REVSTAT – Statistical Journal
Volume 9, Number 2, June 2011, 115–126

A MEASURE OF DEPARTURE FROM AVERAGE MARGINAL HOMOGENEITY FOR SQUARE CONTINGENCY TABLES WITH ORDERED CATEGORIES

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Received: February 2010 Revised: November 2010 Accepted: December 2010

Abstract:

- For the analysis of square contingency tables, Tomizawa, Miyamoto and Ashihara (2003) considered a measure to represent the degree of departure from marginal homogeneity. However, the maximum value of this measure cannot distinguish two kinds of marginal inhomogeneity. This paper proposes a measure which can distinguish two kinds of marginal inhomogeneity for square tables with ordered categories. The measure is constructed using the arc-cosine function of symmetric cumulative probabilities. Especially the proposed measure is useful for representing the degree of departure from marginal homogeneity when the extended marginal homogeneity model holds. Examples are given.

Key-Words:

- *average marginal homogeneity; extended marginal homogeneity; measure; ordinal data.*

AMS Subject Classification:

- 62H17.

1. INTRODUCTION

Consider an $R \times R$ square contingency table with the same row and column classifications. Let p_{ij} denote the probability that an observation will fall in the i-th row and j-th column of the table ($i = 1, ..., R$; $j = 1, ..., R$), and let X and Y denote the row and column variables, respectively. The marginal homogeneity model is defined by

$$\Pr(X = i) = \Pr(Y = i) \qquad \text{for} \quad i = 1, ..., R \ ,$$

namely

$$p_{i\cdot} = p_{\cdot i} \qquad \text{for} \quad i = 1, ..., R \ ,$$

where $p_{i\cdot} = \sum_{k=1}^{R} p_{ik}$ and $p_{\cdot i} = \sum_{k=1}^{R} p_{ki}$. See, for example, Stuart (1955), Bishop, Fienberg and Holland (1975, p. 294).

Let

$$G_{1(i)} = \sum_{s=1}^{i} \sum_{t=i+1}^{R} p_{st} \qquad \big[= \Pr(X \le i,\, Y \ge i+1)\big] \ ,$$

and

$$G_{2(i)} = \sum_{s=i+1}^{R} \sum_{t=1}^{i} p_{st} \qquad \big[= \Pr(X \ge i+1,\, Y \le i)\big] \ ,$$

for $i = 1, ..., R-1$. Then, by considering the difference between the cumulative marginal probabilities, $F_i^X - F_i^Y$ for $i = 1, ..., R-1$, where $F_i^X = \Pr(X \le i)$ and $F_i^Y = \Pr(Y \le i)$, we see that the marginal homogeneity model may also be expressed as

$$G_{1(i)} = G_{2(i)} \qquad \text{for} \quad i = 1, ..., R-1 \ .$$

Namely, this model also states that the cumulative probability that an observation will fall in row category i or below and column category $i+1$ or above is equal to the cumulative probability that the observation falls in column category i or below and row category $i+1$ or above for $i = 1, ..., R-1$.

When the marginal homogeneity model does not hold, we are interested in measuring the degree of departure from the marginal homogeneity model.

For square contingency tables with ordered categories, Tomizawa, Miyamoto and Ashihara (2003) proposed the following measure $\Gamma^{(\lambda)}$ to represent the degree of departure from marginal homogeneity: assuming that $\{G_{1(i)} + G_{2(i)} \ne 0\}$, for $\lambda > -1$,

$$\Gamma^{(\lambda)} = \frac{\lambda(\lambda+1)}{2^{\lambda}-1} \sum_{i=1}^{R-1} \big(G_{1(i)}^{*} + G_{2(i)}^{*}\big)\, I_i^{(\lambda)}\bigg(\big\{G_{1(i)}^{c}, G_{2(i)}^{c}\big\};\, \bigg\{\frac{1}{2}, \frac{1}{2}\bigg\}\bigg) \ ,$$

where

$$\Delta = \sum_{i=1}^{R-1} \left(G_{1(i)} + G_{2(i)}\right),$$

$$G^*_{1(i)} = \frac{G_{1(i)}}{\Delta}, \qquad G^*_{2(i)} = \frac{G_{2(i)}}{\Delta},$$

