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REGULAR A-OPTIMAL SPRING BALANCE WEIGHING DESIGNS

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Abstract:

- The problem of indicating an A-optimal spring balance weighing design providing that the measurement errors have different variances and are uncorrelated is considered. The lowest bound of the trace of the inverse information matrix is given and the conditions determining the optimal design are also presented. The incidence matrices of balanced incomplete block designs and group divisible designs are used in constructions of the regular A-optimal spring balance weighing design.

Key-Words:

- *A-optimal design; balanced incomplete block design; group divisible design; spring balance weighing design.*

AMS Subject Classification:

- 62K05, 62K15.

1. INTRODUCTION

In several fields of the experiments, specially in the theory of spectroscopy, metrology, dynamical system theory, computational mechanics and 2^n fractional factorial designs we determine unknown measurements of p objects using n operations according to the linear model

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e} \;, \tag{1.1}$$

where $\mathbf{y}$ is an $n\times 1$ random vector of the observations. The design matrix $\mathbf{X} = (x_{ij})$ usually called weighing matrix belongs to the class $\mathbf{\Phi}_{n\times p}(0,1)$, which denotes the class of $n\times p$ matrices of known elements $x_{ij} = 0$ or 1 according as in the i^{th} weighing operation the j^{th} object is not placed on the pan or is placed. $\mathbf{w}$ is a $p\times 1$ vector of unknown weights of objects and $\mathbf{e}$ is an $n\times 1$ random vector of errors. We assume, that there are no systematic errors, the variances of errors are not equal and the errors are uncorrelated, i.e. $\mathbf{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\,\mathbf{G}$, where $\mathbf{0}_n$ denotes the $n\times 1$ vector with zero elements everywhere, $\mathbf{G}$ is the known $n\times n$ diagonal positive definite matrix.

For the estimation of individual unknown weights of objects we use normal equations $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}\mathbf{w} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$. Any spring balance weighing design is said to be singular or nonsingular, depending on whether the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is singular or nonsingular, respectively. It is obvious, that if $\mathbf{G}$ is the known positive definite matrix then the matrix $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular if and only if the matrix $\mathbf{X}'\mathbf{X}$ is nonsingular, i.e. if and only if $\mathbf{X}$ is full column rank $\mathrm{r}(\mathbf{X}) = p$. However, if $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is nonsingular, then the generalized least squares estimator of $\mathbf{w}$ is given by $\hat{\mathbf{w}} = (\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ and the variance matrix of $\hat{\mathbf{w}}$ is $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$. A more complete theory may be obtained in literature[1].

In many problems cases the weighing designs, the A-optimal design is considered. For given variance matrix of the errors $\sigma^2\mathbf{G}$, the A-optimal design is the design $\mathbf{X}$ for which, the sum of variances of estimators for unknown parameters is minimal, i.e. $\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$ is minimal in $\mathbf{\Phi}_{n\times p}(0,1)$. Moreover, the design for which the sum of variances of estimators for parameters attains the lowest bound in $\mathbf{\Phi}_{n\times p}(0,1)$ is called the regular A-optimal design. Let note, in the set of design matrices $\mathbf{\Phi}_{n\times p}(0,1)$, the regular A-optimal design may not exist, whereas A-optimal design always exists. The concept of the A-optimality was shown in many papers[2].

[1] See, for instance, Raghavarao ([12]) and Banerjee ([1]).

[2] See, Jacroux and Notz ([8]), Shah and Sinha ([14]), Pukelsheim ([11]), Ceranka and Graczyk ([2]), Ceranka *et al.* ([3], [4]), Masaro and Wong ([10]) and Graczyk ([6], [7]).

2. REGULAR A-OPTIMAL DESIGNS

For any experimental setting, i.e. for fixed n, p and $\mathbf{G}$, there is always a number of designs available for using. In each class of available designs, the regular A-optimal design is considered. Furthermore, the main difficulty in carrying out the construction is that each form of $\mathbf{G}$ requires the specific investigations. That's why we consider the experimental situation we determine unknown measurements of p objects in $n = \sum_{i=1}^{h} n_s$ measurement operations under model 1.1. It is assumed that n_s measurements are taken in different h conditions or at different h installations. So, the variance matrix of errors $\sigma^2\mathbf{G}$ is given by the matrix $\mathbf{G}$

