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A NOTE ON SECOND ORDER CONDITIONS IN EXTREME VALUE THEORY: LINKING GENERAL AND HEAVY TAIL CONDITIONS

Authors: M. Isabel Fraga Alves
– CEAUL, DEIO, Faculty of Science, University of Lisbon, Portugal
isabel.alves@fc.ul.pt

M. Ivette Gomes
– CEAUL, DEIO, Faculty of Science, University of Lisbon, Portugal
ivette.gomes@fc.ul.pt

Laurens de Haan
– Department of Economics, Erasmus University Rotterdam, The Netherlands
ldhaan@few.eur.nl

Cláudia Neves
– UIMA, Department of Mathematics, University of Aveiro, Portugal
claudia.neves@ua.pt

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Abstract:

- Second order conditions ruling the rate of convergence in any first order condition involving regular variation and assuring a unified extreme value limiting distribution function for the sequence of maximum values, linearly normalized, have appeared in several contexts whenever researchers are working either with a general tail, i.e., $\gamma \in \mathbb{R}$, or with heavy tails, with an extreme value index $\gamma > 0$. In this paper we shall clarify the link between the second order parameters, say ρ and $\widetilde{\rho}$ that have appeared in the two above mentioned set-ups, i.e., for a general tail and for heavy tails, respectively. We illustrate the theory with some examples and, for heavy tails, we provide a link with a third order framework.

Key-Words:

- *extreme value index; regular variation; semi-parametric estimation.*

AMS Subject Classification:

- Primary 62G32, 62E20, 26A12.

1. INTRODUCTION

Let $X_1, X_2, ..., X_n$ be an independent, identically distributed (i.i.d.) sample from an unknown distribution function (d.f.) F. It is well-known from Gnedenko's seminal work (Gnedenko, 1943) that if there exist normalizing constants $a_n > 0$, $b_n \in \mathbb{R}$ and a non-degenerate d.f. G such that, for all x,

$$\lim_{n\to\infty} P\Big\{a_n^{-1}\big(\max(X_1, ..., X_n) - b_n\big) \le x\Big\} = G(x) ,$$

G is, up to scale and location, an *Extreme Value* d.f., dependent on a shape parameter $\gamma \in \mathbb{R}$, and given by

$$G_\gamma(x) := \begin{cases} \exp\big(-(1+\gamma x)^{-1/\gamma}\big), & 1+\gamma x > 0 \quad \text{if } \gamma \neq 0 \\ \exp\big(-\exp(-x)\big), & x \in \mathbb{R} \quad \text{if } \gamma = 0 . \end{cases} \tag{1.1}$$

We then say that F is in the domain of attraction for maxima of the d.f. G_γ in (1.1) and write $F \in \mathcal{D}_{\mathcal{M}}(G_\gamma)$.

2. FIRST AND SECOND ORDER CONDITIONS

2.1. A general tail ($\gamma \in \mathbb{R}$)

The following *extended regular variation* property (de Haan, 1984), denoted ERV_γ, is a well-known necessary and sufficient condition for $F \in \mathcal{D}_{\mathcal{M}}(G_\gamma)$:

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{a(t)} = \begin{cases} \dfrac{x^\gamma - 1}{\gamma} & \text{if } \gamma \neq 0 \\ \ln x & \text{if } \gamma = 0 , \end{cases} \tag{2.1}$$

for every $x > 0$ and some positive measurable function a. For the case $\gamma > 0$ we see easily from (2.9) that we can choose $a(t) = \gamma U(t)$.

Apart from the first order condition in (2.1), we shall consider the most common second order condition, specifying the rate of convergence in (2.1). We shall assume the existence of a function $A(t)$, possibly not changing in sign and tending to zero as $t \to \infty$, such that

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a(t)} - \frac{x^\gamma - 1}{\gamma}}{A(t)} = H_{\gamma,\rho}(x) := \frac{1}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma - 1}{\gamma}\right) \tag{2.2}$$

for all $x > 0$, where $\rho \le 0$ is also a *second order* parameter controlling the speed of convergence of maximum values, linearly normalized, towards the limit law in

(1.1), for a general $\gamma \in \mathbb{R}$. We then say that the function U is of *second order extended regular variation, and use the notation* $U \in 2ERV_{\gamma,\rho}$. In (2.2), the cases $\gamma = 0$ and $\rho = 0$ are obtained by continuity arguments. More specifically, we can write

$$H_{\gamma,\rho}(x) = \begin{cases} \dfrac{1}{\rho}\left(\dfrac{x^\rho - 1}{\rho} - \ln x\right) & \text{if } \gamma = 0,\ \rho \neq 0 \\ \dfrac{1}{\gamma}\left(x^\gamma \ln x - \dfrac{x^\gamma - 1}{\gamma}\right) & \text{if } \gamma \neq 0,\ \rho = 0 \\ \dfrac{\ln^2 x}{2} & \text{if } \gamma = \rho = 0\ . \end{cases}$$

We remark that $|A| \in RV_\rho$. For a large variety of models we have $\rho < 0$ thus making sensible to simplify (2.2). We now state:

Proposition 2.1 (Gomes and Neves, 2007)**.** *Let us assume that there exist* $a(\cdot)$ *and* $A(\cdot)$ *such that* (2.2) *holds, with* $\rho < 0$*. Then, there exist* $a_0(\cdot)$ *and* $A_0(\cdot)$ *such that*

$$\lim_{t\to\infty} \frac{\frac{U(tx)-U(t)}{a_0(t)} - \frac{x^\gamma - 1}{\gamma}}{A_0(t)} = \frac{x^{\gamma+\rho} - 1}{\gamma+\rho} \tag{2.3}$$

with

$$A_0(t) = A(t)/\rho\ , \qquad a_0(t) = a(t)\big(1 - A_0(t)\big)\ . \tag{2.4}$$

From Theorem A in Draisma de Haan, Peng and Pereira (1999), with slight additions in Ferreira, de Haan and Peng (2003) and in de Haan and Ferreira (2006), we state the following:

Theorem 2.1. *Suppose the right endpoint* $x^F := U(\infty) > 0$ *and there exist* $a(\cdot)$ *and* $A(\cdot)$ *such that* (2.2) *holds, with* $\rho \leq 0$, $\gamma \neq \rho$*. Define*

$$\overline{A}(t) := \left(\frac{a(t)}{U(t)} - \gamma_+\right), \qquad \gamma_+ := \max(0, \gamma)\ . \tag{2.5}$$

Then for $\gamma + \rho < 0$

$$l := \lim_{t\to\infty}\left(U(t) - \frac{a(t)}{\gamma}\right) \ \text{exists and is finite} \tag{2.6}$$

and the following holds

$$\overline{A}(t) \underset{t\to\infty}{\longrightarrow} 0 \qquad \text{and} \qquad \frac{\overline{A}(t)}{A(t)} \underset{t\to\infty}{\longrightarrow} c\ ,$$

with

$$c = \begin{cases} 0 & \text{if } \gamma < \rho \leq 0 \\ \dfrac{\gamma}{\gamma+\rho} & \text{if } 0 \leq -\rho < \gamma \text{ or } (0 < \gamma < -\rho \text{ and } l = 0) \\ \pm\infty & \text{if } \gamma + \rho = 0 \text{ or } (0 < \gamma < -\rho \text{ and } l \neq 0) \text{ or } \rho < \gamma \leq 0\ . \end{cases} \tag{2.7}$$