$$G^c_{1(i)} = \frac{G_{1(i)}}{G_{1(i)} + G_{2(i)}}, \qquad G^c_{2(i)} = \frac{G_{2(i)}}{G_{1(i)} + G_{2(i)}},$$

with

$$I_i^{(\lambda)}(\cdot;\cdot) = \frac{1}{\lambda(\lambda+1)}\left[G^c_{1(i)}\left\{\left(\frac{G^c_{1(i)}}{1/2}\right)^{\lambda} - 1\right\} + G^c_{2(i)}\left\{\left(\frac{G^c_{2(i)}}{1/2}\right)^{\lambda} - 1\right\}\right],$$

and the value at $\lambda = 0$ is taken to be the limit as $\lambda \to 0$. Note that $I_i^{(\lambda)}(\cdot;\cdot)$ is the Cressie and Read (1984) power-divergence between two distributions (also see Read and Cressie, 1988, p. 15).

The measure $\Gamma^{(\lambda)}$ has characteristics that:

(i) it lies between 0 and 1;

(ii) $\Gamma^{(\lambda)} = 0$ if and only if the marginal homogeneity model holds;

(iii) $\Gamma^{(\lambda)} = 1$ if and only if the degree of departure from marginal homogeneity is maximum (that is, $G_{1(i)} = 0$ (then $G_{2(i)} > 0$) or $G_{2(i)} = 0$ (then $G_{1(i)} > 0$) for all $i = 1, ..., R-1$).

However, using the measure $\Gamma^{(\lambda)}$, we cannot distinguish two kinds of marginal inhomogeneity, namely, that the marginal inhomogeneity is which of

(i) $G_{1(i)} = 0$ (then $F_i^X < F_i^Y$) for all $i = 1, ..., R-1$,

or

(ii) $G_{2(i)} = 0$ (then $F_i^X > F_i^Y$) for all $i = 1, ..., R-1$.

Since these two kinds of marginal inhomogeneity indicate the opposite different maximum departures from marginal homogeneity, we are now interested in proposing a measure which can take the different values for them.

The purpose of this paper is to propose such a measure which can distinguish two kinds of marginal inhomogeneity for square contingency tables with ordered categories. We note that Tahata, Yamamoto, Nagatani and Tomizawa (2009) investigated average symmetry. In the present paper, we consider the average marginal homogeneity using a similar ideas to Tahata *et al.* (2009) and using as example the same data.

2. A MEASURE FOR MARGINAL HOMOGENEITY

Consider an $R \times R$ table with ordered categories. Let

$$\Delta = \sum_{i=1}^{R-1} \left(G_{1(i)} + G_{2(i)}\right) ,$$

and

$$G^*_{1(i)} = \frac{G_{1(i)}}{\Delta} , \quad G^*_{2(i)} = \frac{G_{2(i)}}{\Delta} , \qquad \text{for} \quad i = 1, ..., R-1 .$$

Assuming that $\left\{G_{1(i)} + G_{2(i)} \neq 0\right\}$, consider a measure defined by

$$\Psi = \frac{4}{\pi} \sum_{i=1}^{R-1} \left(G^*_{1(i)} + G^*_{2(i)}\right) \left(\theta_i - \frac{\pi}{4}\right) ,$$

where

$$\theta_i = \cos^{-1}\left(\frac{G_{1(i)}}{\sqrt{G^2_{1(i)} + G^2_{2(i)}}}\right) .$$

Noting that the range of θ_i is $0 \leq \theta_i \leq \pi/2$, we see that the measure Ψ lies between -1 and 1. The measure Ψ has characteristics that:

(i) $\Psi = -1$ if and only if $G_{2(i)} = 0$ (then $F_i^X > F_i^Y$) for all $i = 1, ..., R-1$, [marginal inhomogeneity with all probabilities zero of lower left triangle (say, L-marginal inhomogeneity)];

(ii) $\Psi = 1$ if and only if $G_{1(i)} = 0$ (then $F_i^X < F_i^Y$) for all $i = 1, ..., R-1$, [marginal inhomogeneity with all probabilities zero of upper right triangle (say, U-marginal inhomogeneity)].

