknown variance. This initial estimate is continuously updated using the new incoming samples as they are collected as long as the estimated value of σ^2 does not change, i.e. is not detected using the control chart. The control chart and the plotting statistic is based on the in-control distribution of the process. The two sample test statistic for testing the hypothesis at time r that the two independent samples (the measurements of the r^{th} sample alone and the measurements of the first $r-1$ samples combined) are from normal distributions with the same unknown variance, is based on the statistic

$$U_r^* = \frac{S_r^2}{S_{r-1}^{2\text{pooled}}} \qquad \text{for} \quad r = 2, 3, \ldots\,,$$

(2.1)

$$\text{where} \quad S_{r-1}^{2\text{pooled}} = \frac{\sum_{i=1}^{r-1} n_i S_i^2}{\sum_{i=1}^{r-1} n_i} \quad \text{and} \quad S_i^2 = \frac{1}{n_i} \sum_{k=1}^{n_i} (X_{ik} - \mu_i)^2 \quad \text{for } i = 1, 2, \ldots, r\,.$$

[Take note: μ_i denotes the known population mean of sample i.]

The focus will be on the part where the process is out-of-control, i.e. encountered a shift, since the exact distribution of the plotting statistic is then unknown. To simplify the notation used in expression (2.1), following a change in the process variance between samples $\kappa-1$ and κ, define the random variable

$$U_0^* = U_\kappa^* = \frac{S_\kappa^2}{S_{\kappa-1}^{2\text{pooled}}}\,. \tag{2.2}$$

The subscript of the random variable U_0^* indicates the number of samples after the parameter has changed, with zero indicating that it is the first sample after the process encountered a permanent upward or downward step shift in the variance.

Note that, the three scenarios can theoretically occur with equal probability as there would be no reason to expect (without additional information such as expert knowledge about the process being monitored) that the mean would sustain a change prior to the variance (and vice versa). In fact, it might be more realistic to argue that in practice the mean and variance would change simultaneously in the event of a "special cause" as such an event might change the entire underlying process generating distribution and hence both the location and variability might be affected. Having said the aforementioned, the likelihood of the three scenarios will most likely depend on the interaction between the underlying process distribution and the special causes that may occur. The focus of this article is on scenario 2 since the results for the other scenarios follow by means of simplifications (by setting the noncentrality parameter equal to zero) and will be shown as remarks.

Suppose that the process variance has changed between samples (time periods) $\kappa - 1$ and $\kappa > 1$ from σ^2 to $\sigma_1^2 = \lambda\sigma^2$ where λ is unknown, $\lambda \neq 1$ and $\lambda > 0$, but the process mean also encountered an unknown sustained shift between samples (time periods) $h - 1$ and h where $1 < h < \kappa$. Note that, in practice h, κ and λ would be unknown (but deterministic) values. Consider the sample variance, i.e. S_i^2, before and after the shifts in the process mean and variance took place:

Before the shift in the mean:

$$\text{Samples:} \quad i = 1, 2, ..., h-1 \,.$$
$$\text{Distribution:} \quad X_{ik} \sim N\big(\mu_0, \sigma^2\big) \,.$$

$$\begin{aligned} S_i^2 &= \frac{1}{n_i}\sum_{k=1}^{n_i}(X_{ik} - \mu_0)^2 \;, \\ \frac{n_i S_i^2}{\sigma^2} &\sim \chi^2_{n_i} \;. \end{aligned}$$

After the shift in the mean:

$$\text{Samples:} \quad i = h, ..., \kappa - 1 \,.$$
$$\text{Distribution:} \quad X_{ik} \sim N\big(\mu_1 = \mu_0 + \xi_0\sigma, \, \sigma^2\big) \,.$$

[Take note: The observer is unaware of the shift in the process mean and therefore still wrongly assumes $X_{ik} \sim N(\mu_0, \sigma^2)$.
This is the key to the noncentral case because the plotting statistic and transformations (see Section 6) depends on the in-control distribution.]

$$\begin{aligned} S_i^2 &= \frac{1}{n_i}\sum_{k=1}^{n_i}(X_{ik} - \mu_0)^2 \;, \\ n_i S_i^2 &= \sum_{k=1}^{n_i}(X_{ik} - \mu_1 + \mu_1 - \mu_0)^2 \;, \\ \frac{n_i S_i^2}{\sigma^2} &= \sum_{k=1}^{n_i}\left(\frac{X_{ik} - \mu_1}{\sigma} + \frac{\mu_1 - \mu_0}{\sigma}\right)^2 \\ &= \sum_{k=1}^{n_i}(Z_{ik} + \xi_0)^2 \qquad \text{where} \quad Z_{ik} \sim N(0,1) \\ &\sim \chi'^2_{n_i}\left(\sum_{k=1}^{n_i}\xi_0^2\right) = \chi'^2_{n_i}(\delta_i) \qquad \text{where} \quad \delta_i = \sum_{k=1}^{n_i}\xi_0^2 = n_i\xi_0^2 > 0 \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{with} \quad \xi_0 = \frac{\mu_1 - \mu_0}{\sigma} \;. \end{aligned}$$

After the shift in the mean and variance:

Samples: $i = \kappa, \kappa + 1, \dots$.
Distribution: $X_{ik} \sim N\big(\mu_1 = \mu_0 + \xi_1\sigma_1,\, \sigma_1^2 = \lambda\sigma^2\big)$.

[Take note: The observer is unaware of the shifts in the process parameters and therefore still wrongly assumes $X_{ik} \sim N(\mu_0, \sigma^2)$.]

$$\begin{aligned}
S_i^2 &= \frac{1}{n_i}\sum_{k=1}^{n_i}(X_{ik} - \mu_0)^2 \ , \\
n_i S_i^2 &= \sum_{k=1}^{n_i}(X_{ik} - \mu_1 + \mu_1 - \mu_0)^2 \ , \\
\frac{n_i S_i^2}{\sigma_1^2} &= \sum_{k=1}^{n_i}\left(\frac{X_{ik} - \mu_1}{\sigma_1} + \frac{\mu_1 - \mu_0}{\sigma_1}\right)^2 \\
&= \sum_{k=1}^{n_i}(Z_{ik} + \xi_1)^2 \qquad \text{where} \quad Z_{ik} \sim N(0,1) \\
&\sim \chi'^2_{n_i}\left(\sum_{k=1}^{n_i}\xi_1^2\right) = \chi'^2_{n_i}(\delta_i) \qquad \text{where} \quad \delta_i = \sum_{k=1}^{n_i}\xi_1^2 = n_i\xi_1^2 > 0 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{with} \quad \xi_1 = \frac{\mu_1 - \mu_0}{\sigma_1} \ .
\end{aligned}$$

Remark 2.1.

(i) $\chi^2_{n_i}$ denotes a central χ^2 random variable with degrees of freedom n_i (see Johnson *et al.* (1995), Chapter 18).

(ii) $\chi'^2_{n_i}(\delta_i)$ denotes a noncentral χ^2 random variable with degrees of freedom n_i and noncentrality parameter δ_i (see Johnson *et al.* (1995), Chapter 29).

(iii) The degrees of freedom is assumed to be n_i, since the mean is not estimated because it is assumed that the mean is a fixed / deterministic value before and after the shift. In case the mean is unknown and has to be estimated too, the degrees of freedom changes from n_i to $n_i - 1$ and the μ_0 would be replaced by $\hat{\mu}_0$, i.e. an estimate of μ_0.

(iv) The shift in the mean, before the variance changed, is modelled as follows: $\xi_0 = \frac{\mu_1 - \mu_0}{\sigma}$, i.e. $\mu_1 = \mu_0 + \xi_0\sigma$.

(v) The shift in the mean, after the variance changed, is modelled as follows: $\xi_1 = \frac{\mu_1 - \mu_0}{\sigma_1}$, i.e. $\mu_1 = \mu_0 + \xi_1\sigma_1$.

(vi) The pivotal quantity $\frac{n_i S_i^2}{\sigma_1^2} \sim \chi'^2_{n_i}(\delta_i)$ after the shift in the variance reduces to a central chi-squared random variable if the process mean did not change, i.e. when $\mu_1 = \mu_0$ (see Adamski *et al.* (2012)).