2.1.1. Heavy tails $(\gamma > 0)$

The most typical first order condition for heavy tails, i.e., for the case $\gamma > 0$ in (1.1), comes also from Gnedenko (1943). For any real τ, let us denote by RV_τ the class of regularly varying functions with an index of regular variation τ, i.e., positive measurable functions g such that $\lim_{t\to\infty} g(tx)/g(t) = x^\tau$ for all $x > 0$. Then, for $\gamma > 0$,

$$F \in \mathcal{D}_{\mathcal{M}}(G_\gamma) \iff \overline{F} = 1 - F \in RV_{-1/\gamma} \,. \tag{2.8}$$

Equivalently, and with U standing for a quantile type function associated to F and defined by $U(t) := \big(1/(1-F)\big)^{\leftarrow}(t) = \inf\big\{x\colon F(x) \geq 1 - \frac{1}{t}\big\}$, de Haan (1970) established that

$$F \in \mathcal{D}_{\mathcal{M}}(G_\gamma) \iff U \in RV_\gamma \,. \tag{2.9}$$

To measure the rate of convergence in (2.9), it is then sensible to consider one of the following conditions:

$$\begin{aligned} &\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^\gamma}{\widetilde{A}(t)} = x^\gamma \, \frac{x^{\widetilde{\rho}} - 1}{\widetilde{\rho}} \iff \\ &\qquad\iff \lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{\widetilde{A}(t)} = \frac{x^{\widetilde{\rho}} - 1}{\widetilde{\rho}} \,, \end{aligned} \tag{2.10}$$

for all $x > 0$, where $\widetilde{\rho} \leq 0$ is a *second order* parameter controlling the speed of convergence of maximum values, linearly normalized, towards the limit law in (1.1) pertaining to $\gamma > 0$. Under these circumstances, we say that the function U is of *regular variation of second order*, and use the notation $U \in 2RV(\gamma, \widetilde{\rho})$. We remark that $|\widetilde{A}| \in RV_{\widetilde{\rho}}$.

3. THE LINK BETWEEN THE SECOND ORDER CONDITION FOR A HEAVY AND FOR A GENERAL TAIL

The following results hold with any measurable (eventually) positive function U.

Lemma 3.1. *If* (2.1) *holds for some* $\gamma \in \mathbb{R}$*, then the auxiliary function* $a(t)$ *in* (2.1) *is of regular variation at infinity with index* γ*, i.e.,* $a \in RV_\gamma$ *and*

$$\lim_{t\to\infty} \frac{a(t)}{U(t)} = \gamma_+ := \max(0, \gamma) \,.$$

Moreover, if $\gamma>0$, both functions a and U belong to RV_γ; if $\gamma<0$, then $x^F = U(\infty)<\infty$, $\lim_{t\to\infty} a(t)/\big(x^F - U(t)\big) = -\gamma$ and $x^F - U \in RV_\gamma$.

Furthermore, with $\gamma_- := \min(\gamma,0)$, and provided that $U(\infty)>0$,

$$\lim_{t\to\infty} \frac{\ln U(tx)-\ln U(t)}{a(t)/U(t)} = \frac{x^{\gamma_-}-1}{\gamma_-}, \qquad \text{for every } x>0 . \tag{3.1}$$

Proof: The first part of the lemma comes from Theorems 1.9 and 1.10 in Geluk and de Haan (1987). The limit in (3.1) follows easily when we distinguish between the cases $\gamma>0$ and $\gamma\le 0$. □

For the derivation of asymptotic properties of semi-parametric estimators of γ, a topic out of the scope of this paper, it is important to know, for all $x>0$, not only the rate of convergence of $\ln U(tx)-\ln U(t)$, but also of $U(tx)/U(t)$ and of $U(t)/U(tx)$, as $t\to\infty$. We shall now see in more detail for the different relevant subspaces of the semi-plane $(\gamma,\rho)\in\mathbb{R}\times\mathbb{R}_0^-$, the limiting behaviour, as $t\to\infty$, of $U(tx)/U(t)$ and $U(t)/U(tx)$. The limit behavior of $\ln U(tx)-\ln U(t)$ has been analyzed e.g. in Appendix B of de Haan and Ferreira (2006).

Lemma 3.2. *Assume that (2.2) holds, i.e., $U\in 2ERV_{\gamma,\rho}$. Then, we may write*

$$\frac{U(tx)}{U(t)} = x^{\gamma_+} + \overline{A}(t)\left[\frac{x^\gamma-1}{\gamma} + A(t)\,\overline{a}(x,t;\gamma,\rho)\,\big(1+o(1)\big)\right], \tag{3.2}$$

where

$$\overline{a}(x,t;\gamma,\rho) = \begin{cases} \dfrac{\ln^2 x}{2} & \text{if } \gamma=\rho=0 \\[2mm] \dfrac{1}{\gamma}\left(x^\gamma \ln x - \dfrac{x^\gamma-1}{\gamma}\right) & \text{if } \gamma<\rho=0 \\[2mm] \dfrac{1}{\rho}\left(\dfrac{x^{\gamma+\rho}-1}{\gamma+\rho} - \dfrac{x^\gamma-1}{\gamma}\right) & \text{if } \gamma\le 0,\ \ \rho<0 \\[2mm] \dfrac{\gamma}{\rho\,\overline{A}(t)}\left(\dfrac{x^{\gamma+\rho}-1}{\gamma+\rho} - \dfrac{x^\gamma-1}{\gamma}\right) & \text{if } \gamma>0,\ \ \rho<0 \\[2mm] \dfrac{1}{\overline{A}(t)}\left(x^\gamma \ln x - \dfrac{x^\gamma-1}{\gamma}\right) & \text{if } \rho=0<\gamma\ . \end{cases}$$

Proof: Directly from (2.2), we get

$$\frac{U(tx)}{U(t)} - 1 = \frac{a(t)}{U(t)}\left\{\frac{x^\gamma-1}{\gamma} + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right)\big(1+o(1)\big)\right\} .$$

With the notation in (2.5), i.e., $a(t)/U(t) = \gamma_+ + \overline{A}(t)$, we may write

$$\begin{aligned}\frac{U(tx)}{U(t)} - 1 \;&=\; \gamma_+\left(\frac{x^\gamma-1}{\gamma}\right) + \overline{A}(t)\left(\frac{x^\gamma-1}{\gamma}\right) \\ &\quad + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right)\big(\gamma_+ + \overline{A}(t)\big)\big(1+o(1)\big)\end{aligned}$$

and (3.2) follows for any $\rho < 0$.