In addition, $\Psi = 0$ indicates that the weighted average of $\{\theta_i - \frac{\pi}{4}\}$ equals zero. Thus when $\Psi = 0$, we shall refer to this structure as the average marginal homogeneity. We note that if the marginal homogeneity holds then the average marginal homogeneity holds, but the converse does not hold.

Therefore, using the measure Ψ, we can see whether the average marginal homogeneity departs toward the L-marginal inhomogeneity or toward the U-marginal inhomogeneity. As the measure Ψ approaches -1, the departure from the average marginal homogeneity becomes greater toward the L-marginal inhomogeneity. While as the Ψ approaches 1, it becomes greater toward the U-marginal inhomogeneity.

3. RELATIONSHIPS BETWEEN THE MEASURE AND SOME MODELS

First, we consider the relationship between the measure Ψ and the extended marginal homogeneity model. The extended marginal homogeneity model considered by Tomizawa (1984), is defined by

$$G_{1(i)} = \tau \, G_{2(i)} \qquad \text{for} \quad i = 1, ..., R-1 \ .$$

A special case of this model obtained by putting $\tau = 1$ is the marginal homogeneity model. If the extended marginal homogeneity model holds true, then the measure Ψ can be expressed as

$$\Psi \ = \ \frac{4}{\pi} \cos^{-1}\left(\frac{\tau}{\sqrt{\tau^2+1}}\right) - 1 \ . \tag{1}$$

Therefore, $\Psi = 0$ if and only if the marginal homogeneity model holds, i.e., $\tau = 1$, thus $G_{1(i)} = G_{2(i)}$ for $i = 1, ..., R-1$. As the value of τ approaches the infinity, the measure Ψ approaches –1. While as the value of τ approaches zero, Ψ approaches 1. Thus when the extended marginal homogeneity model holds in a table, the measure Ψ represents the degree of departure from marginal homogeneity toward the L-marginal inhomogeneity or toward the U-marginal inhomogeneity.

Next, consider the conditional symmetry model (McCullagh, 1978) defined by

$$p_{ij} = \tau \, p_{ji} \qquad \text{for} \quad i < j \ .$$

This model implies the extended marginal homogeneity model. Therefore, if the conditional symmetry model holds true, then the measure Ψ can also be expressed as (1).

Therefore for comparisons in several tables, if it can be estimated that there is a structure of extended marginal homogeneity or conditional symmetry in each table, then the measure Ψ would be adequate for representing and comparing the degree of departure from the marginal homogeneity toward the L-marginal inhomogeneity and U-marginal inhomogeneity.

The measure Ψ should be applied to the ordinal data of square tables with the same row and column classifications because the Ψ is not invariant under arbitrary similar permutations of row and column categories.

4. APPROXIMATE CONFIDENCE INTERVAL FOR THE MEASURE

Let n_{ij} denote the observed frequency in the i-th row and j-th column of the table ($i=1,...,R$; $j=1,...,R$). Assuming that a multinomial distribution applies to the $R\times R$ table, we shall consider the approximate variance for estimated measure and large-sample confidence interval for the measure Ψ using delta method, the descriptions of which are given by, e.g., Bishop *et al.* (1975, Sec. 14.6). The sample version of Ψ, i.e., $\hat{\Psi}$, is given by Ψ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$, where $\hat{p}_{ij} = n_{ij}/n$ and $n = \sum\sum n_{ij}$. Using delta method, $\sqrt{n}(\hat{\Psi} - \Psi)$ has asymptotically (as $n \to \infty$) a normal distribution with mean zero and variance,

$$\sigma^2[\Psi] = \sum\sum_{k<l} \left(p_{kl} D_{kl}^2 + p_{lk} D_{lk}^2\right),$$