Following a change in the variance between samples $\kappa-1$ and κ, define the following random variable:

$$\begin{aligned} U_0^* &= \frac{S_\kappa^2}{S_{\kappa-1}^{2\text{pooled}}} = \sum_{i=1}^{\kappa-1} n_i \times \frac{S_\kappa^2}{\sum_{i=1}^{h-1} n_i S_i^2 + \sum_{i=h}^{\kappa-1} n_i S_i^2} \\ &= \frac{\sum_{i=1}^{\kappa-1} n_i}{n_\kappa} \times \frac{\frac{n_\kappa S_\kappa^2}{\sigma_1^2} \times \frac{\sigma_1^2}{\sigma^2}}{\sum_{i=1}^{h-1} \frac{n_i S_i^2}{\sigma^2} + \sum_{i=h}^{\kappa-1} \frac{n_i S_i^2}{\sigma^2}} = \frac{\sum_{i=1}^{\kappa-1} n_i}{n_\kappa} \times \frac{\lambda\, \chi_{n_\kappa}'^2(\delta_\kappa)}{\sum_{i=1}^{\kappa-1} \chi_{n_i}'^2(\delta_i)} \ , \end{aligned}$$

where $\lambda = \frac{\sigma_1^2}{\sigma^2}$ indicates the unknown size of the shift in the variance

$$\text{and } \delta_i = \begin{cases} 0 & \text{for } i = 1, ..., h-1 \,, \\ n_i \xi_0^2 > 0 \quad \text{with } \xi_0 = \frac{\mu_1 - \mu_0}{\sigma} & \text{for } i = h, ..., \kappa - 1 \,, \\ n_\kappa \xi_1^2 > 0 \quad \text{with } \xi_1 = \frac{\mu_1 - \mu_0}{\sigma_1} & \text{for } i = \kappa \,. \end{cases}$$

[Take note: $\sum_{i=1}^{h-1} \chi_{n_i}^2 \stackrel{d}{=} \sum_{i=1}^{h-1} \chi_{n_i}'^2(0)$.]

In general, at sample $\kappa + j$, where $\kappa > 1$ and $j = 1, 2, ..., p$, we define the following sequence of random variables (all based on the two sample test statistic for testing the equality of variances):

$$\begin{aligned} U_j^* &= \frac{S_{\kappa+j}^2}{S_{\kappa+j-1}^{2\text{pooled}}} \\ &= \sum_{i=1}^{\kappa+j-1} n_i \times \frac{S_{\kappa+j}^2}{\sum_{i=1}^{h-1} n_i S_i^2 + \sum_{i=h}^{\kappa-1} n_i S_i^2 + \sum_{i=\kappa}^{\kappa+j-1} n_i S_i^2} \\ &= \frac{\sum_{i=1}^{\kappa+j-1} n_i}{n_{\kappa+j}} \times \frac{\frac{n_{\kappa+j} S_{\kappa+j}^2}{\sigma_1^2} \times \frac{\sigma_1^2}{\sigma^2}}{\sum_{i=1}^{h-1} \frac{n_i S_i^2}{\sigma^2} + \sum_{i=h}^{\kappa-1} \frac{n_i S_i^2}{\sigma^2} + \sum_{i=\kappa}^{\kappa+j-1} \frac{n_i S_i^2}{\sigma_1^2} \times \frac{\sigma_1^2}{\sigma^2}} \\ &= \frac{\sum_{i=1}^{\kappa+j-1} n_i}{n_{\kappa+j}} \times \frac{\lambda\, \chi_{n_{\kappa+j}}'^2(\delta_{\kappa+j})}{\sum_{i=1}^{\kappa-1} \chi_{n_i}'^2(\delta_i) + \lambda \sum_{i=\kappa}^{\kappa+j-1} \chi_{n_i}'^2(\delta_i)} \qquad \text{with } \lambda = \frac{\sigma_1^2}{\sigma^2} \ , \end{aligned}$$

$$\text{where } \delta_i = \begin{cases} 0 & \text{for } i = 1, ..., h-1 \,, \\ n_i \xi_0^2 > 0 \quad \text{with } \xi_0 = \frac{\mu_1 - \mu_0}{\sigma} & \text{for } i = h, ..., \kappa - 1 \,, \\ n_i \xi_1^2 > 0 \quad \text{with } \xi_1 = \frac{\mu_1 - \mu_0}{\sigma_1} & \text{for } i = \kappa, ..., \kappa + j \,. \end{cases}$$

To simplify matters going forward and for notational purposes we omit the factors $\sum_{i=1}^{\kappa-1} n_i/n_\kappa$ and $\sum_{i=1}^{\kappa+j-1} n_i/n_{\kappa+j}$, respectively, in U_0^* and U_j^*, since they do not contain any random variables, and also drop the * superscript, and therefore the random variables of interest are:

$$\begin{aligned} U_0 &= \frac{\lambda W_0}{X} \ , \\ U_j &= \frac{\lambda W_j}{X + \lambda \sum_{k=0}^{j-1} W_k} \ , \qquad j = 1, 2, ..., p \quad \text{and} \quad \lambda > 0 \ , \end{aligned} \tag{2.3}$$

where

$\lambda \;=\; \frac{\sigma_1^2}{\sigma^2}$ indicates the unknown size of the shift in the variance,

$X \;=\; \sum_{i=1}^{\kappa-1}\chi_{n_i}'^2(\delta_i) \sim \chi_a'^2(\delta_a)$, i.e. X is a noncentral chi-squared random variable with degrees of freedom, $a=\sum_{i=1}^{\kappa-1} n_i$ and noncentrality parameter $\delta_a = \sum_{i=h}^{\kappa-1}\delta_i$, $h<\kappa$ where $\delta_i = n_i\,\xi_0^2$ with $\xi_0 = \frac{\mu_1-\mu_0}{\sigma}$,

$W_i \;\sim\; \chi_{v_i}'^2(\delta_i)$, i.e. W_i is a noncentral chi-squared random variable with degrees of freedom $v_i = n_{\kappa+i}$ and noncentrality parameter $\delta_i = n_{\kappa+i}\,\xi_1^2$ with $\xi_1 = \frac{\mu_1-\mu_0}{\sigma_1}$, $i=0,1,...,p$.

Take note that X represents the sum of $\kappa-1$ independent noncentral χ^2 random variables, i.e. $\chi_{n_1}'^2, ..., \chi_{n_{\kappa-1}}'^2$ since we assume the samples are independent.

Remark 2.2. Scenarios 1 and 3 can be obtained as follows:

(i) When the process mean and variance change simultaneously (scenario 1), i.e. $h=\kappa$, then $\delta_a = 0$. The superscript (S1) in the expressions that follow indicate scenario 1 as discussed and shown in Figure 1. From (2.3) it then follows that

$$U_0^{(S1)} = \frac{\lambda W_0}{X}\,,$$
$$U_j^{(S1)} = \frac{\lambda W_j}{X+\lambda\sum_{k=0}^{j-1}W_k}\,, \qquad j=1,2,...,p \quad \text{and} \quad \lambda>0\,,$$

where

$$X = \textstyle\sum_{i=1}^{\kappa-1}\chi_{n_i}^2 \sim \chi_a^2 \quad \text{with } a=\sum_{i=1}^{\kappa-1}n_i\,,$$
$$W_i \sim \chi_{v_i}'^2(\delta_i) \;\; \text{with } v_i = n_{\kappa+i},\; \delta_i = n_{\kappa+i}\xi_1^2 \text{ and } \xi_1 = \tfrac{\mu_1-\mu_0}{\sigma_1},\; i=0,1,...,p\,.$$

(ii) For scenario 3, the process variance has changed between samples (time periods) $\kappa-1$ and $\kappa>1$, but the process mean encountered a sustained shift between samples (time periods) $h-1$ and h where $h>\kappa$, i.e. the mean changed after the variance. The random variables in (2.3) will change as follows:

$$U_0^{(S3)} = \frac{\lambda W_0}{X}\,,$$
$$U_j^{(S3)} = \frac{\lambda W_j}{X+\lambda\sum_{k=0}^{j-1}W_k}\,, \qquad j=1,2,...,p \quad \text{and} \quad \lambda>0\,,$$

where

$$X = \textstyle\sum_{i=1}^{\kappa-1}\chi_{n_i}^2 \sim \chi_a^2 \quad \text{with } a=\sum_{i=1}^{\kappa-1}n_i\,,$$
$$W_i \sim \chi_{v_i}'^2(\delta_i) \;\; \text{with } v_i = n_{\kappa+i},\; \delta_i = \begin{cases} 0 & \text{for } i=0,1,...,h-1\,, \\ n_{\kappa+i}\xi_1^2 \text{ and } \xi_1 = \frac{\mu_1-\mu_0}{\sigma_1} & \text{for } i=h,h+1,...,p\,. \end{cases}$$

(iii) If the process mean remains unchanged and only the process variance encountered a sustained shift, the components X and W_i in (2.3) will reduce to central chi-squared random variables. The joint distribution of the random variables (2.3) will then be the generalized multivariate beta distribution derived by Adamski *et al.* (2012), with the only difference being the degrees of freedom of the chi-squared random variables. This shows that the solution to the run-length distribution of a Q-chart used to monitor the parameter θ in the $\mathrm{Exp}(\theta)$ distribution (when θ is unknown) is similar to the solution to the run-length distribution when monitoring the variance with a Q-chart in case of a $N(\mu_0, \sigma^2)$ distribution where μ_0 is known and σ^2 is unknown.