If $\rho = 0$ and $\gamma \neq 0$, then, also directly from (2.2), and by continuity arguments,

$$\frac{U(tx)}{U(t)} - 1 \;=\; \frac{a(t)}{U(t)}\left\{\frac{x^\gamma-1}{\gamma} + \frac{A(t)}{\gamma}\left(x^\gamma \ln x - \frac{x^\gamma-1}{\gamma}\right)\big(1+o(1)\big)\right\},$$

and things work as before, with $A(t)/\rho$ replaced by $A(t)/\gamma$ and $\frac{x^{\gamma+\rho}-1}{\gamma+\rho}$ replaced by $x^\gamma \ln x$. The case $\gamma = \rho = 0$ comes again directly from (2.2) and by continuity arguments. □

Theorem 3.1. *Let $U \in ERV_{\gamma,\rho}$ as introduced in* (2.2). *Let c be the limit in* (2.7).

(i) *If $\gamma > 0$,*

$$\lim_{t\to\infty} \frac{\frac{U(t)}{U(tx)} - x^{-\gamma}}{\widetilde{A}(t)} \;=\; K_{\gamma,\rho}(x) \;:=\; \begin{cases} -x^{-\gamma}\,\dfrac{x^\rho-1}{\rho} & \text{if } \; c = \dfrac{\gamma}{\gamma+\rho} \\[2ex] -x^{-\gamma}\,\dfrac{x^{-\gamma}-1}{-\gamma} & \text{if } \; c = \pm\infty\,, \end{cases} \tag{3.3}$$

for all $x > 0$, where, with $\overline{A}(t)$ given in (2.5),

$$\widetilde{A}(t) \;:=\; \begin{cases} \dfrac{\gamma\, A(t)}{\gamma+\rho} & \text{if } \; c = \dfrac{\gamma}{\gamma+\rho} \\[2ex] \overline{A}(t) & \text{if } \; c = \pm\infty\,. \end{cases} \tag{3.4}$$

Necessarily, $|\widetilde{A}| \in RV_{\widetilde{\rho}}$, with

$$\widetilde{\rho} \;=\; \begin{cases} \rho & \text{if } \; c = \dfrac{\gamma}{\gamma+\rho} \\[2ex] -\gamma & \text{if } \; c = \pm\infty\,. \end{cases} \tag{3.5}$$

(ii) *If $\gamma \le 0$, we need further to assume that $\gamma \neq \rho$. Then,*

$$\lim_{t\to\infty} \frac{\frac{U(t)}{a^*(t)}\left(1 - \frac{U(t)}{U(tx)}\right) - \frac{x^\gamma-1}{\gamma}}{A^*(t)} = K^*_{\gamma,\rho}(x) \;=\; \begin{cases} x^\gamma \ln x & \text{if } \; \gamma<\rho=0 \\[1ex] \dfrac{x^{\gamma+\rho}-1}{\gamma+\rho} & \text{if } \; \gamma<\rho<0 \\[2ex] \dfrac{x^{2\gamma}-1}{2\,\gamma} & \text{if } \; \rho<\gamma<0 \\[2ex] \ln^2 x & \text{if } \; \rho<\gamma=0\,, \end{cases} \tag{3.6}$$

where

$$A^*(t) \;=\; \begin{cases} \dfrac{A(t)}{\gamma} & \textit{if} \;\; \gamma<\rho=0 \\ \dfrac{A(t)}{\rho} & \textit{if} \;\; \gamma<\rho<0 \\ -\dfrac{2\,\overline{A}(t)}{\gamma} & \textit{if} \;\; \rho<\gamma<0 \\ -\overline{A}(t) & \textit{if} \;\; \rho<\gamma=0 \end{cases} \tag{3.7}$$

and

$$a^*(t) \;=\; \begin{cases} a(t) & \textit{if} \;\; \rho<\gamma=0 \\ a(t)\,\big(1-A^*(t)\big) & \textit{otherwise}\;. \end{cases} \tag{3.8}$$

Necessarily, $|A^*| \in RV_{\rho^*}$*, with*

$$\rho^* \;=\; \begin{cases} \rho & \textit{if} \;\; \gamma<\rho\leq 0 \\ \gamma & \textit{if} \;\; \rho<\gamma\leq 0\;. \end{cases} \tag{3.9}$$

Proof: We shall consider the cases (i) and (ii) separately.

Case (i): If $\gamma > 0$, $\rho < 0$ and (2.2) holds, i.e., $U \in 2ERV_{\gamma,\rho}$, we have from (3.2),

$$\frac{U(tx)}{U(t)} - x^\gamma \;=\; \overline{A}(t)\left(\frac{x^\gamma-1}{\gamma}\right) + \frac{\gamma\,A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right) + o\big(A(t)\big)\;.$$

If $c=\pm\infty$, then $A(t)=o\left(\overline{A}(t)\right)$,

$$\frac{U(tx)}{U(t)} - x^\gamma \;=\; x^\gamma\left(\frac{x^{-\gamma}-1}{-\gamma}\right)\overline{A}(t) + o\big(\overline{A}(t)\big) \quad\text{and}\quad \frac{\frac{U(tx)}{U(t)}-x^\gamma}{\overline{A}(t)} \underset{t\to\infty}{\longrightarrow} x^\gamma\left(\frac{x^{-\gamma}-1}{-\gamma}\right)\;.$$

If $c=\gamma/(\gamma+\rho)$, we get $\overline{A}(t)=\frac{\gamma\,A(t)}{\gamma+\rho}\big(1+o(1)\big)$. Since in this region $\gamma\neq-\rho$, we may further write

$$\begin{aligned} \frac{U(tx)}{U(t)} - x^\gamma \;&=\; x^\gamma\left(\overline{A}(t)\left(\frac{x^{-\gamma}-1}{-\gamma}\right) + \frac{\gamma\,A(t)}{\gamma+\rho}\left(\frac{x^\rho-1}{\rho} - \frac{x^{-\gamma}-1}{-\gamma}\right) + o\big(A(t)\big)\right) \\ &=\; \frac{\gamma\,A(t)}{\gamma+\rho}\,x^\gamma\left(\frac{x^\rho-1}{\rho}\right) + o\big(A(t)\big)\;. \end{aligned}$$

Consequently,

$$\lim_{t\to\infty}\frac{\frac{U(tx)}{U(t)}-x^\gamma}{\widetilde{A}(t)} \;=\; \begin{cases} x^\gamma\,\dfrac{x^\rho-1}{\rho} & \text{if}\;\; c=\dfrac{\gamma}{\gamma+\rho} \\ x^\gamma\,\dfrac{x^{-\gamma}-1}{-\gamma} & \text{if}\;\; c=\pm\infty \end{cases} \;=\; -x^{2\gamma}\,K_{\gamma,\rho}(x)\;, \tag{3.10}$$

with $K_{\gamma,\rho}(x)$ and $\widetilde{A}(t)$ defined in (3.3) and (3.4), respectively. Finally, (3.10), together with the fact that

$$\frac{U(tx)}{U(t)} - x^\gamma \;=\; -x^\gamma\,\frac{U(tx)}{U(t)}\left(\frac{U(t)}{U(tx)} - x^{-\gamma}\right) \;=\; -x^{2\gamma}\left(\frac{U(t)}{U(tx)} - x^{-\gamma}\right)\big(1+o(1)\big)\,,$$

leads us to the limit in (3.3), with $\widetilde{A}(t)$ and $\widetilde{\rho}$ given in (3.4) and (3.5), respectively.