where for $k < l$,

$$D_{kl} = \frac{4}{\pi\Delta} \sum_{i=k}^{l-1} \left[\cos^{-1}\left(\frac{G_{1(i)}}{\sqrt{G_{1(i)}^2 + G_{2(i)}^2}}\right) - \frac{G_{2(i)}\left(G_{1(i)} + G_{2(i)}\right)}{G_{1(i)}^2 + G_{2(i)}^2}\right] - \frac{(l-k)(\Psi+1)}{\Delta},$$

$$D_{lk} = \frac{4}{\pi\Delta} \sum_{i=k}^{l-1} \left[\cos^{-1}\left(\frac{G_{1(i)}}{\sqrt{G_{1(i)}^2 + G_{2(i)}^2}}\right) + \frac{G_{1(i)}\left(G_{1(i)} + G_{2(i)}\right)}{G_{1(i)}^2 + G_{2(i)}^2}\right] - \frac{(l-k)(\Psi+1)}{\Delta}.$$

Let $\hat{\sigma}^2[\Psi]$ denote $\sigma^2[\Psi]$ with $\{p_{ij}\}$ replaced by $\{\hat{p}_{ij}\}$. $\hat{\sigma}[\Psi]/\sqrt{n}$ is an estimated standard error for $\hat{\Psi}$, $\hat{\Psi} \pm z_{p/2}\, \hat{\sigma}[\Psi]/\sqrt{n}$ is an approximate $100(1-p)\%$ confidence interval for Ψ, where $z_{p/2}$ is the percentage point from the standard normal distribution that corresponds to a two-tail probability equal to p.

The maximum likelihood estimates of expected frequencies under each of the marginal homogeneity, extended marginal homogeneity and average marginal homogeneity models can be obtained using the Newton–Raphson methods to the log-likelihood equations. The marginal homogeneity, extended marginal homogeneity and average marginal homogeneity models can be tested for goodness-of-fit by, e.g., the likelihood ratio chi-squared statistic with $R-1$, $R-2$, and 1 degrees of freedom, respectively.

5. ANALYSIS OF DATA

5.1. Analysis of Table 1(a)

Consider the data in Table 1(a) taken from Stuart (1955). These are data on unaided distance vision of 7477 women aged 30 to 39 employed in Royal Ordnance factories in Britain from 1943 to 1946. These data have been analyzed by many statisticians, e.g., including Stuart (1955), Caussinus (1965), Bishop *et al.* (1975, p. 284), McCullagh (1978), Goodman (1979), Agresti (1983), Tomizawa (1993), and Tomizawa and Tahata (2007), etc.

Table 1: The unaided vision data of
(a) 7477 women in Britain (from Stuart, 1955),
(b) 3242 men in Britain (from Stuart, 1953),
(c) 4746 students in Japan (from Tomizawa, 1984).

(a) Women in Britain

Right eye grade	Left eye grade				Total
	Best (1)	Second (2)	Third (3)	Worst (4)	
Best (1)	1520	266	124	66	1976
Second (2)	234	1512	432	78	2256
Third (3)	117	362	1772	205	2456
Worst (4)	36	82	179	492	789
Total	1907	2222	2507	841	7477

(b) Men in Britain

Right eye grade	Left eye grade				Total
	Best (1)	Second (2)	Third (3)	Worst (4)	
Best (1)	821	112	85	35	1053
Second (2)	116	494	145	27	782
Third (3)	72	151	583	87	893
Worst (4)	43	34	106	331	514
Total	1052	791	919	480	3242

(c) Students in Japan

Right eye grade	Left eye grade				Total
	Best (1)	Second (2)	Third (3)	Worst (4)	
Best (1)	1291	130	40	22	1483
Second (2)	149	221	114	23	507
Third (3)	64	124	660	185	1033
Worst (4)	20	25	249	1429	1723
Total	1524	500	1063	1659	4746

We see from Table 2 that for the data in Table 1(a), the value of estimated measure $\hat{\Psi}$ is -0.102 and all values in the confidence interval for Ψ are negative. Therefore, the average marginal homogeneity for the women's right and left eyes departs toward the L-marginal inhomogeneity. Table 3 gives the values of likelihood ratio chi-squared statistic for testing goodness-of-fit of each model.