$$\mathbf{G} = \begin{bmatrix} g_1^{-1}\mathbf{I}_{n_1} & \mathbf{0}_{n_1}\mathbf{0}'_{n_2} & \cdots & \mathbf{0}_{n_1}\mathbf{0}'_{n_h} \\ \mathbf{0}_{n_1}\mathbf{0}'_{n_1} & g_2^{-1}\mathbf{I}_{n_2} & \cdots & \mathbf{0}_{n_2}\mathbf{0}'_{n_h} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{n_h}\mathbf{0}'_{n_1} & \mathbf{0}_{n_h}\mathbf{0}'_{n_2} & \cdots & g_h^{-1}\mathbf{I}_{n_h} \end{bmatrix}, \tag{2.1}$$

where $g_s > 0$ denotes the factor of precision, $s = 1, 2, ..., h$. Consequently, according to the form of $\mathbf{G}$ we write the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ as

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \cdots \\ \mathbf{X}_h \end{bmatrix} \tag{2.2}$$

where $\mathbf{X}_s$ is $n_s \times p$ design matrix of any spring balance weighing design.

The Lemma given below presented in [9] is required for determining the regular A-optimal design.

Lemma 2.1. *Let $\mathbf{\Pi}$ be the the set of all $p\times p$ permutation matrices and let $\mathbf{M}$ be a $p\times p$ matrix. If $\bar{\mathbf{M}} = \frac{1}{p!}\sum_{\mathbf{P}\in\mathbf{\Pi}} \mathbf{P}'\mathbf{M}\mathbf{P}$ then $\bar{\mathbf{M}} = \left(\frac{\mathrm{tr}(\mathbf{M})}{p} - \frac{Q(\mathbf{M})}{p(p-1)}\right)\mathbf{I}_p + \frac{Q(\mathbf{M})}{p(p-1)}\mathbf{1}_p\mathbf{1}'_p$, where $\mathrm{tr}(\mathbf{M})$ is the trace of $\mathbf{M}$, $Q(\mathbf{M})$ denotes the sum of the off-diagonal elements of $\mathbf{M}$ and $\mathbf{1}_p$ is the vector of ones. Moreover, $\mathrm{tr}(\mathbf{M}) = \mathrm{tr}(\bar{\mathbf{M}})$ and $Q(\mathbf{M}) = Q(\bar{\mathbf{M}})$.*

From now on, we assume that $\mathbf{G}$ is taken into consideration in the form (2.1).

Theorem 2.1. *In any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in (2.2) with the variance matrix of errors $\sigma^2\mathbf{G}$*

$$\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} \geq \begin{cases} \dfrac{4(p^2-2p+2)}{p\,\mathrm{tr}(\mathbf{G}^{-1})} & \text{if } p \text{ is even}, \\[2ex] \dfrac{4p^3}{(p+1)^2\,\mathrm{tr}(\mathbf{G}^{-1})} & \text{if } p \text{ is odd}. \end{cases} \tag{2.3}$$

Proof: For $\mathbf{X}$ in (2.2) and $\mathbf{G}$ in (2.1), we obtain $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \sum_{s=1}^{h} g_s \mathbf{X}_s' \mathbf{X}_s$ and moreover $\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \sum_{z=1}^{p} \frac{1}{\mu_z}$, where μ_z is the eigenvalue of $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$. Next, we consider the matrix $\bar{\mathbf{M}} = \alpha\,\mathbf{I}_p + \beta\,\mathbf{1}_p\mathbf{1}_p'$. From [8], $\bar{\mathbf{M}}$ has eigenvalues α with the multiplicity $p-1$ and $\alpha + p\beta$ with the multiplicity 1. Based on Lemma 2.1, $\bar{\mathbf{M}} = \frac{p\,\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p'\mathbf{M}\mathbf{1}_p}{p(p-1)}\mathbf{I}_p + \frac{\mathbf{1}_p'\mathbf{M}\mathbf{1}_p - \mathrm{tr}(\mathbf{M})}{p(p-1)}\mathbf{1}_p\mathbf{1}_p'$. The eigenvalues of $\bar{\mathbf{M}}$ are $\mu_1 = \frac{1}{p(p-1)}\big(p\,\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p'\mathbf{M}\mathbf{1}_p\big)$ with the multiplicity $p-1$ and $\mu_2 = \frac{1}{p}\mathbf{1}_p'\mathbf{M}\mathbf{1}_p$ with the multiplicity 1. Taking $\mathbf{M} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ we have $\mathrm{tr}(\mathbf{M}) = \sum_{s=1}^{h} g_s \mathbf{k}_s' \mathbf{1}_{n_s}$ and $\mathbf{1}_p'\mathbf{M}\mathbf{1}_p = \sum_{s=1}^{h} g_s \mathbf{k}_s'\mathbf{k}_s$ so we obtain