If $\gamma > 0$ and $\rho = 0$, we get, again from (3.2),

$$\begin{aligned}\frac{U(tx)}{U(t)} - x^\gamma \;&=\; \overline{A}(t)\left(\frac{x^\gamma-1}{\gamma}\right) + A(t)\left(x^\gamma \ln x - \frac{x^\gamma-1}{\gamma}\right) + o\big(A(t)\big)\\ &=\; x^\gamma\left(\overline{A}(t)\left(\frac{x^{-\gamma}-1}{-\gamma}\right) + A(t)\left(\ln x - \frac{x^{-\gamma}-1}{-\gamma}\right) + o\big(A(t)\big)\right).\end{aligned}$$

But if $\gamma > 0$ and $\rho = 0$, then $c = \gamma/(\gamma+\rho) = 1$, $\overline{A}(t) = A(t) + o\big(A(t)\big)$, and

$$\frac{U(tx)}{U(t)} - x^\gamma \;=\; A(t)\,x^\gamma \ln x + o\big(A(t)\big)\,.$$

Consequently, (3.3) holds, with $\widetilde{A}(t) = A(t) \equiv \gamma\,A(t)/(\gamma+\rho)$ and $\widetilde{\rho} = \rho = 0$.

Case (ii): If $\gamma < \rho = 0$, we get, again from (3.2),

$$\begin{aligned}\frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) \;&=\; \frac{x^\gamma-1}{\gamma} + \frac{A(t)}{\gamma}\left(x^\gamma \ln x - \frac{x^\gamma-1}{\gamma}\right) + o\big(A(t)\big)\\ &=\; \frac{x^\gamma-1}{\gamma}\left(1 - \frac{A(t)}{\gamma}\right) + \frac{A(t)}{\gamma}\,x^\gamma \ln x + o\big(A(t)\big)\end{aligned}$$

and with $a^*(t) = a(t)\left(1 - \frac{A(t)}{\gamma}\right)$,

$$\frac{U(t)}{a^*(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \frac{x^\gamma-1}{\gamma} + \frac{A(t)}{\gamma}\,x^\gamma \ln x + o\big(A(t)\big)\,.$$

Consequently, (3.6), (3.7), (3.8) and (3.9) follow in this region of the (γ,ρ)-plane.

If $\gamma < \rho < 0$, $a(t)/U(t) \equiv \overline{A}(t) = o(A(t))$, and again from (3.2),

$$\begin{aligned}\frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) \;&=\; \frac{x^\gamma-1}{\gamma} + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho} - \frac{x^\gamma-1}{\gamma}\right) + o\big(A(t)\big)\\ &=\; \frac{x^\gamma-1}{\gamma}\left(1 - \frac{A(t)}{\rho}\right) + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho}-1}{\gamma+\rho}\right) + o\big(A(t)\big)\end{aligned}$$

and with $a^*(t) = a(t)\left(1 - \frac{A(t)}{\rho}\right)$,

$$\frac{U(t)}{a^*(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \frac{x^\gamma - 1}{\gamma} + \frac{A(t)}{\rho}\left(\frac{x^{\gamma+\rho} - 1}{\gamma+\rho}\right) + o\big(A(t)\big) ,$$

and the results in the proposition hold.

If $\rho < \gamma \le 0$, $A(t) = o\left(\overline{A}(t)\right)$, and also from (3.2), we get

$$\frac{U(t)}{U(tx)} = 1 - \overline{A}(t)\left(\frac{x^\gamma - 1}{\gamma}\right) + \overline{A}^2(t)\left(\frac{x^\gamma - 1}{\gamma}\right)^2 \big(1 + o(1)\big) . \tag{3.11}$$

Consequently, for $\gamma < 0$, since $\left(\frac{x^\gamma-1}{\gamma}\right)^2 = \frac{2}{\gamma}\left(\frac{x^{2\gamma}-1}{2\gamma} - \frac{x^\gamma-1}{\gamma}\right)$

$$\begin{aligned} \frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) &= \frac{x^\gamma - 1}{\gamma} - \frac{2\,\overline{A}(t)}{\gamma}\left(\frac{x^{2\gamma} - 1}{2\,\gamma} - \frac{x^\gamma - 1}{\gamma}\right)\big(1 + o(1)\big) \\ &= \frac{x^\gamma - 1}{\gamma}\left(1 + \frac{2\,\overline{A}(t)}{\gamma}\right) - \frac{2\,\overline{A}(t)}{\gamma}\left(\frac{x^{2\gamma} - 1}{2\,\gamma}\right)\big(1 + o(1)\big) , \end{aligned}$$

and with $a^*(t) = a(t)\left(1 + \frac{2\,\overline{A}(t)}{\gamma}\right)$,

$$\frac{U(t)}{a^*(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \frac{x^\gamma - 1}{\gamma} - \frac{2\,\overline{A}(t)}{\gamma}\left(\frac{x^{2\gamma} - 1}{2\,\gamma}\right)\big(1 + o(1)\big) ,$$

and the results in the proposition follow.

If $\rho < \gamma = 0$, then from (3.11), we get

$$\frac{U(t)}{a(t)}\left(1 - \frac{U(t)}{U(tx)}\right) = \ln x - \overline{A}(t)\,\ln^2 x\,\big(1 + o(1)\big) ,$$

and the result in the proposition follows as well. □

Corollary 3.1. *Under the conditions and notations of Proposition 2.1, and for* $\gamma > 0$,

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{\widetilde{A}(t)} = \widetilde{K}_{\gamma,\rho}(x) := \begin{cases} \dfrac{x^\rho - 1}{\rho} & \text{if } c = \dfrac{\gamma}{\gamma+\rho} \\[2ex] \dfrac{x^{-\gamma} - 1}{-\gamma} & \text{if } c = \pm\infty , \end{cases} \tag{3.12}$$

for every $x > 0$, *and with* $\widetilde{A}$ *provided in* (3.4).

Proof: The proof follows immediately from relation (3.3). □

Remark 3.1. Note that the second order condition in (3.12) is the usual second order condition for heavy tails, i.e., the second order condition provided in (2.10).

Remark 3.2. Note next that the region $\{(\gamma,\rho)\colon 0<\gamma<-\rho \text{ and } l\neq 0\}$ in the (γ,ρ)-plane, jointly with the line $\rho=-\gamma$, are transformed in the line $\widetilde{\rho}=-\gamma$ in the $(\gamma,\widetilde{\rho})$-plane. There we have $c=\pm\infty$. Outside that line we have $c=\gamma/(\gamma+\widetilde{\rho})=\gamma/(\gamma+\rho)$.