3. THE EXACT DENSITY FUNCTION

In this section the joint distribution of the random variables U_0, U_1, U_2 (see (2.3)) is derived, i.e. the first three random variables following a change in the variance. In section 4 the multivariate extension is considered with a detailed proof. The reason for this unorthodox presentation of results is to first demonstrate the different marginals for the trivariate case.

Theorem 3.1. *Let X, W_i with $i = 0,1,2$ be independent noncentral chi-squared random variables with degrees of freedom a and v_i and noncentrality parameters δ_a and δ_i with $i = 0,1,2$, respectively. Let $U_0 = \frac{\lambda W_0}{X}$, $U_1 = \frac{\lambda W_1}{X+\lambda W_0}$ and $U_2 = \frac{\lambda W_2}{X+\lambda W_0+\lambda W_1}$ (see (2.3)) and $\lambda > 0$. The joint density of (U_0, U_1, U_2) is given by*

$$
\begin{aligned}
& f(u_0,u_1,u_2) \\
& \quad = \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+\delta_2}{2}\right)} \lambda^{\frac{a}{2}} \, \Gamma\!\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma\!\left(\frac{a}{2}\right)\Gamma\!\left(\frac{v_0}{2}\right)\Gamma\!\left(\frac{v_1}{2}\right)\Gamma\!\left(\frac{v_2}{2}\right)} \, u_0^{\frac{v_0}{2}-1} u_1^{\frac{v_1}{2}-1} u_2^{\frac{v_2}{2}-1} (1+u_0)^{\frac{v_1}{2}+\frac{v_2}{2}} \\
& \quad \quad \times (1+u_1)^{\frac{v_2}{2}} \Big[\lambda + u_0 + u_1(1+u_0) + u_2(1+u_0)(1+u_1)\Big]^{-\left(\frac{a+v_0+v_1+v_2}{2}\right)} \qquad (3.1) \\
& \quad \quad \times \Psi_2^{(4)}\left[\frac{a+v_0+v_1+v_2}{2}; \frac{a}{2}, \frac{v_0}{2}, \frac{v_1}{2}, \frac{v_2}{2}; \frac{\lambda\delta_a}{2z}, \frac{\delta_0 u_0}{2z}, \frac{\delta_1 u_1(1+u_0)}{2z}, \frac{\delta_2 u_2 (1+u_0)(1+u_1)}{2z}\right], \\
& \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad u_j > 0\,, \quad j = 0,1,2\,,
\end{aligned}
$$

where $z = \lambda + u_0 + u_1(1+u_0) + u_2(1+u_0)(1+u_1)$ and $\Psi_2^{(4)}$ the confluent hypergeometric function in four variables (see Sánchez et al. (2006) or Srivastava & Kashyap (1982)).

Proof: The expression for the joint density of (U_0, U_1, U_2) is obtained by setting $p = 2$ in (4.1) and applying result A.2 of Sanchez *et al.* (2006). □

Remark 3.1.

(i) For the special case when $\lambda = 1$ (i.e. the process variance did not encounter a shift although the mean did), this trivariate density (3.1) simplifies to

$$f(u_0, u_1, u_2)$$

$$= \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+\delta_2}{2}\right)} \Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{v_0}{2}\right)\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} u_0^{\frac{v_0}{2}-1} u_1^{\frac{v_1}{2}-1} u_2^{\frac{v_2}{2}-1} (1+u_0)^{\frac{v_1}{2}+\frac{v_2}{2}} (1+u_1)^{\frac{v_2}{2}}$$
$$\times \left[(1+u_0)(1+u_1)(1+u_2)\right]^{-\left(\frac{a+v_0+v_1+v_2}{2}\right)}$$
$$\times \Psi_2^{(4)}\left[\frac{a+v_0+v_1+v_2}{2}; \frac{a}{2}, \frac{v_0}{2}, \frac{v_1}{2}, \frac{v_2}{2}; \frac{\delta_a}{2y}, \frac{\delta_0 u_0}{2y}, \frac{\delta_1 u_1 (1+u_0)}{2y}, \frac{\delta_2 u_2 (1+u_0)(1+u_1)}{2y}\right],$$

where $y = (1+u_0)(1+u_1)(1+u_2)$.

(ii) When the shift in the mean and the variance occurs simultaneously (scenario 1), we have that $\delta_a = 0$, and it follows that the trivariate density (3.1) is given by

$$f(u_0, u_1, u_2)$$

$$= \frac{e^{-\left(\frac{\delta_0+\delta_1+\delta_2}{2}\right)} \lambda^{\frac{a}{2}} \Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{v_0}{2}\right)\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} u_0^{\frac{v_0}{2}-1} u_1^{\frac{v_1}{2}-1} u_2^{\frac{v_2}{2}-1} (1+u_0)^{\frac{v_1}{2}+\frac{v_2}{2}} (1+u_1)^{\frac{v_2}{2}}$$
$$\times \left[\lambda + u_0 + u_1(1+u_0) + u_2(1+u_0)(1+u_1)\right]^{-\left(\frac{a+v_0+v_1+v_2}{2}\right)}$$
$$\times \Psi_2^{(3)}\left[\frac{a+v_0+v_1+v_2}{2}; \frac{v_0}{2}, \frac{v_1}{2}, \frac{v_2}{2}; \frac{\delta_0 u_0}{2z}, \frac{\delta_1 u_1 (1+u_0)}{2z}, \frac{\delta_2 u_2 (1+u_0)(1+u_1)}{2z}\right],$$

where $z = \lambda + u_0 + u_1(1+u_0) + u_2(1+u_0)(1+u_1)$ with $\Psi_2^{(3)}$ the confluent hypergeometric function in three variables.

(iii) When monitoring the variance and the mean did not change, i.e. $\delta_a = \delta_0 = \delta_1 = \delta_2 = 0$, the trivariate density (3.1) simplifies to the generalized multivariate beta distribution, derived by Adamski *et al.* (2012):

$$f(u_0, u_1, u_2)$$

$$= \frac{\lambda^{\frac{a}{2}} \Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{v_0}{2}\right)\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} u_0^{\frac{v_0}{2}-1} u_1^{\frac{v_1}{2}-1} u_2^{\frac{v_2}{2}-1} (1+u_0)^{\frac{v_1}{2}+\frac{v_2}{2}} (1+u_1)^{\frac{v_2}{2}}$$
$$\times \left[\lambda + u_0 + u_1(1+u_0) + u_2(1+u_0)(1+u_1)\right]^{-\left(\frac{a+v_0+v_1+v_2}{2}\right)}.$$

3.1. Bivariate cases

Theorem 3.2. *Let X, W_i with $i=0,1,2$ be independent noncentral chi-squared random variables with degrees of freedom a and v_i and noncentrality parameters δ_a and δ_i with $i=0,1,2$, respectively. Let $U_0=\frac{\lambda W_0}{X}$, $U_1=\frac{\lambda W_1}{X+\lambda W_0}$ and $U_2=\frac{\lambda W_2}{X+\lambda W_0+\lambda W_1}$ and $\lambda>0$.*

(**a**) *The joint density of (U_0,U_1) is given by*

$$f(u_0,u_1) \tag{3.2}$$

$$= \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1}{2}\right)}\lambda^{\frac{a}{2}}\,\Gamma\left(\frac{a+v_0+v_1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{v_0}{2}\right)\Gamma\left(\frac{v_1}{2}\right)}\, u_0^{\frac{v_0}{2}-1}\, u_1^{\frac{v_1}{2}-1}(1+u_0)^{\frac{v_1}{2}}\left[\lambda+u_0+u_1(1+u_0)\right]^{-\left(\frac{a+v_0+v_1}{2}\right)}$$
$$\times\, \Psi_2^{(3)}\left[\frac{a+v_0+v_1}{2};\frac{a}{2},\frac{v_0}{2},\frac{v_1}{2};\frac{\lambda\,\delta_a}{2\left[\lambda+u_0+u_1(1+u_0)\right]},\frac{\delta_0\,u_0}{2\left[\lambda+u_0+u_1(1+u_0)\right]},\frac{\delta_1 u_1\,(1+u_0)}{2\left[\lambda+u_0+u_1(1+u_0)\right]}\right],$$
$$u_j>0\,,\quad j=0,1\,.$$

