REVSTAT – Statistical Journal
Volume 5, Number 2, June 2007, 137–161

ON THE EXCESS DISTRIBUTION OF SUMS OF RANDOM VARIABLES IN BIVARIATE EV MODELS

Authors: Michael Falk
– Institute of Mathematics, University of Würzburg, Germany
falk@mathematik.uni-wuerzburg.de

Received: July 2006 Accepted: January 2007

Abstract:

- Let (U,V) be a random vector following a bivariate extreme value distribution (EVD) with reverse exponential margins. It is known that the excess distribution $F_c(t) = P\big(U+V > c\,t \mid U+V > c\big)$ of $U+V$ converges to $F(t)=t^2$ as the threshold c increases if U,V are independent, and to $F(t)=t$, $t\in[0,1]$, elsewhere. We investigate the limit of the excess distribution of $aU+bV$ in case of an EVD with arbitrary margins and with arbitrary scale parameters $a,b>0$. It turns out that the limiting excess df may have a different behavior. For Fréchet margins, independence of U,V does not affect the limit excess distribution, whereas for Gumbel and reverse Weibull margins it does. Unless for Gumbel margins, the limit excess distribution is independent of a,b.
Interpreting a,b as weights and U,V as risks, $aU+bV$ can be viewed as a (short) linear portfolio. The fact that the limiting excess distribution of $aU+bV$ does not depend on a,b, unless for Gumbel margins, implies that risk measures such as the expected shortfall $E\big(aU+bV \mid aU+bV < c\big)$ might fail for multivariate extreme value models.

Key-Words:

- *univariate extreme value distribution; multivariate extreme value distribution; sums of random variables; excess distribution; Pickands dependence function; linear portfolio; risk measure; expected shortfall.*

AMS Subject Classification:

- 60G70.

1. INTRODUCTION

Let (X, Y) be a random vector (rv), whose distribution function (df) is a bivariate extreme value df (EVD) G with reverse exponential margins, i.e., G is max-stable

$$G^n\Big(\frac{x}{n}, \frac{y}{n}\Big) = G(x, y)\ , \qquad x, y \le 0, \quad n \in \mathbb{N}\ ,$$

and satisfies

$$G(x, 0) = G(0, x) = P(X \le x) = P(Y \le x) = \exp(x)\ , \qquad x \le 0\ .$$

It is well-known that G can be represented as

$$G(x, y) = \exp\bigg((x + y)\, D\Big(\frac{x}{x + y}\Big)\bigg)\ , \qquad x, y \le 0\ , \tag{1.1}$$

where $D\colon [0, 1] \to [1/2, 1]$ is a *Pickands dependence function*; see, for example, Sections 4.3, 6.1, 6.2 in Falk *et al.* (2004). A Pickands dependence function is characterized by the two properties

$$D \text{ is convex}\ , \tag{1.2}$$

$$\max(z, 1 - z) \le D(z) \le 1\ , \qquad z \in [0, 1]\ , \tag{1.3}$$

i.e., $G(x, y) = \exp\big((x + y)\, D(x/(x + y))\big)$, $x, y \le 0$, defines an EVD G with reverse exponential margins if, and only if the function $D\colon [0, 1] \to [1/2, 1]$ satisfies condition (1.2) and (1.3) (see Falk (2006)).

A popular example is, with $\lambda \in [1, \infty]$,

$$D(z) = \Big(z^\lambda + (1 - z)^\lambda\Big)^{1/\lambda}\ , \qquad z \in [0, 1]\ ,$$

which yields the Gumbel type B df $G(x, y) = \exp\big(-\big(|x|^\lambda + |y|^\lambda\big)^{1/\lambda}\big)$, $x, y \le 0$, with the convention $D(z) = \max(z, 1 - z)$ if $\lambda = \infty$.

Note that the case of independence of X, Y is in general characterized by the constant dependence function $D = 1$, in which case $G(x, y) = \exp(x + y)$, $x, y \le 0$. A major problem in the statistical analysis of given data $(x_1, y_1), ...,$ (x_n, y_n), is the decision whether the data were generated by rvs (X_i, Y_i) with independent margins X_i, Y_i, see, for example, Dupuis and Tawn (2001).