Remark 3.3. For $\gamma>0$, the rate of convergence in (3.1), i.e., the rate of convergence of $\big(\ln U(tx)-\ln U(t)\big)/\big(a(t)/U(t)\big)-\ln x$ towards zero, is measured by $\widetilde{A}(t)$ in (3.4) only if $\rho\neq 0$. If $\rho=0$, the rate of convergence in (3.1) can be of a smaller order than $\widetilde{A}(t)$ as may be seen in Example 4.1.
For $\gamma\leq 0$, Lemma 3.2 gives rise to (3.1) in a similar way as it yields Corollary 3.1.

4. EXAMPLES AND SOME ADDITIONAL COMMENTS

Example 4.1. (A model with $\rho=\widetilde{\rho}=0$ and $\gamma>0$). Consider the model $U(t)=t^\gamma(1+\ln t)$. Then

$$U(tx)-U(t) \;=\; \gamma\, t^\gamma(\ln t+1)\left(\frac{x^\gamma-1}{\gamma}+\frac{x^\gamma\ln x}{\gamma(\ln t+1)}\right), \qquad x>0\,,$$

i.e., $\rho=0$ in (2.2), since

$$\frac{\frac{U(tx)-U(t)}{\gamma\, t^\gamma(\ln t+1)}-\frac{x^\gamma-1}{\gamma}}{1/\big(\gamma(\ln t+1)\big)} \;=\; x^\gamma\ln x\,.$$

Notice that $H_{\gamma,0}(x)=\gamma^{-1}\big(x^\gamma\ln x-(x^\gamma-1)/\gamma\big)$, meaning that (2.2) is equivalent to

$$\frac{\frac{U(tx)-U(t)}{a(t)\,(1-A(t)/\gamma)}-\frac{x^\gamma-1}{\gamma}}{A(t)/\gamma} \;\underset{t\to\infty}{\longrightarrow}\; x^\gamma\ln x\,,$$

as stated in (2.3). Consequently we should choose

$$A(t)=\frac{1}{\ln t+1}\in RV_0\,, \qquad a(t)\left(1-\frac{1}{\gamma(\ln t+1)}\right) \;=\; \gamma\, t^\gamma(\ln t+1)\,.$$

Theorem 2.1 yields $c=1$ while Theorem 3.1 determines $\widetilde{\rho}=0$ and $\widetilde{A}(t)=A(t)$. Indeed, we have

$$\text{(4.1)}\qquad \ln U(tx)-\ln U(t)-\gamma\ln x \;=\; \frac{\ln x}{\ln t+1}+\frac{\ln^2 x}{2(\ln t+1)^2}+o\!\left(\frac{1}{\ln^2 t}\right),$$

as $t \to \infty$, thus making suitable to take $A(t) = (\ln t + 1)^{-1}$ in the left hand side of

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \gamma \ln x}{A(t)} = \ln x \ , \qquad x > 0$$

and (3.12) holds in fact with $\widetilde{A}(t) = A(t)$. Furthermore, after a few manipulations of (4.1), we get

$$\frac{\frac{\ln U(tx) - \ln U(t)}{\gamma\left(1+\frac{1}{\gamma(\ln t+1)}\right)} - \ln x}{\frac{1}{2\ln^2 t}} \underset{t\to\infty}{\longrightarrow} \ln^2 x \ .$$

Therefore, the rate of convergence in (3.1) is of the order of $1/\ln^2 t = o\big(A(t)\big)$, as mentioned in Remark 3.3.

Example 4.2. For the Fréchet model, $F(x) = \exp\left(-x^{-1/\gamma}\right)$, $x \geq 0$ ($\gamma > 0$), we get successively,

$$\begin{aligned} U(t) &= \left(-\ln\left(1-\frac{1}{t}\right)\right)^{-\gamma} \\ &= t^\gamma\left(1+\frac{1}{2t}+\frac{1}{3t^2}+o\big(t^{-2}\big)\right)^{-\gamma} \\ &= t^\gamma\left(1-\frac{\gamma}{2t}+\frac{\gamma(3\gamma-5)}{24t^2}+o\big(t^{-2}\big)\right) . \end{aligned}$$

Hence,

$$U(tx) - U(t) = \begin{cases} \gamma\, t^\gamma\left(\dfrac{x^\gamma-1}{\gamma} - \dfrac{\gamma-1}{2t}\left(\dfrac{x^{\gamma-1}-1}{\gamma-1}\right) + o\big(t^{-1}\big)\right) & \text{if } \gamma \neq 1 \\ t\left((x-1) - \dfrac{1}{12t^2}\,(x^{-1}-1) + o\big(t^{-2}\big)\right) & \text{if } \gamma = 1 \ . \end{cases}$$

If we make correspondence with condition (2.3), we see that $\rho = \begin{cases} -1 & \text{if } \gamma \neq 1 \\ -2 & \text{if } \gamma = 1 \ . \end{cases}$

Likewise, (2.4) can be set as

$$a_0(t) = \gamma\, t^\gamma \qquad \text{and} \qquad A_0(t) = \begin{cases} \dfrac{1-\gamma}{2t} & \text{if } \gamma \neq 1 \\ \dfrac{1}{12t^2} & \text{if } \gamma = 1 \ . \end{cases}$$

According to Proposition 2.1, if we choose

$$A(t) = \rho A_0(t) = \begin{cases} \dfrac{\gamma-1}{2t} & \text{if } \gamma \neq 1 \\ -\dfrac{1}{6t^2} & \text{if } \gamma = 1 \end{cases}$$

and

$$a(t) \;=\; \gamma\, t^{\gamma}/\big(1-A_0(t)\big) \;=\; \begin{cases} \dfrac{2\,\gamma\, t^{\gamma+1}}{2\,t+\gamma-1} & \text{if } \gamma\neq 1 \\[2ex] \dfrac{12\, t^{3}}{12\, t^{2}-1} & \text{if } \gamma=1\;, \end{cases}$$

we get the limiting result in (2.2).

We will derive that (3.12) holds for $\widetilde{A}(t)=\gamma/(2\,t)$, with $\widetilde{\rho}=-1\neq\rho=-2$ for $\gamma=1$, and $\widetilde{\rho}=-1=\rho$ for $\gamma\neq 1$. As seen before regarding the limit in (2.2), we have whenever $\gamma\neq 1$,

$$\begin{aligned} U(t) \;&=\; t^{\gamma}\left(1-\frac{\gamma}{2\,t}+\frac{\gamma\,(3\,\gamma-5)}{24\,t^{2}}+o\big(t^{-2}\big)\right)\;;\\ a(t) \;&=\; 2\,\gamma\; t^{\gamma+1}/(2\,t+\gamma+\rho)\;;\\ A(t) \;&=\; -\rho\,(\gamma+\rho)/(2\,t)\;. \end{aligned}$$