$$\begin{aligned}\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} &= \frac{p}{\mathbf{1}_p'\mathbf{M}\mathbf{1}_p} + \frac{p(p-1)^2}{p\,\mathrm{tr}(\mathbf{M}) - \mathbf{1}_p'\mathbf{M}\mathbf{1}_p} \\ &= \frac{p}{\sum_{s=1}^{h} g_s \mathbf{k}_s'\mathbf{k}_s} + \frac{p(p-1)^2}{\sum_{s=1}^{h} g_s (p\mathbf{1}_{n_s} - \mathbf{k}_s)'\,\mathbf{k}_s}\,,\end{aligned} \tag{2.4}$$

where $\mathbf{k}_s = \mathbf{X}_s\mathbf{1}_p$, $s = 1, 2, ..., h$. For even p, minimum of (2.4) is attained if and only if $\mathbf{k}_s = \frac{p}{2}\mathbf{1}_{n_s}$. For odd p, minimum of (2.4) is attained if and only if $\mathbf{k}_s = \frac{p+1}{2}\mathbf{1}_{n_s}$, $s = 1, 2, ..., h$. Hence, we obtain (2.3). □

Definition 2.1. Any $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ given in (2.2) with the variance matrix of errors $\sigma^2\mathbf{G}$ is said to be the regular A-optimal if the equality in (2.3) is satisfied.

Theorem 2.2. *Any $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ in (2.2) with the variance matrix of errors $\sigma^2\mathbf{G}$ is the regular A-optimal spring balance weighing design if and only if*

a) *for even p,* $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{p}{4(p-1)}\,\mathrm{tr}(\mathbf{G}^{-1})\,\mathbf{I}_p + \frac{p-2}{4(p-1)}\,\mathrm{tr}(\mathbf{G}^{-1})\,\mathbf{1}_p\mathbf{1}_p'$,

or

b) *for odd p,* $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{p+1}{4p}\,\mathrm{tr}(\mathbf{G}^{-1})\,(\mathbf{I}_p + \mathbf{1}_p\mathbf{1}_p')$.

Proof: The proof follows naturally into two parts. If p be odd, then from the Theorem 2.1, $\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$ takes minimum if and only if $\mathbf{k}_s = \frac{p}{2}\mathbf{1}_{n_s}$ for each s. Then it is easy to see that $\mathrm{tr}(\mathbf{M}) = \frac{p}{2}\,\mathrm{tr}(\mathbf{G}^{-1})$ and $\mathbf{1}_p'\mathbf{M}\mathbf{1}_p = \frac{p^2}{4}\,\mathrm{tr}(\mathbf{G}^{-1})$. So we have $\alpha = \frac{p\,\mathrm{tr}(\mathbf{G}^{-1})}{4(p-1)}$ and $\beta = \frac{(p-2)\,\mathrm{tr}(\mathbf{G}^{-1})}{4(p-1)}$ and we obtain a). The analogous consideration for odd p, imply that $\alpha = \beta = \frac{(p+1)\,\mathrm{tr}(\mathbf{G}^{-1})}{4p}$ and b) is true that finishes the proof. □

If $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ satisfies the equalities given in Theorem 2.2 then $\mathbf{X}$ is the regular A-optimal design for any $\mathbf{G}$ in (2.1). Hence $\mathbf{X}$ is the regular A-optimal design in the special case when $\mathbf{G} = \mathbf{I}_n$ and

$$\mathrm{tr}(\mathbf{X}'\mathbf{X})^{-1} \geq \begin{cases} \dfrac{4(p^2-2p+2)}{np} & \text{if } p \text{ is even}\,, \\[2ex] \dfrac{4p^3}{n(p+1)^2} & \text{if } p \text{ is odd}\,. \end{cases} \tag{2.5}$$

(2.5) is equivalent to the lowest bound of $\mathrm{tr}(\mathbf{X}'\mathbf{X})^{-1}$ which follows from theorems given in [8]. On the other hand, we assume that $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ is the regular A-optimal design for $\mathbf{G} = \mathbf{I}_n$. Then we can compare two traces $\frac{\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}}{\mathrm{tr}(\mathbf{X}'\mathbf{X})^{-1}} = \frac{\sum_{s=1}^{h} g_s n_s}{n}$. We obtain the following Corollary.