(**b**) *The joint density of (U_0,U_2) is given by*

$$f(u_0,u_2) \tag{3.3}$$

$$= \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+\delta_2}{2}\right)}\lambda^{\frac{a}{2}}\,\Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)\Gamma\left(\frac{a+v_0}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{v_0}{2}\right)\Gamma\left(\frac{v_2}{2}\right)\Gamma\left(\frac{a+v_0+v_1}{2}\right)}\, u_0^{\frac{v_0}{2}-1}(1+u_0)^{-\left(\frac{a+v_0}{2}\right)}\, u_2^{\frac{v_2}{2}-1}$$
$$\times\,(1+u_2)^{-\left(\frac{a+v_0+v_1+v_2}{2}\right)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\sum_{k_5=0}^{\infty}\frac{\left(\frac{a+v_0+v_1+v_2}{2}\right)_{k_1+k_2+k_3+k_4+k_5}}{\left(\frac{a}{2}\right)_{k_1}\left(\frac{v_0}{2}\right)_{k_2}\left(\frac{v_2}{2}\right)_{k_4}}$$
$$\times\,\frac{\left(\frac{a+v_0}{2}\right)_{k_1+k_2+k_5}}{\left(\frac{a+v_0+v_1}{2}\right)_{k_1+k_2+k_3+k_5}\,k_1!\,k_2!\,k_3!\,k_4!\,k_5!}\left(\frac{\lambda\,\delta_a}{2\,(1+u_0)\,(1+u_2)}\right)^{k_1}$$
$$\times\left(\frac{\delta_0\,u_0}{2\,(1+u_0)\,(1+u_2)}\right)^{k_2}\left(\frac{\delta_1}{2\,(1+u_2)}\right)^{k_3}\left(\frac{\delta_2\,u_2}{2\,(1+u_2)}\right)^{k_4}\left(\frac{1-\lambda}{(1+u_0)\,(1+u_2)}\right)^{k_5},$$
$$u_j>0\,,\quad j=0,2\,.$$

(**c**) *The joint density of (U_1,U_2) is given by*

$$f(u_1,u_2) \tag{3.4}$$

$$= \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+\delta_2}{2}\right)}\lambda^{\frac{a}{2}}\,\Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)\Gamma\left(\frac{a+v_0}{2}\right)}\, u_1^{\frac{v_1}{2}-1}(1+u_1)^{-\left(\frac{a+v_0+v_1}{2}\right)}\, u_2^{\frac{v_2}{2}-1}$$
$$\times\,(1+u_2)^{-\left(\frac{a+v_0+v_1+v_2}{2}\right)}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\sum_{k_5=0}^{\infty}\frac{\left(\frac{a+v_0+v_1+v_2}{2}\right)_{k_1+k_2+k_3+k_4+k_5}}{\left(\frac{a}{2}\right)_{k_1}\left(\frac{v_1}{2}\right)_{k_3}\left(\frac{v_2}{2}\right)_{k_4}}\;\times$$

$$\times \frac{\left(\frac{a}{2}\right)_{k_1+k_5}}{\left(\frac{a+v_0}{2}\right)_{k_1+k_2+k_5} k_1!\, k_2!\, k_3!\, k_4!\, k_5!} \left(\frac{\lambda\,\delta_a}{2\,(1+u_1)\,(1+u_2)}\right)^{k_1}$$

$$\times \left(\frac{\delta_0}{2\,(1+u_1)\,(1+u_2)}\right)^{k_2} \left(\frac{\delta_1\,u_1}{2\,(1+u_1)\,(1+u_2)}\right)^{k_3} \left(\frac{\delta_2\,u_2}{2\,(1+u_2)}\right)^{k_4} \left(\frac{1-\lambda}{(1+u_1)\,(1+u_2)}\right)^{k_5},$$

$$u_j > 0\,, \quad j = 1, 2\ .$$

Proof: (**a**) Expanding $\Psi_2^{(4)}(\cdot)$ in equation (3.1) in series form and integrating this trivariate density with respect to u_2, yields

$$f(u_0, u_1)$$

$$= \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+\delta_2}{2}\right)}\lambda^{\frac{a}{2}}\,\Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})}\; u_0^{\frac{v_0}{2}-1}\, u_1^{\frac{v_1}{2}-1}\, (1+u_0)^{\frac{v_1+v_2}{2}}\, (1+u_1)^{\frac{v_2}{2}}$$

$$\times \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty} \frac{\left(\frac{a+v_0+v_1+v_2}{2}\right)_{k_1+k_2+k_3+k_4}}{\left(\frac{a}{2}\right)_{k_1}\left(\frac{v_0}{2}\right)_{k_2}\left(\frac{v_1}{2}\right)_{k_3}\left(\frac{v_2}{2}\right)_{k_4} k_1!k_2!k_3!k_4!} \left(\frac{\lambda\delta_a}{2}\right)^{k_1}$$

$$\times \left(\frac{\delta_0 u_0}{2}\right)^{k_2} \left(\frac{\delta_1 u_1(1+u_0)}{2}\right)^{k_3} \left(\frac{\delta_2(1+u_0)(1+u_1)}{2}\right)^{k_4}$$

$$\times \int_0^{\infty} u_2^{\frac{v_2}{2}+k_4-1}\Big[\lambda+u_0+u_1\,(1+u_0)+u_2(1+u_0)(1+u_1)\Big]^{-\left(\frac{a+v_0+v_1+v_2}{2}+k_1+k_2+k_3+k_4\right)} du_2\ .$$

Take note:

$$\int_0^{\infty} u_2^{\frac{v_2}{2}+k_4-1}\Big[\lambda+u_0+u_1(1+u_0)+u_2(1+u_0)(1+u_1)\Big]^{-\left(\frac{a+v_0+v_1+v_2}{2}+k_1+k_2+k_3+k_4\right)} du_2$$

$$= \Big[\lambda+u_0+u_1(1+u_0)\Big]^{-\left(\frac{a+v_0+v_1+v_2}{2}+k_1+k_2+k_3+k_4\right)}$$

$$\times \int_0^{\infty} u_2^{\frac{v_2}{2}+k_4-1}\left[1+\frac{u_2(1+u_0)(1+u_1)}{\lambda+u_0+u_1\,(1+u_0)}\right]^{-\left(\frac{a+v_0+v_1+v_2}{2}+k_1+k_2+k_3+k_4\right)} du_2\ .$$

Using Gradshteyn and Ryzhik (2007) Eq. 3.194.3 p. 315, the joint density of U_0 and U_1 in expression (3.2) follows after simplification.

Remark 3.2.

(**i**) Alternatively, the proof of this theorem can be derived by substituting $p = 1$ in (4.1).

(**ii**) If $\delta_a = \delta_0 = \delta_1 = 0$, the density simplifies to the bivariate distribution derived by Adamski *et al.* (2012):

$$f(u_0,u_1)$$

$$= \frac{\lambda^{\frac{a}{2}}\Gamma\left(\frac{a+v_0+v_1}{2}\right)}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})\Gamma(\frac{v_1}{2})} u_0^{\frac{v_0}{2}-1}(1+u_0)^{-\left(\frac{a+v_0}{2}\right)} u_1^{\frac{v_1}{2}-1}(1+u_1)^{-\left(\frac{a+v_0+v_1}{2}\right)}$$

$$\times\left[\frac{\lambda+u_0+u_1(1+u_0)}{(1+u_0)(1+u_1)}\right]^{-\left(\frac{a+v_0+v_1}{2}\right)}.$$

This can be rewritten using the binomial series ${}_1F_0(\alpha;z)=(1-z)^{-\alpha}$, for $|z|<1$ (Mathai (1993) p. 25) with $1-z=\frac{\lambda+u_0+u_1(1+u_0)}{(1+u_0)(1+u_1)}$. Therefore

$$f(u_0,u_1)$$

$$= \frac{\Gamma(\frac{a+v_0+v_1}{2})\lambda^{\frac{a}{2}}}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})\Gamma(\frac{v_1}{2})} u_0^{\frac{v_0}{2}-1}(1+u_0)^{-\left(\frac{a+v_0}{2}\right)} u_1^{\frac{v_1}{2}-1}(1+u_1)^{-\left(\frac{a+v_0+v_1}{2}\right)}$$

$$\times\ {}_1F_0\left(\frac{a+v_0+v_1}{2};\frac{1-\lambda}{(1+u_0)(1+u_1)}\right).$$