Then,

$$\begin{aligned} \overline{A}(t) \;=\; \frac{a(t)}{U(t)}-\gamma \;&=\; \frac{2\,\gamma\, t}{2\,t+\gamma+\rho}\left(1+\frac{\gamma}{2\,t}-\frac{\gamma\,(3\,\gamma-5)}{24\,t^{2}}+\frac{\gamma^{2}}{4\,t^{2}}+o\big(t^{-2}\big)\right)-\gamma\\ &=\; \frac{2\,\gamma\, t\,(2\,t+\gamma)}{2\,t\,(2\,t+\gamma+\rho)}-\gamma-\frac{2\,\gamma^{2}\, t\,(9\,\gamma-5)}{24\,t^{2}\,(2\,t+\gamma+\rho)}+o\big(t^{-2}\big)\\ &=\; -\frac{\gamma\,\rho}{2\,t+\gamma+\rho}\left(1+\frac{\gamma\,(9\,\gamma-5)}{12\,\rho\, t}+o\big(t^{-1}\big)\right)\;\underset{t\to\infty}{\longrightarrow}\; 0\;, \end{aligned}$$

and

$$\frac{\overline{A}(t)}{A(t)} \;=\; \frac{2\,\gamma\, t}{(\gamma+\rho)\,(2\,t+\gamma+\rho)}\left(1+\frac{\gamma\,(9\,\gamma-5)}{12\,t}+o\big(t^{-1}\big)\right)\;\underset{t\to\infty}{\longrightarrow}\;\frac{\gamma}{\gamma+\rho}\;.$$

Let us think on

$$\begin{aligned} U(t)-\frac{a(t)}{\gamma} \;&=\; -\frac{U(t)}{\gamma}\,\overline{A}(t)\\ &=\; t^{\gamma}\left(\frac{2\,\rho\, t-\gamma(\gamma+\rho)}{2\,t\,(2\,t+\gamma+\rho)}+\frac{\gamma\,(3\gamma-5)}{24\,t^{2}}+o\big(t^{-2}\big)\right)\underset{t\to\infty}{\longrightarrow} 0=:l,\quad \text{if } 0<\gamma<1\,. \end{aligned}$$

Hence, we conclude that $c=\gamma/(\gamma+\rho)$, for $\gamma\neq 1$.

If we consider the case $\gamma=1$,

$$\begin{aligned} \frac{a(t)}{U(t)} \;&=\; \frac{12\,t^{2}}{12\,t^{2}-1}\left(1+\frac{1}{2\,t}+\frac{1}{12\,t^{2}}+\frac{1}{4\,t^{2}}+o\big(t^{-2}\big)\right)\\ &=\; 1+\frac{1}{2\,t}+\frac{5}{12\,t^{2}}+o\big(t^{-2}\big)\;. \end{aligned}$$

Consequently, and as was expected from Theorem 2.1,

$$\overline{A}(t) \;=\; \frac{a(t)}{U(t)} - 1 \;=\; \frac{1}{2\,t}\left(1 + \frac{5}{6\,t} + o\big(t^{-1}\big)\right) \underset{t\to\infty}{\longrightarrow} 0\ ,$$

$$\frac{\overline{A}(t)}{A(t)} \;=\; -3\,t\left(1 + \frac{5}{6\,t} + o\big(t^{-1}\big)\right) \underset{t\to\infty}{\longrightarrow} -\infty, \qquad \text{i.e.,}\ \ c = -\infty$$

and

$$U(t) - a(t) \;=\; -\frac{1}{2} - \frac{t}{12\,t^2 - 1} - \frac{1}{12\,t} + o\big(t^{-1}\big) \underset{t\to\infty}{\longrightarrow} -\frac{1}{2} = l\ .$$

Since this limit l is different from zero and $\gamma = 1 < -\rho = 2$, we indeed expected to have $c = \pm\infty$, as actually happens. Now, from Theorem 3.1, $\widetilde{\rho} = -\gamma = -1$ and we may choose

$$\widetilde{A}(t) \;=\; \overline{A}(t) \;=\; \frac{1}{2\,t}\left(1 + \frac{5}{6\,t} + o\big(t^{-1}\big)\right)\ ,$$

or more simply $\widetilde{A}(t) = 1/(2\,t)$. Indeed, and as mentioned before for $\gamma = 1$, (3.12) holds true with $\widetilde{A}(t) = \gamma/(2\,t)$ and $\widetilde{\rho} = -1 \neq \rho = -2$.

Example 4.3. Consider the extreme value model with d.f. $G_\gamma(x) = \exp\big(-(1+\gamma\,x)^{-1/\gamma}\big)$, $1+\gamma\,x > 0$, $\gamma \in \mathbb{R}$. For this model,

$$\begin{aligned}
U(t) \;&=\; \frac{\Big(-\ln\big(1-\frac{1}{t}\big)\Big)^{-\gamma} - 1}{\gamma} \\
&=\; \frac{t^\gamma}{\gamma}\left(1 - t^{-\gamma} - \frac{\gamma}{2\,t} + \frac{\gamma\,(3\,\gamma-5)}{24\,t^2} + o\big(t^{-2}\big)\right) \\
&=\; \begin{cases}
-\dfrac{1}{\gamma}\left(1 - t^\gamma + \dfrac{\gamma\,t^{\gamma-1}}{2} + o\big(t^{\gamma-1}\big)\right) & \text{if}\ \ \gamma < 0 \\[2mm]
\ln t - \dfrac{1}{2\,t} + o\big(t^{-1}\big) & \text{if}\ \ \gamma = 0 \\[2mm]
\dfrac{t^\gamma}{\gamma}\left(1 - t^{-\gamma} - \dfrac{\gamma}{2\,t} + o\big(t^{-1}\big)\right) & \text{if}\ \ 0 < \gamma < 1 \\[2mm]
\dfrac{t^\gamma}{\gamma}\left(1 - \dfrac{3}{2\,t} - \dfrac{1}{12\,t^2} + o\big(t^{-2}\big)\right) & \text{if}\ \ \gamma = 1 \\[2mm]
\dfrac{t^\gamma}{\gamma}\left(1 - \dfrac{\gamma}{2\,t} + o\big(t^{-1}\big)\right) & \text{if}\ \ \gamma > 1\ .
\end{cases}
\end{aligned}$$

Then

$$U(t\,x) - U(t) \;=\; \begin{cases}
t^\gamma\left(\dfrac{x^\gamma - 1}{\gamma} - \dfrac{\gamma-1}{2\,t}\left(\dfrac{x^{\gamma-1}-1}{\gamma-1}\right) + o\big(t^{-1}\big)\right) & \text{if}\ \ \gamma \neq 1 \\[3mm]
t^\gamma\left(\dfrac{x^\gamma - 1}{\gamma} + \dfrac{\gamma-2}{12\,t^2}\left(\dfrac{x^{\gamma-2}-1}{\gamma-2}\right) + o\big(t^{-2}\big)\right) & \text{if}\ \ \gamma = 1\ ,
\end{cases}$$

i.e., we may choose, in (2.3),

$$a_0(t) = t^\gamma \quad \text{and} \quad A_0(t) = \begin{cases} -\dfrac{\gamma-1}{2\,t} & \text{if } \gamma \neq 1 \\ -\dfrac{1}{12\,t^2} & \text{if } \gamma = 1 \end{cases}, \quad \text{with } \rho = \begin{cases} -1 & \text{if } \gamma \neq 1 \\ -2 & \text{if } \gamma = 1 \,. \end{cases}$$