Corollary 2.1. *Let $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ and $\mathbf{G}$ be of the form (2.1).*

a) *If $\sum_{s=1}^{h} g_s n_s = n$, then $\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \mathrm{tr}(\mathbf{X}'\mathbf{X})^{-1}$,*

b) *If $\sum_{s=1}^{h} g_s n_s > n$, then $\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} < \mathrm{tr}(\mathbf{X}'\mathbf{X})^{-1}$,*

c) *If $\sum_{s=1}^{h} g_s n_s < n$, then $\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} > \mathrm{tr}(\mathbf{X}'\mathbf{X})^{-1}$.*

3. CONSTRUCTION OF THE REGULAR A-OPTIMAL DESIGNS

It is worth pointing out that the incidence matrices of the block designs may be used for the construction of the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$, then we take $n = b$ and $p = v$. From all possible block designs, in this paper the construction of the regular A-optimal spring balance weighing design based on the incidence matrices of the balanced incomplete block designs is chosen.

Theorem 3.1. *Let v be even and let $\mathbf{N}$ be the incidence matrix of balanced incomplete block design with the parameters $v = 2t$, $b = 2(2t-1)$, $r = 2t-1$, $k = t$, $\lambda = t-1$, $t = 2, 3, \ldots$. Then any $\mathbf{X} \in \mathbf{\Phi}_{hb\times 2t}(0,1)$ in the form $\mathbf{X} = \mathbf{1}_h \otimes \mathbf{N}'$ is the regular A-optimal spring balance weighing design with the variance matrix of errors $\sigma^2\mathbf{G}$.*

Proof: An easy computation shows that the matrix $\mathbf{X} = \mathbf{1}_h \otimes \mathbf{N}'$ satisfies a) of Theorem 2.2. □

Theorem 3.2. *Let v be odd and let $\mathbf{N}$ be the incidence matrix of balanced incomplete block design with the parameters*

a) $v = 2t+1$, $b = 2(2t+1)$, $r = 2(t+1)$, $k = t+1$, $\lambda = t+1$,

or

b) $v = 4t-1$, $b = 4t-1$, $r = 2t$, $k = 2t$, $\lambda = t$,

$t = 1, 2, \ldots$. Then any $\mathbf{X} \in \mathbf{\Phi}_{hb\times v}(0,1)$ in the form $\mathbf{X} = \mathbf{1}_h \otimes \mathbf{N}'$ is the regular A-optimal spring balance weighing design with the variance matrix of errors $\sigma^2\mathbf{G}$.

Proof: This is proved by checking that the matrix $\mathbf{X} = \mathbf{1}_h \otimes \mathbf{N}'$ satisfies b) of Theorem 2.2. □

For an even v, we give the construction of the regular A-optimal spring balance weighing design based on the incidence matrices of group divisible design. Hence, we get

$$\mathbf{X}_s = \begin{bmatrix} \mathbf{N}'_{1s} \\ \mathbf{N}'_{2s} \end{bmatrix}, \qquad s = 1, 2, ..., h , \tag{3.1}$$

where $\mathbf{N}_{uh}$ is the incidence matrix of group divisible design with the parameters v, b_{us}, r_{us}, k_{us}, λ_{1us}, λ_{2us}, $u = 1, 2$, see [13]. Furthermore, let the condition

$$\lambda_{11s} + \lambda_{12s} = \lambda_{21s} + \lambda_{22s} = \lambda_s , \qquad u = 1, 2, \quad s = 1, 2, ..., h , \tag{3.2}$$

be satisfied. Below, we show the parameters of group divisible design which satisfy (3.1). Next, for $n = \sum_{s=1}^{h} \sum_{u=1}^{2} b_{us}$ bus measurements and v objects, based on the incidence matrices of the group divisible designs will be constructed $\mathbf{X} \in \mathbf{\Phi}_{n\times v}(0, 1)$. For the t, q, u given in Lemmas 3.1–3.5, some restrictions derive from the ones given in [5]: $r, k \leq 10$.