(**b**) Expanding $\Psi_2^{(4)}(\cdot)$ in equation (3.1) in series form and integrating the trivariate density (3.1) with respect to u_1, it follows that

$$f(u_0,u_2)$$

$$= \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+\delta_2}{2}\right)}\lambda^{\frac{a}{2}}\,\Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} u_0^{\frac{v_0}{2}-1}(1+u_0)^{\frac{v_1+v_2}{2}} u_2^{\frac{v_2}{2}-1}$$

$$\times\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\frac{\left(\frac{a+v_0+v_1+v_2}{2}\right)_{k_1+k_2+k_3+k_4}}{\left(\frac{a}{2}\right)_{k_1}\left(\frac{v_0}{2}\right)_{k_2}\left(\frac{v_1}{2}\right)_{k_3}\left(\frac{v_2}{2}\right)_{k_4}k_1!k_2!k_3!k_4!}\left(\frac{\lambda\delta_a}{2}\right)^{k_1}\left(\frac{\delta_0u_0}{2}\right)^{k_2}$$

$$\times\left(\frac{\delta_1(1+u_0)}{2}\right)^{k_3}\left(\frac{\delta_2u_2(1+u_0)}{2}\right)^{k_4}\int_0^{\infty}u_1^{\frac{v_1}{2}+k_3-1}(1+u_1)^{\frac{v_2}{2}+k_4}$$

$$\times\Big[\lambda+u_0+u_1(1+u_0)+u_2(1+u_0)(1+u_1)\Big]^{-\left(\frac{a+v_0+v_1+v_2}{2}+k_1+k_2+k_3+k_4\right)}du_1\,.$$

Using Gradshteyn and Ryzhik (2007) Eq. 3.197.5 p. 317 and Eq. 9.131.1 p. 998, it follows that

$$f(u_0,u_2)=\frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+\delta_2}{2}\right)}\lambda^{\frac{a}{2}}\,\Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} u_0^{\frac{v_0}{2}-1}(1+u_0)^{\frac{v_1+v_2}{2}} u_2^{\frac{v_2}{2}-1}$$

$$\times\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\frac{\left(\frac{a+v_0+v_1+v_2}{2}\right)_{k_1+k_2+k_3+k_4}}{\left(\frac{a}{2}\right)_{k_1}\left(\frac{v_0}{2}\right)_{k_2}\left(\frac{v_1}{2}\right)_{k_3}\left(\frac{v_2}{2}\right)_{k_4}k_1!k_2!k_3!k_4!}$$

$$\times\left(\frac{\lambda\delta_a}{2(1+u_0)(1+u_2)}\right)^{k_1}\left(\frac{\delta_0u_0}{2(1+u_0)(1+u_2)}\right)^{k_2}\left(\frac{\delta_1(1+u_0)}{2(1+u_0)(1+u_2)}\right)^{k_3}$$

$$\times\left(\frac{\delta_2 u_2(1+u_0)}{2(1+u_0)(1+u_2)}\right)^{k_4}\frac{\Gamma\left(\frac{v_1}{2}+k_3\right)\Gamma\left(\frac{a+v_0}{2}+k_1+k_2\right)}{\Gamma\left(\frac{a+v_0+v_1}{2}+k_1+k_2+k_3\right)}\left[(1+u_0)(1+u_2)\right]^{-\left(\frac{a+v_0+v_1+v_2}{2}\right)}$$

$$\times\,{}_2F_1\left(\frac{a+v_0+v_1+v_2}{2}+k_1+k_2+k_3+k_4,\frac{a+v_0}{2}+k_1+k_2;\frac{a+v_0+v_1}{2}+k_1+k_2+k_3;\frac{1-\lambda}{(1+u_0)(1+u_2)}\right).$$

Expanding the Gauss hypergeometric function, ${}_2F_1(\cdot)$ (see Gradshteyn and Ryzhik (2007)), in series form, the desired result (3.3) follows after simplification.

(**c**) Proof follows similarly as in (b). □

3.2. Univariate cases

Theorem 3.3. *Let X, W_i with $i=0,1,2$ be independent noncentral chi-squared random variables with degrees of freedom a and v_i and noncentrality parameters δ_a and δ_i with $i=0,1,2$, respectively. Let $U_0=\frac{\lambda W_0}{X}$, $U_1=\frac{\lambda W_1}{X+\lambda W_0}$ and $U_2=\frac{\lambda W_2}{X+\lambda W_0+\lambda W_1}$ and $\lambda>0$. The marginal density of*

(**a**) *U_0 is given by*

$$f(u_0) = \frac{e^{-\left(\frac{\delta_a+\delta_0}{2}\right)}\lambda^{\frac{a}{2}}\Gamma\left(\frac{a+v_0}{2}\right)}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})}u_0^{\frac{v_0}{2}-1}(u_0+\lambda)^{-\left(\frac{a+v_0}{2}\right)}$$
$$\times\,\Psi_2\left(\frac{a+v_0}{2};\frac{a}{2},\frac{v_0}{2};\frac{\lambda\delta_a}{2\,(u_0+\lambda)},\frac{\delta_0 u_0}{2\,(u_0+\lambda)}\right),\qquad u_0>0\,, \tag{3.5}$$

with Ψ_2 the Humbert confluent hypergeometric function of two variables (see Sanchez et al. (2006)),

(**b**) *U_1 is given by*

$$f(u_1) = \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1}{2}\right)}\lambda^{\frac{a}{2}}\Gamma\left(\frac{a+v_0+v_1}{2}\right)}{\Gamma(\frac{v_1}{2})\Gamma\left(\frac{a+v_0}{2}\right)}u_1^{\frac{v_1}{2}-1}(1+u_1)^{-\left(\frac{a+v_0+v_1}{2}\right)}$$
$$\times\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\frac{\left(\frac{a+v_0+v_1}{2}\right)_{k_1+k_2+k_3+k_4}\left(\frac{a}{2}\right)_{k_1+k_4}}{\left(\frac{a}{2}\right)_{k_1}\left(\frac{v_1}{2}\right)_{k_3}\left(\frac{a+v_0}{2}\right)_{k_1+k_2+k_4}k_1!k_2!k_3!k_4!} \tag{3.6}$$
$$\times\left(\frac{\lambda\delta_a}{2\,(1+u_1)}\right)^{k_1}\left(\frac{\delta_0}{2\,(1+u_1)}\right)^{k_2}\left(\frac{\delta_1 u_1}{2\,(1+u_1)}\right)^{k_3}\left(\frac{1-\lambda}{(1+u_1)}\right)^{k_4},$$
$$u_1>0\,,$$

(**c**) *U_2 is given by*

$$f(u_2) = \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+\delta_2}{2}\right)} \lambda^{\frac{a}{2}} \Gamma\left(\frac{a+v_0+v_1+v_2}{2}\right)}{\Gamma(\frac{v_2}{2})\Gamma\left(\frac{a+v_0+v_1}{2}\right)} u_2^{\frac{v_2}{2}-1} (1+u_2)^{-\left(\frac{a+v_0+v_1+v_2}{2}\right)}$$

$$\text{(3.7)} \qquad \times \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\sum_{k_4=0}^{\infty}\sum_{k_5=0}^{\infty} \frac{\left(\frac{a+v_0+v_1+v_2}{2}\right)_{k_1+k_2+k_3+k_4+k_5} \left(\frac{a}{2}\right)_{k_1+k_5}}{\left(\frac{a}{2}\right)_{k_1} \left(\frac{v_2}{2}\right)_{k_4} \left(\frac{a+v_0+v_1}{2}\right)_{k_1+k_2+k_3+k_5} k_1!k_2!k_3!k_4!k_5!}$$

$$\times \left(\frac{\lambda\delta_a}{2(1+u_2)}\right)^{k_1} \left(\frac{\delta_0}{2(1+u_2)}\right)^{k_2} \left(\frac{\delta_1}{2(1+u_2)}\right)^{k_3} \left(\frac{\delta_2 u_2}{2(1+u_2)}\right)^{k_4} \left(\frac{1-\lambda}{(1+u_2)}\right)^{k_5},$$

$$u_2 > 0\,.$$

Proof: (**a**) Using Gradshteyn and Ryzhik (2007) Eq. 3.194.3 p. 315, the result (3.5) follows after simplification.

Remark 3.3. If $\delta_a = \delta_0 = 0$, the density simplifies to the univariate distribution derived by Adamski *et al.* (2012), namely

$$f(u_0) = \frac{\lambda^{\frac{a}{2}} \Gamma\left(\frac{a+v_0}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{v_0}{2}\right)} u_0^{\frac{v_0}{2}-1} (u_0+\lambda)^{-\left(\frac{a+v_0}{2}\right)}\,.$$

(**b**) Using the definition of the beta type II integral function (see Prudnikov *et al.* (1986) Eq. 2.2.4(24) p. 298) yields the desired result.

(**c**) Proof follows similarly as in (b). □

4. MULTIVARIATE EXTENSION

In this section the noncentral generalized multivariate beta type II distribution is proposed.