Since

$$1 - A_0(t) = \begin{cases} \dfrac{2\,t+\gamma-1}{2\,t} & \text{if } \gamma \neq 1 \\ \dfrac{12\,t^2-1}{12\,t^2} & \text{if } \gamma = 1 \,, \end{cases}$$

we get, from (2.4),

$$a(t) = \frac{a_0(t)}{1-A_0(t)} = \begin{cases} \dfrac{2\,t^{\gamma+1}}{2\,t+\gamma-1} & \text{if } \gamma \neq 1 \\ \dfrac{12\,t^3}{12\,t^2-1} & \text{if } \gamma = 1 \end{cases}$$

and

$$A(t) = \rho\, A_0(t) = \begin{cases} \dfrac{\gamma-1}{2\,t} & \text{if } \gamma \neq 1 \\ \dfrac{1}{6\,t^2} & \text{if } \gamma = 1 \,. \end{cases}$$

Then

$$\frac{a(t)}{U(t)} = \begin{cases} -\gamma\, t^\gamma \left(1 + \left(\dfrac{1-\gamma}{2\,t} + t^\gamma\right)\big(1+o(1)\big)\right) & \text{if } \gamma < 0 \\ \dfrac{1}{\ln t}\left(1 + \dfrac{1}{2\,t} + o\big(t^{-1}\big)\right) & \text{if } \gamma = 0 \\ \gamma\Big(1 + t^{-\gamma} + o\big(t^{-\gamma}\big)\Big) & \text{if } 0 < \gamma < 1 \\ 1 + \dfrac{3}{2\,t} + o\big(t^{-1}\big) & \text{if } \gamma = 1 \\ \gamma\left(1 + \dfrac{1}{2\,t} + o\big(t^{-1}\big)\right) & \text{if } \gamma > 1 \,, \end{cases}$$

and consequently,

$$\overline{A}(t) = \frac{a(t)}{U(t)} - \gamma_+ = \begin{cases} -\gamma\, t^\gamma \big(1+o(1)\big) & \text{if } \gamma < 0 \\ \dfrac{1}{\ln t}\big(1+o(1)\big) & \text{if } \gamma = 0 \\ \gamma\, t^{-\gamma} + o\big(t^{-\gamma}\big) & \text{if } 0 < \gamma < 1 \\ \dfrac{3}{2\,t} + o\big(t^{-1}\big) & \text{if } \gamma = 1 \\ \dfrac{\gamma}{2\,t} + o\big(t^{-1}\big) & \text{if } \gamma > 1 \,. \end{cases}$$

Then

$$\frac{\overline{A}(t)}{A(t)} = \begin{cases} -\dfrac{2\,\gamma\, t^{\gamma+1}}{\gamma-1}\,\big(1+o(1)\big) & \text{if} \;\; \gamma<0 \\ -\dfrac{2\,t}{\ln t}\,\big(1+o(1)\big) & \text{if} \;\; \gamma=0 \\ \dfrac{2\,\gamma}{\gamma-1}\; t^{1-\gamma}\big(1+o(1)\big) & \text{if} \;\; 0<\gamma<1 \\ -9\,t\,\big(1+o(1)\big) & \text{if} \;\; \gamma=1 \\ \dfrac{\gamma}{\gamma-1}\,\big(1+o(1)\big) & \text{if} \;\; \gamma>1 \end{cases} \underset{t\to\infty}{\longrightarrow}$$

$$\underset{t\to\infty}{\longrightarrow} \begin{cases} 0 & \text{if} \;\; \gamma<-1 \\ -\infty & \text{if} \;\; -1<\gamma\le 1 \\ \dfrac{\gamma}{\gamma-1} = \dfrac{\gamma}{\gamma+\rho} & \text{if} \;\; \gamma>1 \end{cases} =: \;\; c\;.$$

Note that for $\gamma=\rho=-1$ we get a finite limit $\overline{A}(t)/A(t) \underset{t\to\infty}{\longrightarrow} -1$ and different from $\gamma/(\gamma+\rho)=1/2$.

Let us now compute for $0<\gamma<-\rho$,

$$U(t) - \frac{a(t)}{\gamma} = \begin{cases} \dfrac{t^\gamma}{\gamma}\,\Big(-t^{-\gamma}+o\big(t^{-\gamma}\big)\Big) & \text{if} \;\; 0<\gamma<1 \\ t\,\bigg(-\dfrac{3}{2\,t}+o\big(t^{-1}\big)\bigg) & \text{if} \;\; \gamma=1 \end{cases} \underset{t\to\infty}{\longrightarrow}$$

$$\underset{t\to\infty}{\longrightarrow} \begin{cases} -\dfrac{1}{\gamma} & \text{if} \;\; 0<\gamma<1 \\ -\dfrac{3}{2} & \text{if} \;\; \gamma=1 \end{cases} =: \;\; l\;,$$

in agreement with Theorem 2.1. Note however that $l\neq 0$ for all $0<\gamma<-\rho$ and $c=\pm\infty$ for all region $0<\gamma<-\rho$.

On another side, for heavy tails, i.e., for $\gamma>0$,

$$\frac{\ln U(tx)-\ln U(t)-\gamma\ln x}{\widetilde{A}(t)} \underset{t\to\infty}{\longrightarrow} \frac{x^{\widetilde{\rho}}-1}{\widetilde{\rho}}\,, \qquad \widetilde{\rho} = \begin{cases} -\gamma & \text{if} \;\; 0<\gamma\le 1 \\ \rho=-1 & \text{if} \;\; \gamma>1\,, \end{cases}$$

$$\widetilde{A}(t) = \begin{cases} \gamma\, t^{-\gamma} & \text{if} \;\; 0<\gamma<1 \\ \dfrac{3}{2\,t} & \text{if} \;\; \gamma=1 \\ \dfrac{\gamma}{2\,t} & \text{if} \;\; \gamma>1 \end{cases} = \overline{A}(t)\,\big(1+o(1)\big)\,,$$

now in agreement with the results in Corollary 3.1.

Example 4.4. The most common heavy-tailed models with $\widetilde{\rho} = -\gamma$ and $0 < \gamma < -\rho$ (then necessarily with $l \neq 0$), are such that

$$U(t) \;=\; C\,t^\gamma \Big(1 + A\,t^{-\gamma} + B\,t^{-2\gamma} + o\big(t^{-2\gamma}\big)\Big)\;, \qquad A, B \neq 0, \quad C > 0\;.$$

For these models,

$$U(tx) - U(t) \;=\; C\,\gamma\,t^\gamma \left(\frac{x^\gamma - 1}{\gamma} - B\,t^{-2\gamma}\left(\frac{x^{-\gamma}-1}{-\gamma}\right) + o\big(t^{-2\gamma}\big)\right)\;,$$

and

$$\frac{\frac{U(tx)-U(t)}{C\,\gamma\,t^\gamma} - \frac{x^\gamma - 1}{\gamma}}{-B\,t^{-2\gamma}} \;\underset{t\to\infty}{\longrightarrow}\; \frac{x^{-\gamma}-1}{-\gamma}\;,$$

i.e., $\rho + \gamma = -\gamma$, or equivalently, $\rho = -2\,\gamma$. Then, (2.2) holds, provided that we choose

$$a(t) \;=\; \frac{C\,\gamma\,t^\gamma}{1 + B\,t^{-2\gamma}}\;, \qquad A(t) \;=\; 2\,B\,\gamma\,t^{-2\gamma}$$

and

$$\frac{a(t)}{U(t)} \;=\; \gamma\Big(1 - A\,t^{-\gamma} - \big(2\,B - A^2\big)\;t^{-2\gamma} + o\big(t^{-2\gamma}\big)\Big)\;.$$