Lemma 3.1. *Let $v = 4$. If the parameters of group divisible designs are equal to*

a) $b_{1s} = 2(3t+1)$, $r_{1s} = 3t+1$, $k_{1s} = 2$, $\lambda_{11s} = t+1$, $\lambda_{21s} = t$, $t = 1, 2, 3$ *and* $b_{2s} = 2(3q+2)$, $r_{2s} = 3q+2$, $k_{2s} = 2$, $\lambda_{12s} = q$, $\lambda_{22s} = q+1$, $q = 0, 1, 2$,

b) $b_{1s} = 2(3t+2)$, $r_{1s} = 3t+2$, $k_{1s} = 2$, $\lambda_{11s} = t+2$, $\lambda_{21s} = t$, $t = 1, 2$ *and* $b_{2s} = 2(3q+4)$, $r_{2s} = 3q+4$, $k_{2s} = 2$, $\lambda_{12s} = q$, $\lambda_{22s} = q+2$, $q = 0, 1, 2$,

c) $b_{1s} = 2(u+3)$, $r_{1s} = u+3$, $k_{1s} = 2$, $\lambda_{11s} = u+1$, $\lambda_{21s} = 1$ *and* $b_{2s} = 4u$, $r_{2s} = 2u$, $k_{2s} = 2$, $\lambda_{12s} = 0$, $\lambda_{22s} = u$, $u = 1, 2, 3, 4, 5$,

d) $b_{1s} = 16$, $r_{1s} = 8$, $k_{1s} = 2$, $\lambda_{11s} = 0$, $\lambda_{21s} = 4$ *and* $b_{2s} = 2(3u+4)$, $r_{2s} = 3u+4$, $k_{2s} = 2$, $\lambda_{12s} = u+4$, $\lambda_{22s} = u$, $u = 1, 2$,

e) $b_{1s} = 18$, $r_{1s} = 9$, $k_{1s} = 2$, $\lambda_{11s} = 5$, $\lambda_{21s} = 2$ *and* $b_{2s} = 6(u+2)$, $r_{2s} = 3(u+2)$, $k_{2s} = 2$, $\lambda_{12s} = u$, $\lambda_{22s} = u+3$, $u = 0, 1$,

then for any matrix in (3.1), the condition (3.2) is satisfied.

Lemma 3.2. *Let $v = 6$. If the parameters of group divisible designs are equal to*

a) $b_{1s} = 4t$, $r_{1s} = 2t$, $k_{1s} = 3$, $\lambda_{11s} = 0$, $\lambda_{21s} = t$ *and* $b_{2s} = 6t$, $r_{2s} = 3t$, $k_{2s} = 3$, $\lambda_{12s} = 2t$, $\lambda_{22s} = t$, $t = 1, 2, 3$,

b) $b_{1s} = 2(2t+5)$, $r_{1s} = 2t+5$, $k_{1s} = 3$, $\lambda_{11s} = t+1$, $\lambda_{21s} = t+2$ *and* $b_{2s} = 6t$, $r_{2s} = 3t$, $k_{2s} = 3$, $\lambda_{12s} = t+1$, $\lambda_{22s} = t$, $t = 1, 2$,

c) $b_{1s} = 12$, $r_{1s} = 6$, $k_{1s} = 3$, $\lambda_{11s} = 4$, $\lambda_{21s} = 2$ *and* $b_{2s} = 2(5t+4)$, $r_{2s} = 5t+4$, $k_{2s} = 3$, $\lambda_{12s} = 2t$, $\lambda_{22s} = 2(t+1)$, $t = 0, 1$,

d) $b_{1s} = 16$, $r_{1s} = 8$, $k_{1s} = 3$, $\lambda_{11s} = 4$, $\lambda_{21s} = 3$ *and* $b_{2s} = 2(5t+2)$, $r_{2s} = 5t+2$, $k_{2s} = 3$, $\lambda_{12s} = t+2$, $\lambda_{22s} = 2t+1$, $t = 0, 1$,

then for any matrix in (3.1), the condition (3.2) is satisfied.

Lemma 3.3. *Let* $v = 8$. *If the parameters of group divisible designs are equal to*

a) $b_{1s} = 4(t+1)$, $r_{1s} = 2(t+1)$, $k_{1s} = 4$, $\lambda_{11s} = 0$, $\lambda_{21s} = t+1$ *and* $b_{2s} = 4(6-t)$, $r_{2s} = 2(6-t)$, $k_{2s} = 4$, $\lambda_{12s} = 6$, $\lambda_{22s} = 5-t$, $t = 1, 2, 3$,

b) $b_{1s} = 2(3t+2)$, $r_{1s} = 3t+2$, $k_{1s} = 4$, $\lambda_{11s} = t+1$, $\lambda_{21s} = t+2$ *and* $b_{2s} = 6(4-t)$, $r_{2s} = 3(4-t)$, $k_{2s} = 4$, $\lambda_{12s} = 4-t$, $\lambda_{22s} = 5-t$, $t = 1, 2$,

then for any matrix in (3.1), the condition (3.2) is satisfied.