Theorem 4.1. *Let X, W_i with $i = 0, 1, 2, ..., p$ be independent noncentral chi-squared random variables with degrees of freedom a and v_i and noncentrality parameters δ_a and δ_i with $i = 0, 1, 2, ..., p$, respectively. Let $U_0 = \frac{\lambda W_0}{X}$, and $U_j = \frac{\lambda W_j}{X+\lambda\sum_{k=0}^{j-1} W_k}$ where $j = 1, 2, ..., p$, and $\lambda > 0$. The joint density of $(U_0, U_1, ..., U_p)$*

is given by

$$f\left(u_0,u_1,...,u_p\right)$$

$$= \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+...+\delta_p}{2}\right)}\Gamma\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}\right)\lambda^{\frac{a}{2}}}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{v_0}{2}\right)\cdots\Gamma\left(\frac{v_p}{2}\right)}\left(\prod_{j=0}^{p}u_j^{\frac{v_j}{2}-1}\right)\left(\prod_{k=0}^{p-1}\left(1+u_k\right)^{\sum_{j=k+1}^{p}\frac{v_j}{2}}\right)$$

$$\times\left(\lambda+u_0+\sum_{j=1}^{p}u_j\prod_{k=0}^{j-1}\left(1+u_k\right)\right)^{-\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}\right)} \tag{4.1}$$

$$\times\,\Psi_2^{(p+2)}\left[\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2};\frac{a}{2},\frac{v_0}{2},...,\frac{v_p}{2};\frac{\lambda\delta_a}{2z},\frac{\delta_0u_0}{2z},\frac{\delta_1u_1(1+u_0)}{2z},...,\frac{\delta_pu_p\prod_{k=0}^{j-1}(1+u_k)}{2z}\right],$$

$$u_j>0,\quad j=1,2,...,p\ ,$$

where $z=\lambda+u_0+\sum_{j=1}^{p}u_j\prod_{k=0}^{j-1}\left(1+u_k\right)$ *and* $\Psi_2^{(p+2)}$ *the confluent hypergeometric function in* $p+2$ *variables.*

Proof: The joint density of $X,W_0,W_1,...,W_p$ is

$$f\left(x,w_0,w_1,...,w_p\right)$$

$$= \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+...+\delta_p}{2}\right)}}{2^{\frac{a+v_0+...+v_p}{2}}\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{v_0}{2}\right)\cdots\Gamma\left(\frac{v_p}{2}\right)}\,{}_0F_1\left(\frac{a}{2};\frac{\delta_ax}{4}\right)\,{}_0F_1\left(\frac{v_0}{2};\frac{\delta_0w_0}{4}\right)\,{}_0F_1\left(\frac{v_1}{2};\frac{\delta_1w_1}{4}\right)$$
$$\times\,{}_0F_1\left(\frac{v_2}{2};\frac{\delta_2w_2}{4}\right)\cdots{}_0F_1\left(\frac{v_p}{2};\frac{\delta_pw_p}{4}\right)$$
$$\times\,x^{\frac{a}{2}-1}w_0^{\frac{v_0}{2}-1}w_1^{\frac{v_1}{2}-1}w_2^{\frac{v_2}{2}-1}\cdots w_p^{\frac{v_p}{2}-1}e^{-\frac{1}{2}\left(x+w_0+w_1+w_2+...+w_p\right)}$$

where ${}_0F_1\left(a;z\right)=\sum_{j=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a+j)}\frac{z^j}{j!}=\sum_{j=0}^{\infty}\frac{z^j}{(a)_jj!}$ where $\left(\alpha\right)_j$ is the Pochhammer coefficient defined as $\left(\alpha\right)_j=\alpha\left(\alpha+1\right)\cdots\left(\alpha+j-1\right)=\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}$ (see Johnson *et al.* (1995), Chapter 1).

Let $U=X$, $U_0=\frac{\lambda W_0}{X}$ and $U_j=\frac{\lambda W_j}{X+\lambda\sum_{k=0}^{j-1}W_k}$ where $j=1,2,...,p$.

This gives the inverse transformation: $X=U$, $W_0=\frac{1}{\lambda}U_0U$ and $W_j=\frac{1}{\lambda}U_j\left(U+\lambda\sum_{k=0}^{j-1}W_k\right)=\frac{1}{\lambda}U_jU\prod_{k=0}^{j-1}\left(1+U_k\right)$ where $j=1,2,...,p$, with Jacobian

$$J\left(x,w_0,..,w_p\to u,u_0,..,u_p\right)=\frac{u}{\lambda}\prod_{j=1}^{p}\frac{u\prod_{k=0}^{j-1}(1+u_k)}{\lambda}=\left(\frac{u}{\lambda}\right)^{p+1}\prod_{k=0}^{p-1}\left(1+u_k\right)^{p-k}.$$

Thus, the joint density of $U, U_0, U_1, ..., U_p$ is

$$f(u, u_0, .., u_p) = \frac{e^{-\left(\frac{\delta_a+\delta_0+\delta_1+...+\delta_p}{2}\right)}\lambda^{\left(-\sum_{j=0}^{p}\frac{v_j}{2}\right)}}{2^{\frac{a+v_0+...+v_p}{2}}\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})...\Gamma(\frac{v_p}{2})}\, {}_0F_1\left(\frac{a}{2};\frac{\delta_a u}{4}\right)\, {}_0F_1\left(\frac{v_0}{2};\frac{\delta_0 u_0 u}{4\lambda}\right)$$

$$\times\left(\prod_{j=1}^{p} {}_0F_1\left(\frac{v_j}{2};\frac{\delta_j u_j u\prod_{k=0}^{j-1}(1+u_k)}{4\lambda}\right)\right) u^{\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}-1} u_0^{\frac{v_0}{2}-1}\left(\prod_{j=1}^{p} u_j^{\frac{v_j}{2}-1}\right)$$

$$\times\left(\prod_{k=0}^{p-1}(1+u_k)^{\sum_{j=k+1}^{p}\frac{v_j}{2}}\right) e^{-\frac{u}{2}\left(1+\frac{u_0}{\lambda}+\sum_{j=1}^{p}\frac{u_j}{\lambda}\prod_{k=0}^{j-1}(1+u_k)\right)}.$$

Note that $\prod_{j=1}^{p}\left[\prod_{k=0}^{j-1}(1+u_k)\right]^{\frac{v_j}{2}-1} = \prod_{k=0}^{p-1}(1+u_k)^{\sum_{j=k+1}^{p}\frac{v_j}{2}-(p-k)}$. Expanding the ${}_0F_1(\cdot)$ expressions in series form, integrating with respect to u and using the definition of the gamma integral function (see Prudnikov *et al.* (1986) Eq. 2.3.3(1), p. 322) yields the result (4.1). □

Remark 4.1. If $\delta_a = \delta_0 = \delta_1 = ... = \delta_p = 0$, the distribution with density given in (4.1) simplifies to the multivariate distribution derived by Adamski *et al.* (2012)

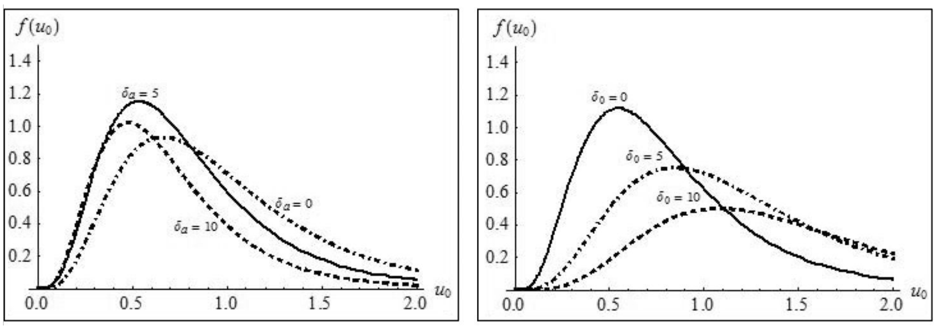
$$f(u_0, u_1, ..., u_p) = \frac{\Gamma\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}\right)\lambda^{\frac{a}{2}}}{\Gamma(\frac{a}{2})\Gamma(\frac{v_0}{2})...\Gamma(\frac{v_p}{2})}\left(\prod_{j=0}^{p} u_j^{\frac{v_j}{2}-1}\right)\left(\prod_{k=0}^{p-1}(1+u_k)^{\sum_{j=k+1}^{p}\frac{v_j}{2}}\right)$$

$$\times\left(\lambda+u_0+\sum_{j=1}^{p}u_j\prod_{k=0}^{j-1}(1+u_k)\right)^{-\left(\frac{a}{2}+\sum_{j=0}^{p}\frac{v_j}{2}\right)}.$$

5. SHAPE ANALYSIS

In this section the shape of the univariate and bivariate marginal densities will be illustrated and the influence of the noncentrality parameters will be investigated.