Table 2: The estimates of Ψ, estimated approximate standard errors for $\hat{\Psi}$, and approximate 95% confidence intervals for Ψ, applied to Tables 1(a), 1(b) and 1(c).

Applied data	Estimated measure	Standard error	Confidence interval
Table 1(a)	-0.102	0.029	$(-0.160, -0.045)$
Table 1(b)	0.038	0.044	$(-0.048, +0.123)$
Table 1(c)	0.128	0.040	$(+0.049, +0.206)$

We see from Table 3 that each model of marginal homogeneity and average marginal homogeneity fits the data in Table 1(a) poorly, but the extended marginal homogeneity model fits these data well. So we can see from the estimated measure that the degree of departure from marginal homogeneity for the vision data in Table 1(a) is estimated to be 10.2 percent of the maximum departure toward the L-marginal inhomogeneity. This indicates that the right eye is better than her left eye for all women.

Table 3: The values of likelihood ratio chi-squared statistic for the models of marginal homogeneity, average marginal homogeneity and extended marginal homogeneity, applied to Tables 1(a), 1(b) and 1(c).

Table 1(a)		
Applied models	degrees of freedom	Likelihood ratio chi-square
Marginal homogeneity	3	11.99^*
Average marginal homogeneity	1	11.98^*
Extended marginal homogeneity	2	0.005

Table 1(b)		
Applied models	degrees of freedom	Likelihood ratio chi-square
Marginal homogeneity	3	3.68
Average marginal homogeneity	1	0.73
Extended marginal homogeneity	2	2.94

Table 1(c)		
Applied models	degrees of freedom	Likelihood ratio chi-square
Marginal homogeneity	3	11.18^*
Average marginal homogeneity	1	9.94^*
Extended marginal homogeneity	2	0.56

$*$ means significant at the 0.05 level.

5.2. Analysis of Table 1(b)

Consider the data in Table 1(b) taken from Stuart (1953). These are data on unaided distance vision of 3242 men in Britain.

We see from Table 2 that for the data in Table 1(b), the value of measure $\hat{\Psi}$ is 0.038 and the confidence interval for Ψ includes zero. So this would indicate that there is a structure of average marginal homogeneity in the data in Table 1(b). Also we see from Table 3 that the marginal homogeneity model fits these data well, and each model of average marginal homogeneity and extended marginal homogeneity also fits these data well. Therefore, it is estimated that there is a structure of marginal homogeneity for the data in Table 1(b), and also the estimated measure $\hat{\Psi}$ would indicate it.

5.3. Analysis of Table 1(c)

Consider the data in Table 1(c) taken from Tomizawa (1984). These are data on unaided distance vision of 4746 students aged 18 to about 25 including about 10% women in Faculty of Science and Technology, Science University of Tokyo in Japan examined in April 1982.

For the data in Table 1(c), we see from Table 2 that the value of $\hat{\Psi}$ is 0.128 and all values in the confidence interval for Ψ are positive. Therefore, the average marginal homogeneity for the students' right and left eyes departs toward the U-marginal inhomogeneity. This is a contrast to the women's vision data in Table 1(a). We see from Table 3 that each model of marginal homogeneity and average marginal homogeneity fits the data in Table 1(c) poorly, but the extended marginal homogeneity model fits these data well. So we can see from the estimated measure that the degree of departure from marginal homogeneity for the vision data in Table 1(c) is estimated to be 12.8 percent of the maximum departure toward the U-marginal inhomogeneity. This indicates that the left eye is better than his/her right eye for all students.

In addition, when we compare the data in Tables 1(a) and 1(c) using the estimated measure $\hat{\Psi}$, the degree of departure from the marginal homogeneity for the right and left eyes is greater in the students data in Table 1(c) than in the women data in Table 1(a) (see Table 2). Since the $\hat{\Psi}$ is negative for the women vision data and positive for the students vision data, a woman's right eye tends to be greater than her left eye, and a student's left eye tends to be greater than his/her right eye.

ACKNOWLEDGMENTS

The authors would like to express their thanks to anonymous referee for his/her helpful suggestions and comments.

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