Lemma 3.4. *Let* $v = 10$. *If the parameters of group divisible designs are equal to* $b_{1s} = 8t$, $r_{1s} = 4t$, $k_{1s} = 5$, $\lambda_{11s} = 0$, $\lambda_{21s} = 2t$ *and* $b_{2s} = 10t$, $r_{2s} = 5t$, $k_{2s} = 5$, $\lambda_{12s} = 4t$, $\lambda_{22s} = 2t$, $t = 1, 2$, *then for any matrix in (3.1), the condition (3.2) is fulfilled.*

Lemma 3.5. *If the parameters of group divisible designs are equal to*

a) $v = 2(2u+1)$, $b_1 = 4u$, $r_1 = 2u$, $k_1 = 2u+1$, $\lambda_{11} = 0$, $\lambda_{21} = u$ *and* $v = b_2 = 2(2u+1)$, $r_2 = k_2 = 2u+1$, $\lambda_{12} = 2u$, $\lambda_{22} = u$, $u = 1, 2, 3, 4$,

b) $v = 4(u+1)$, $b_1 = 2(2u+1)$, $r_1 = 2u+1$, $k_1 = 2(u+1)$, $\lambda_{11} = 2u+1$, $\lambda_{21} = u$ *and* $v = b_2 = 4(u+1)$, $r_2 = k_2 = 2(u+1)$, $\lambda_{12} = 0$, $\lambda_{22} = u+1$, $u = 1, 2, 3, 4$,

then for a matrix $\mathbf{X} = \mathbf{X}_s$ *in (3.1), the condition (3.2) is true.*

Lemmas given above are essential to construct the design $\mathbf{X}$. For a given number of objects $p = v$ and $n = \sum_{s=1}^{h}(b_{1s} + b_{2s})$ measurements, we choose appropriate number h of matrices $\mathbf{X}_s$ satisfying conditions in Lemmas 3.1–3.5 and in the result we form the design matrix $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$. Thus we obtain the theorem.

Theorem 3.3. *Any* $\mathbf{X} \in \mathbf{\Phi}_{n\times p}(0,1)$ *in (2.2) for* $\mathbf{X}_s$ *in (3.1), where* $\mathbf{N}_{uh}$ *is the incidence matrix of group divisible design with the parameters given in Lemmas 3.1–3.5,* $u = 1, 2$, *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *is the regular A-optimal spring balance weighing design.*

Proof: It is easy to verify that for the matrix $\mathbf{X}$ the condition a) of Theorem 2.2 is satisfied. □

Note 3.1. The criterion of A-optimality is interpreted as minimizing the sum of the variances of estimators of unknown measurements of objects. Some design matrices are better than others in the sense that the sum of the variances of estimators of unknown measurements of objects is smaller. The design $\mathbf{X}$ for which $\mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1}$ attains the lower bound is called regular A-optimal. In any class $\mathbf{\Phi}_{n\times p}(0,1)$, regular A-optimal spring balance weighing design $\mathbf{X}_R$ may exist, whereas A-optimal spring balance weighing design $\mathbf{X}$ exists always. From statistical point of view, the sum of variances in regular A-optimal spring balance weighing design $\mathbf{X}_R$ is so small if it is possible and in this sense design $\mathbf{X}_R$ is the best one. Moreover, if in the class $\mathbf{\Phi}_{n\times p}(0,1)$, the regular A-optimal spring balance weighing design does not exist then determined lower bound of the variance of the sum of estimators may be used for indicating the design which is the closest to the best one $\mathbf{X}_R$.