Panels (i) and (ii) of Figure 2 illustrate the effect of the noncentrality parameters δ_a and δ_0 on the univariate marginal density of U_0 (see equation (3.5)).



(**i**) Role of δ_a for $\delta_0 = 2$. (**ii**) Role of δ_0 for $\delta_a = 0$.

Figure 2: The marginal density function for different values of the parameters δ_a and δ_0 for $\lambda = 1.5$, $\kappa = 3$, $a = 20$ and $\nu_0 = 10$.

In terms of the process control application the parameters can be interpreted as follows:

a: pooled number of observations before the shift in the unknown variance took place,

v_0: sample size at time period κ, the first sample following the shift in the variance; the shift in the variance took place between samples $\kappa - 1$ and κ,

δ_a: noncentrality parameter that quantifies the change in the mean before the change in the variance took place,

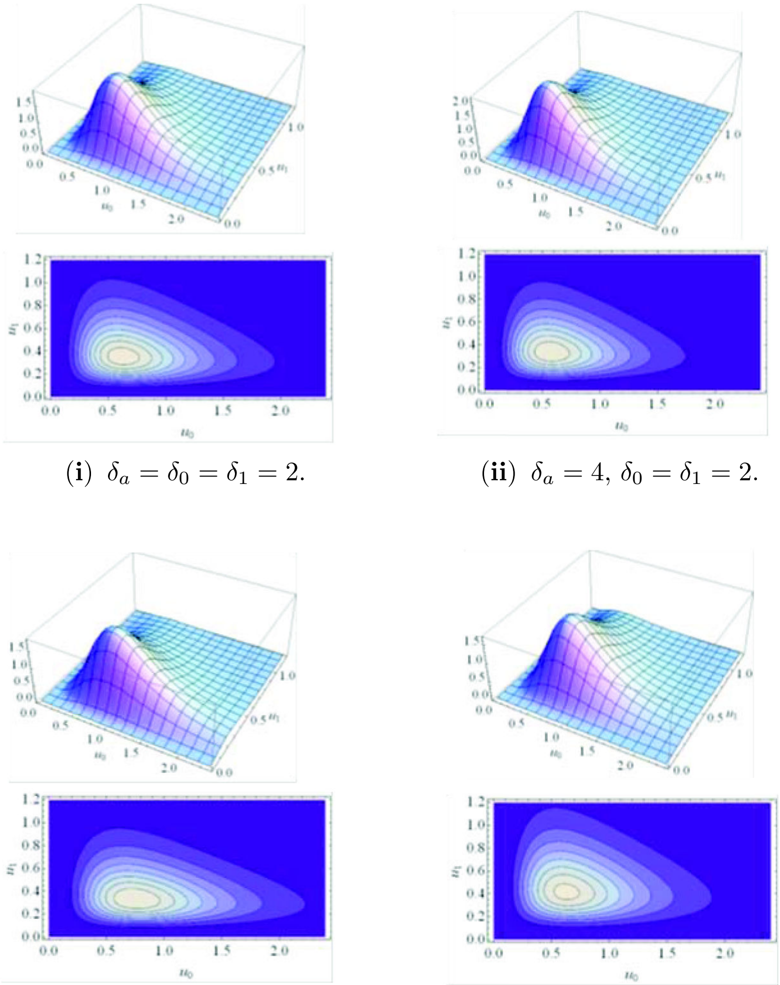
[Take note: if the mean and variance changes simultaneously, then $\delta_a = 0$.]

δ_0: noncentrality parameter that quantifies the change in the mean after the change in the variance took place,

λ: size of the unknown shift in the variance.

Panel (i) shows the effect of δ_a; we observe that as δ_a increases the density initially moves towards the vertical axis and then towards the horizontal axis. In panel (ii) the density moves towards the horizontal axis for bigger values of δ_0. The influence of the parameters a, v_0 and λ on the marginal density is discussed in detail in Adamski *et al.* (2012).

Panels (i) to (iv) of Figure 3 illustrate the effect of the noncentrality parameters δ_a, δ_0 and δ_1 on the bivariate density of U_0, U_1 (see equation (3.2)) for $\lambda = 1.5$, $\kappa = 3$, $a = 20$, $v_0 = v_1 = 10$. For $\lambda < 1$ the pattern is similar. The effect of λ is addressed in Adamski *et al.* (2012).



(i) $\delta_a = \delta_0 = \delta_1 = 2.$ **(ii)** $\delta_a = 4,\ \delta_0 = \delta_1 = 2.$

(iii) $\delta_a = 2,\ \delta_0 = 4,\ \delta_1 = 2.$ **(iv)** $\delta_a = \delta_0 = 2,\ \delta_1 = 4.$

Figure 3: The bivariate density of U_0, U_1.

6. PROBABILITY CALCULATIONS

A practical example (based on simulated data) of calculating the probability that a control chart will signal after the process variance and mean encountered a sustained shift, is considered.

At time period κ the plotting statistic for the Q-chart is constructed by calculating the statistic $U_0^* = \frac{S_\kappa^2}{S_{\kappa-1}^{2\,pooled}}$, transforming this statistic to obtain a standard normal random variable, denoted $Q\left(U_0^*\right)$ and plotting $Q\left(U_0^*\right)$ on a Shewhart-type chart where the control limits are $UCL/LCL = \pm 3$ and the centerline is $CL = 0$ (see Human and Chakraborti (2010)). When transforming the statistic to a normal random variable, Q-charts make use of the classical probability integral transformation theorem (see Quesenberry (1991)).

The marginal density of U_0 can be used to determine the probability of detecting the shift in the process variance immediately, i.e. when collecting the first sample after the shift took place. Once a shift in the process parameter occurred, the run-length is the number of samples collected from time κ (i.e. first sample after the change) until an out-of-control signal is observed (i.e. a plotting statistic plots on or outside the control limits). The discrete random variable defining the run-length is called the run-length random variable and typically denoted by N. The distribution of N is called the run-length distribution. The probability of detecting a shift immediately, in other words, the probability of a run-length of one, is the likelihood that a signal is obtained at time κ. The probability that the run-length is one, is one minus the probability that the random variable, U_0^*, plots between the control limits,

$$\text{(6.1)} \qquad \Pr(N=1) \;=\; 1-\int_{LCL}^{UCL} f(u_0^*)\,du_0^* \;=\; 1-\int_{LCL_\kappa}^{UCL_\kappa} f(u_0)\,du_0 \;.$$

Take note that the difference between the random variables U_0^* and U_0 (refer to the definitions on page 7 and 8 of the introduction) will be incorporated in the control limits of the control chart.

When the process is in-control, i.e. $\lambda = 1$ and the process mean did not encounter a shift, $U_0^* = \dfrac{W_0/n_\kappa}{X/\sum_{i=1}^{\kappa-1} n_i} \sim F\left(n_\kappa, \sum_{i=1}^{\kappa-1} n_i\right)$, then the Q plotting statistic is given by $Q\left(U_0^*\right) = \Phi^{-1}\left[F\left(U_0^*\right)\right]$ and the control limits UCL_κ and LCL_κ are determined as follows:

$$\begin{aligned}
& -3 < \Phi^{-1}\left[F\left(U_0^*\right)\right] < 3 \\
\Longleftrightarrow\quad & \Phi(-3) < F\left(U_0^*\right) < \Phi(3) \\
\Longleftrightarrow\quad & F^{-1}\left[\Phi(-3)\right] < U_0^* < F^{-1}\left[\Phi(3)\right] \\
\Longleftrightarrow\quad & \frac{F^{-1}\left[\Phi(-3)\right]}{\frac{\sum_{i=1}^{\kappa-1} n_i}{n_\kappa}} < U_0 < \frac{F^{-1}\left[\Phi(3)\right]}{\frac{\sum_{i=1}^{\kappa-1} n_i}{n_\kappa}}
\end{aligned}$$

where

$F(\cdot)$ denotes the cumulative distribution function of the F distribution,

$F^{-1}(\cdot)$ denotes the inverse of the cumulative distribution function of the F distribution,

$\Phi(\cdot)$ denotes the standard normal cumulative distribution function,

$\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal cumulative distribution function.

Therefore $UCL_\kappa = \frac{F^{-1}[\Phi(3)]}{\frac{\sum_{i=1}^{\kappa-1} n_i}{n_\kappa}}$ and $LCL_\kappa = \frac{F^{-1}[\Phi(-3)]}{\frac{\sum_{i=1}^{\kappa-1} n_i}{n_\kappa}}$. Note that LCL_κ and UCL_κ depend on κ whereas LCL and UCL equals -3 and 3, respectively (regardless the value of κ).