Consequently, with $\overline{A}(t)$, l and c provided in (2.5), (2.6) and (2.7), respectively,

$$\overline{A}(t) \;=\; -A\,\gamma\,t^{-\gamma}\,\big(1 + O(t^{-\gamma})\big)\;, \qquad \frac{\overline{A}(t)}{A(t)} \;=\; -\frac{A}{2\,B\,t^{-\gamma}}\,\big(1 + O(t^{-\gamma})\big) \underset{t\to\infty}{\longrightarrow} \pm\infty\;,$$

i.e., $c = \pm\infty$ and

$$U(t) - \frac{a(t)}{\gamma} \;=\; C\,t^\gamma\Big(A\,t^{-\gamma} + 2\,B\,t^{-2\gamma} + o\big(t^{-2\gamma}\big)\Big) \underset{t\to\infty}{\longrightarrow} AC \neq 0\;,$$

i.e., $l = AC \neq 0$, as mentioned at the very beginning of this example. Indeed, we could also have written

$$U(t) \;=\; l + C\,t^\gamma\Big(1 + B\,t^{-2\gamma} + o\big(t^{-2\gamma}\big)\Big), \qquad \text{as} \;\; t \to \infty\;.$$

5. THE SECOND ORDER CONDITION FOR A GENERAL TAIL, HEAVY TAIL AND A THIRD ORDER FRAMEWORK

Note that for heavy-tailed models, the second order condition (2.2) implies a third order behaviour of the function $\ln U(t)$, whenever we are in the region $0 < \gamma \le -\rho$, and $l \neq 0$, a region where $A(t) = o\big(\overline{A}(t)\big)$. Also, since $\big|\overline{A}\big| \in RV_{-\gamma}$, $|A| \in RV_\rho$ and $\overline{A}^2 \in RV_{-2\gamma}$, then A dominates $\overline{A}^2$ if $\rho > -2\,\gamma$, but $\overline{A}^2$ dominates A if $\rho < -2\,\gamma$. From the Proof of Theorem 3.1, Case (i), the third order behaviour of $\ln U(t)$ may be written as

$$\lim_{t\to\infty} \frac{\frac{\ln U(tx) - \ln U(t) - \gamma\,\ln x}{\overline{A}(t)} - \frac{x^{-\gamma}-1}{-\gamma}}{\widetilde{B}(t)} \;=\; H_{\widetilde{\rho},\widetilde{\eta}}(x)\;,$$

where H is defined in (2.2),

$$\widetilde{B}(t) := \begin{cases} -\overline{A}(t) & \text{if } \ 0<\gamma<-\dfrac{\rho}{2} \\ \gamma\,\dfrac{A(t)}{\overline{A}(t)} & \text{if } \ -\dfrac{\rho}{2}<\gamma<-\rho \end{cases}$$

and the second and third order parameters $\widetilde{\rho}$ and $\widetilde{\eta}$, respectively, are given by

$$\widetilde{\rho}=-\gamma\ , \qquad \widetilde{\eta} = \begin{cases} -\gamma & \text{if } \ 0<\gamma<-\dfrac{\rho}{2} \\ \gamma+\rho & \text{if } \ -\dfrac{\rho}{2}<\gamma<-\rho\ . \end{cases}$$

Note that in the region $-\rho/2 < \gamma < -\rho$ we get $\widetilde{\rho} \neq \widetilde{\eta}$.

Remark 5.1. For the case $\gamma = -\rho/2$, excluded from this note, everything depends on the relative behaviour of A and $\overline{A}^2$, both regularly varying functions with the same index of regular variation ρ.

Note also that the situation $\widetilde{\eta}=\widetilde{\rho}$ is the one that most often happens in practice, for standard heavy-tailed models like Fréchet, Burr, the Generalized Pareto and Student's t d.f.'s. For these d.f.'s, (2.2) holds with $\rho=-2\,\gamma$. However, if we induce a shift $l\neq 0$ in the above mentioned models, this relation between γ and ρ no longer exists and we may cover all region $0<\gamma<-\rho$.

Finally, we mention that for the extreme value model with d.f. G_γ, we get

$$\rho=-1\ , \qquad \widetilde{\rho}=-\gamma \qquad \text{and} \qquad \widetilde{\eta} = \begin{cases} -\gamma & \text{if } \ 0\leq\gamma\leq 1/2 \\ \gamma-1 & \text{if } \ 1/2<\gamma<1\ . \end{cases}$$

For more details on the way the third order framework may be used in Statistics of Extremes, see, for heavy tails, Gomes, de Haan and Peng (2002) and Fraga Alves, Gomes and de Haan (2003a), papers dealing with the estimation of the second order parameter ρ, and Gomes, Caeiro and Figueiredo (2004), a paper dealing with reduced bias extreme value index estimation. For details on the general third order framework, see Fraga Alves, de Haan and Lin (2003b, Appendix; 2006).

As a final remark, we would like to emphasise the importance of all these conditions in Statistics of Extremes. The first order conditions in (2.1), (2.8) and (2.9), together with additional light conditions on k, the number of top order statistics used in the estimation of a first order parameter, enable us to guarantee consistency of semi-parametric estimators of such a parameter. The primary first order parameter is the extreme value index γ, but we can refer other relevant parameters of extreme events, like high quantiles, return periods or probabilities of exceedances of high levels, among others. To obtain a Central Limit Theorem for these estimators, or consistency of any estimator of a second order parameter, like the shape second order parameters ρ or $\widetilde{\rho}$, discussed in this paper, it is convenient to assume a second order condition, like the ones in (2.2) and (2.10).

For the derivation of an asymptotic non-degenerate behaviour of estimators of second order parameters, we further need to assume a third order condition, ruling the rate of convergence in (2.2) or in (2.10). Such a type of condition is also quite useful for the study of any second-order reduced-bias estimators, particularly if we want to have information on the bias of such estimators. For details on this type of extreme value index estimators and the importance of third order conditions see, for instance, the most recent papers on the subject (Caeiro, Gomes and Pestana, 2005; Gomes, de Haan and Henriques Rodrigues, 2007b; Gomes, Martins and Neves, 2007c). In these papers, the adequate external estimation of second order parameters leads to reduced-bias estimators with the same asymptotic variance as the (biased) classical estimator for heavy tails, the Hill estimator (Hill, 1975). For overviews on second-order reduced-bias estimation see Reiss and Thomas (Chapter 6) and Gomes, Canto e Castro, Fraga Alves and Pestana (2007a).

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