4. EXAMPLES

Example 4.1. To present theory given above, let us consider $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ and let p be even. Among all possible variance matrices $\sigma^2\mathbf{G}$ we take matrix $\mathbf{G}$ in the form

a) $\mathbf{G}_1 = \begin{bmatrix} a\mathbf{I}_{\frac{n}{2}} & \mathbf{0}_{\frac{n}{2}}\mathbf{0}'_{\frac{n}{2}} \\ \mathbf{0}_{\frac{n}{2}}\mathbf{0}'_{\frac{n}{2}} & \frac{1}{a}\mathbf{I}_{\frac{n}{2}} \end{bmatrix}$, $a>0$. We have $\mathrm{tr}(\mathbf{X}'\mathbf{G}_1^{-1}\mathbf{X})^{-1} = \frac{8a(p^2-2p+2)}{np(a^2+1)}$ and $\mathrm{tr}(\mathbf{X}'\mathbf{X})^{-1} - \mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \frac{4(p^2-2p+2)\,(a-1)^2}{np(a^2+1)} \geq 0$. Hence, for $a\neq 1$, $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ with the variance matrix $\sigma^2\mathbf{G}$ is regular A-optimal spring balance weighing design, whereas $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ with $\sigma^2\mathbf{I}_n$ is A-optimal design. For $a=1$, $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ is regular A-optimal spring balance weighing design with the variance matrix $\sigma^2\mathbf{G}_1$ and $\sigma^2\mathbf{I}_n$.

b) $\mathbf{G}_2 = \begin{bmatrix} \frac{1}{a}\mathbf{I}_{\frac{n}{2}} & \mathbf{0}_{\frac{n}{2}}\mathbf{0}'_{\frac{n}{2}} \\ \mathbf{0}_{\frac{n}{2}}\mathbf{0}'_{\frac{n}{2}} & \frac{1}{b}\mathbf{I}_{\frac{n}{2}} \end{bmatrix}$, $a,b>0$. We have $\mathrm{tr}(\mathbf{X}'\mathbf{G}_2^{-1}\mathbf{X})^{-1} = \frac{8(p^2-2p+2)}{np(a+b)}$ and

$$\mathrm{tr}(\mathbf{X}'\mathbf{X})^{-1} - \mathrm{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \frac{4(p^2-2p+2)\,(a+b-2)}{np(a+b)}\,. \tag{4.1}$$

If $a+b-2>0$ then $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ with the variance matrix $\sigma^2\mathbf{G}_2$ is regular A-optimal spring balance weighing design, whereas $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ with $\sigma^2\mathbf{I}_n$ is A-optimal design. If $a+b-2=0$ then $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ with the variance matrix $\sigma^2\mathbf{G}_2$ and with $\sigma^2\mathbf{I}_n$ is regular A-optimal design. If $a+b-2<0$ then $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ with the variance matrix $\sigma^2\mathbf{I}_n$ is regular A-optimal spring balance weighing design, whereas $\mathbf{X}\in\mathbf{\Phi}_{n\times p}(0,1)$ with $\sigma^2\mathbf{G}_2$ is A-optimal design.

Example 4.2. As numerical example, let us consider $\mathbf{X} \in \mathbf{\Phi}_{12\times 4}(0,1)$. From Theorem 3.1, we construct the incidence matrix $\mathbf{N}$ of balanced incomplete block design with the parameters $v=4,\ b=6,\ r=3,\ k=2,\ \lambda=1$ as

$$\mathbf{N} = \begin{bmatrix} 1&1&1&0&0&0 \\ 1&0&0&1&1&0 \\ 0&1&0&1&0&1 \\ 0&0&1&0&1&1 \end{bmatrix} \text{ and next we form the design matrix } \mathbf{X} = \begin{bmatrix} 1&1&0&0 \\ 1&0&1&0 \\ 1&0&0&1 \\ 0&1&1&0 \\ 0&1&0&1 \\ 0&0&1&1 \\ 1&1&0&0 \\ 1&0&1&0 \\ 1&0&0&1 \\ 0&1&1&0 \\ 0&1&0&1 \\ 0&0&1&1 \end{bmatrix}.$$

Hence $\operatorname{tr}(\mathbf{X}'\mathbf{X})^{-1} = \frac{5}{6}$. Next, we consider possible forms of the matrix $\mathbf{G}$. For example

$$\text{if } \mathbf{G} = \begin{bmatrix} 2\mathbf{I}_6 & \mathbf{0}_6\mathbf{0}_6' \\ \mathbf{0}_6\mathbf{0}_6' & \frac{1}{2}\mathbf{I}_6 \end{bmatrix} \quad \text{then} \quad \operatorname{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \frac{4}{6} < \operatorname{tr}(\mathbf{X}'\mathbf{X})^{-1} ,$$

$$\text{if } \mathbf{G} = \begin{bmatrix} 2\mathbf{I}_6 & \mathbf{0}_6\mathbf{0}_6' \\ \mathbf{0}_6\mathbf{0}_6' & \frac{2}{3}\mathbf{I}_6 \end{bmatrix} \quad \text{then} \quad \operatorname{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \frac{5}{6} = \operatorname{tr}(\mathbf{X}'\mathbf{X})^{-1} ,$$