The probability that the run-length is two can be calculated by defining the following two events:

Let $A = \{LCL_\kappa < U_0 < UCL_\kappa\}$ and $B = \{LCL_{\kappa+1} < U_1 < UCL_{\kappa+1}\}$. Then,

$$\begin{aligned} \Pr(N=2) &= \Pr(A \cap B^C) \;=\; \Pr(A) - \Pr(A \cap B) \\ &= \int_{LCL_\kappa}^{UCL_\kappa} f(u_0)\,du_0 - \int_{LCL_{\kappa+1}}^{UCL_{\kappa+1}} \int_{LCL_\kappa}^{UCL_\kappa} f(u_0,u_1)\,du_0\,du_1 \;. \end{aligned} \tag{6.2}$$

The run-length probabilities for higher values of N can be calculated in a similar fashion.

Consider the following data set to illustrate the control chart and the use of the proposed density functions to determine the run-length probabilities. Twenty samples of size 5 were generated. The first 10 samples were generated from a $N(10,1)$ distribution. Between samples 10 and 11 the process mean and variance encountered a sustained shift, therefore the last 10 samples were generated from a $N(11.5,1.5)$ distribution. Take note that when calculating the sample variance, the practitioner is unaware of the change in the mean. The simulated data set is given in Table 1 and the control chart in Figure 4. Note that there is no plotting

Table 1: Simulated data set.

Sample (i)	X_{i1}	X_{i2}	X_{i3}	X_{i4}	X_{i5}	Sample variance	U_i^*	Plotting statistic
1	9.672	10.328	9.061	9.606	10.471	0.295	NA	NA
2	12.064	10.689	10.332	8.618	9.949	1.352	4.581	−1.553
3	10.085	10.603	9.986	9.634	10.866	0.251	0.305	1.276
4	8.653	9.371	8.830	10.893	11.324	1.226	1.940	−1.049
5	10.187	9.001	9.234	10.266	8.701	0.676	0.865	0.054
6	9.560	10.738	10.617	8.831	12.225	1.487	1.957	−1.173
7	8.387	11.668	9.590	10.064	9.204	1.238	1.405	−0.672
8	11.194	9.137	7.822	10.723	11.034	1.701	1.825	−1.110
9	10.181	10.413	11.128	10.890	8.524	0.889	0.865	0.033
10	10.761	9.953	11.530	9.330	10.034	0.674	0.666	0.388
11	11.175	11.963	13.257	12.327	13.857	7.227	7.382	−4.012
12	12.850	12.132	11.727	10.362	11.309	3.499	2.262	−1.548
13	9.964	12.275	10.585	11.670	11.529	2.129	1.245	−0.525
14	11.955	11.450	12.625	11.627	11.306	3.434	1.971	−1.312
15	11.981	12.890	11.306	11.725	9.372	3.470	1.863	−1.216
16	10.184	8.689	11.045	11.428	11.687	1.546	0.785	0.160
17	10.651	10.974	10.282	11.372	9.324	0.758	0.390	1.055
18	10.375	12.098	10.711	11.556	9.884	1.496	0.799	0.134
19	10.480	11.489	12.726	12.910	10.191	3.677	1.984	−1.350
20	12.701	11.517	10.126	11.659	11.727	3.069	1.575	−0.936

statistic that corresponds to sample number / time 1 as this sample is used to obtain an initial estimate of the process variance. The process is effectively monitored from sample 2 onwards. This process is declared out-of-control at sample number 11 since this is the first sample where a plotting statistic plots on or outside the control limits.

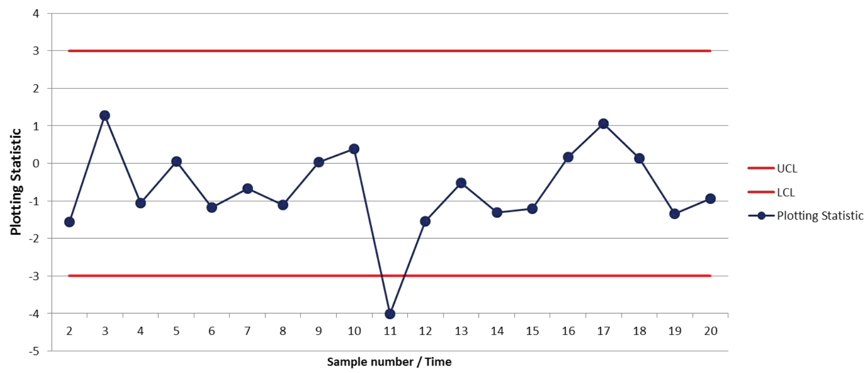


Figure 4: Control chart.

The software package Mathematica was used to calculate the probabilities by using the summation form of the Humbert function in equation (3.5). Based on the information of the simulated data set, we have (i) $v_i = n_i = n = 5$ (equal sample sizes at each point in time), (ii) $\kappa = 11$, (iii) $\delta_a = 0$ (i.e. the mean and variance changed simultaneously between sample number 10 and 11), (iv) $\delta_0 = 5$ and (v) $\lambda = 1.5$. The probability of detecting the shift in the process variance immediately at time period 11 is calculated using (6.1):

$$\begin{aligned}\Pr(N=1) &= 1 - \int_{LCL_{\kappa=11}}^{UCL_{\kappa=11}} f(u_0)\,du_0 = 1 - \int_{0.004632685}^{0.470157314} f(u_0)\,du_0 \\ &= 0.177224\end{aligned}$$

where

$$\begin{aligned}UCL_{\kappa=11} &= \frac{F_{5,50}^{-1}\left[\Phi\left(3\right)\right]}{10} = \frac{F_{5,50}^{-1}\left[0.998650102\right]}{10} = \frac{4.701573136}{10} \\ &= 0.470157314\ ,\end{aligned}$$

$$\begin{aligned}LCL_{\kappa=11} &= \frac{F_{5,50}^{-1}\left[\Phi\left(-3\right)\right]}{10} = \frac{F_{5,50}^{-1}\left[0.001349898\right]}{10} = \frac{0.04632684922}{10} \\ &= 0.004632685\ .\end{aligned}$$

The probability of detecting the shift in the process variance at time period 12 is calculated using (6.2):

$$\Pr(N\!=\!2) \;=\; \int_{0.004632685}^{0.470157314} f(u_0)\,du_0 \;-\; \int_{0.004221604}^{0.420758373}\int_{0.004632685}^{0.470157314} f(u_0,u_1)\,du_0\,du_1$$
$$= \;0.090598\;.$$

These run-length probabilities can then be used to estimate the average run-length (ARL) using the formula $E(N) = ARL = \sum_{k=1}^{\infty} k\,\Pr(N\!=\!k) \approx \sum_{k=1}^{M} k\,\Pr(N\!=\!k)$. The accuracy of the ARL estimate will depend on the cut-off M. The probabilities can be evaluated using the multivariate density function in (4.1) or using Monte Carlo simulation. The evaluation of high dimensional multiple integrals become increasingly more complex (i.e. time consuming and resource intensive) as the dimension increases and is beyond the scope of this article.

Table 2 summarises the effect of the different parameters on the probability to detect the shift in the variance immediately.

Table 2: Probabilities for different parameter values.

Role of	δ_a	δ_0	n_i	κ	λ	$\Pr(N\!=\!1)$	Comment
$\lambda = \dfrac{\sigma_1^2}{\sigma^2}$	0	5	5	11	0.5 1 1.5	0.015147 0.048686 0.177224	The larger the step shift, the higher the probability.
κ	0	5	5	3 5 11	1.5	0.057861 0.110475 0.177224	The more historical samples available before the shift took place, the higher the probability.
n	0	5	1 5 10	11	1.5	0.166158 0.177224 0.171251	The larger the sample size, the probability initially increases and then decreases.
δ_0	0	0 2 5	5	11	1.5	0.015941 0.060114 0.177224	The larger δ_0 (i.e. the relative change in the mean), the higher the probability.

7. CONCLUDING REMARKS

Adamski *et al.* (2012) recently introduced a new generalized multivariate beta distribution with density in closed form, where a distribution is needed for the run-length of a Q-chart that monitors the process mean when measurements are from an exponential distribution with unknown parameter. In this paper the distributions are proposed for the case when measurements from each sample are independent and identically distributed normal random variables and we are monitoring the unknown spread when the known mean encountered a sustained shift. We have generalized the proposed model to the multivariate case and we hope that the results presented in this paper will be useful in the Statistical Process Control field.

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