$$\text{if } \mathbf{G} = \begin{bmatrix} \frac{1}{3}\mathbf{I}_6 & \mathbf{0}_6\mathbf{0}_6' \\ \mathbf{0}_6\mathbf{0}_6' & \frac{1}{2}\mathbf{I}_6 \end{bmatrix} \quad \text{then} \quad \operatorname{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \frac{2}{6} < \operatorname{tr}(\mathbf{X}'\mathbf{X})^{-1} ,$$

$$\text{if } \mathbf{G} = \begin{bmatrix} 2\mathbf{I}_6 & \mathbf{0}_6\mathbf{0}_6' \\ \mathbf{0}_6\mathbf{0}_6' & \frac{3}{2}\mathbf{I}_6 \end{bmatrix} \quad \text{then} \quad \operatorname{tr}(\mathbf{X}'\mathbf{G}^{-1}\mathbf{X})^{-1} = \frac{10}{7} > \operatorname{tr}(\mathbf{X}'\mathbf{X})^{-1} .$$

As you can see, in some cases the sum of the variances of estimators is smaller for the design $\mathbf{X}$ with $\mathbf{G}$ than for this design with $\mathbf{I}_n$. In the other ones it is inversely. Interestingly enough, it depends on the experimental conditions and the assumptions related to the variances of the errors: are they equal and $\operatorname{Var}(\mathbf{e}) = \sigma^2\mathbf{I}_n$ or are they different and $\operatorname{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$. Into practice, the choice of the design $\mathbf{X}$ with $\operatorname{Var}(\mathbf{e}) = \sigma^2\mathbf{I}_n$ or $\sigma^2\mathbf{G}$ is conditional and depends on the experimental conditions. We choose one of these ones. Next, we can assess for which variance matrix of errors the sum of variances of errors is smaller. If the experimental conditions require the design with greater sum of variances of errors we can determine how far we are from the lowest bound considering the difference (4.1).

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REFERENCES

[1] Banerjee, K.S. (1975). *Weighing Designs for Chemistry, Medicine, Economics, Operations Research, Statistics*, Marcel Dekker Inc., New York.

[2] Ceranka, B. and Graczyk, M. (2004). A-optimal chemical balance weighing design, *Folia Facultatis Scientiarum Naturalium Universitatis Masarykianae Brunensis, Mathematica*, **15**, 41–54.

[3] Ceranka, B.; Graczyk, M. and Katulska, K. (2006). A-optimal chemical balance weighing design with nonhomogeneity of variances of errors, *Statistics and Probability Letters*, **76**, 653–665.

[4] Ceranka, B.; Graczyk, M. and Katulska, K. (2007). On certain A-optimal chemical balance weighing designs, *Computational Statistics and Data Analysis*, **51**, 5821–5827.

[5] Clatworthy, W.H. (1973). *Tables of Two-Associate-Class Partially Balanced Design*, NBS Applied Mathematics Series 63.

[6] Graczyk, M. (2011). A-optimal biased spring balance weighing design, *Kybernetika*, **47**, 893–901.

[7] Graczyk, M. (2012). Notes about A-optimal spring balance weighing design, *Journal of Statistical Planning and Inference*, **142**, 781–784.

[8] Jacroux, M. and Notz, W. (1983). On the optimality of spring balance weighing designs, *The Annals of Statistics*, **11**, 970–978.

[9] Katulska, K. and Rychlińska, E. (2010). On regular E-optimality of spring balance weighing designs, *Colloquium Biometricum*, **40**, 165–176.

[10] Masaro, J. and Wong, C.S. (2008). Robustness of A-optimal designs, *Linear Algebra and its Applications*, **429**, 1392–1408.

[11] Pukelsheim, F. (1983). *Optimal Design of Experiment*, John Wiley and Sons, New York.

[12] Raghavarao, D. (1971). *Constructions and Combinatorial Problems in Designs of Experiment*, John Wiley Inc., New York.

[13] Raghavarao, D. and Padgett, L.V. (2005). *Block Designs, Analysis, Combinatorics and Applications*, Series of Applied Mathematics 17, Word Scientific Publishing Co. Pte. Ltd.

[14] Shah, K.R. and Sinha, B.K. (1989). *Theory of Optimal Designs*, Springer-Verlag, Berlin, Heidelberg.