It was observed in Falk and Michel (2006) that the sum $X + Y$ over a high threshold has excellent ability to discriminate between independence and dependence, i.e., between the case of the constant dependence function $D = 1$

and a nonconstant D. Precisely, it was observed in Falk and Michel (2006) that for $t \in [0,1]$

$$P\Big(X+Y > ct \mid X+Y > c\Big) \underset{c\uparrow 0}{\longrightarrow} \begin{cases} t^2 & \text{if } D=1\,, \\ t & \text{elsewhere}\,. \end{cases} \tag{1.4}$$

The excess distribution of the sum $X+Y$ over a high threshold approaches, consequently, either the df $F(t) = t^2$, $t \in [0,1]$, in case of independence of X, Y, or, elsewhere, the uniform distribution on $[0,1]$.

This observation was used in Falk and Michel (2006) to define a test for independence of X, Y, which is derived from the Neyman–Pearson test for the binary testing problem $F(t) = t^2$ against $F(t) = t$, $t \in [0,1]$, based on n independent copies of (X, Y). It was shown that this test has excellent performance and is able to detect deviations from the constant dependence function $D = 1$ which are of order $O(n^{-1/2})$.

The problem suggests itself, whether the characterization of independence and dependence of X, Y via the limiting excess distribution in (1.4) remains valid, if the rv (X, Y) with EVD G with reverse exponential margins is replaced by a rv (U, V), which follows an *arbitrary* EVD. This will be investigated in the present paper, where our investigations include arbitrary scale parameters $a, b > 0$ as well, i.e., we consider the excess distribution of $aU + bV$ over a high threshold with underlying arbitrary EVD. It turns out that the limit df of the excess distribution of the sum depends heavily on the marginal dfs: In some cases independence of U and V affects the limit, in other cases it does not. The main results can be summarized as follows, where it is generally assumed that the joint df of (U, V) is a bivariate EVD.

Reverse Weibull Margins: Suppose that U, V both follow a reverse Weibull df: $P(U \le x) = \exp\big(-(-x)^{\alpha_1}\big)$, $P(V \le x) = \exp\big(-(-x)^{\alpha_2}\big)$, $x \le 0$, $\alpha_1, \alpha_2 > 0$. Then we obtain for $a, b > 0$ and $t \in [0,1]$ (see Theorem 3.1)

$$P\Big(aU + bV > tc \mid aU + bV > c\Big) \underset{c\uparrow 0}{\longrightarrow} \tag{1.5}$$

$$\underset{c\uparrow 0}{\longrightarrow} \begin{cases} t^{\alpha_1+\alpha_2} & \text{if } U, V \text{ are independent}\,, \\ t^{\max(\alpha_1,\alpha_2)} & \text{elsewhere}\,. \end{cases}$$

The special case $\alpha_1 = \alpha_2 = a = b = 1$ was established in Falk and Michel (2006). The limit excess df of $aU + bV$ is, therefore, determined by independence or dependence of U, V, but it is not affected by the scale parameters $a, b > 0$.

Fréchet Margins: Suppose that U, V both follow a Fréchet df: $P(U \le x) = \exp(-x^{-\alpha_1})$, $P(V \le x) = \exp(-x^{-\alpha_2})$, $x > 0$, $\alpha_1, \alpha_2 > 0$. Then we have for $a, b > 0$ and $t \ge 1$

$$P\Big(aU + bV > tc \mid aU + bV > c\Big) \underset{c\to\infty}{\longrightarrow} t^{-\min(\alpha_1,\alpha_2)}\,. \tag{1.6}$$

In the case $\alpha_1 = \alpha_2$ and dependence of U, V, the preceding result requires an additional weak condition on the underlying Pickands dependence function, see Theorem 3.3 and 3.2 for details.

In case of Fréchet margins, the limiting excess df of $aU + bV$ is, consequently, invariant under dependence and independence of U, V and it is not affected by the choice of the scale parameters $a, b > 0$.

Gumbel Margins: If U, V both follow the Gumbel df $F(x) = \exp(-e^{-x})$, $x \in \mathbb{R}$, then we obtain for $a, b > 0$ and $t > 0$

$$(1.7) \qquad P\Big(aU + bV > c + t \mid aU + bV > c\Big) \underset{c\to\infty}{\longrightarrow}$$
$$\underset{c\to\infty}{\longrightarrow} \begin{cases} \exp\Big(-\dfrac{t}{\max(a,b)}\Big) & \text{if } U, V \text{ are independent}\,, \\ \exp\Big(-\dfrac{t}{a+b}\Big) & \text{elsewhere}\,, \end{cases}$$

see Theorem 3.4. In case of Gumbel margins, dependence and independence of U, V determine, consequently, the limiting excess df of $aU + bV$. But different to the other two cases above, it depends on the scale factors $a, b > 0$ as well.

The cases of mixed margins is determined by that df among the two dfs involved, which has a heavier tail, see Theorem 3.5, 3.6 and 3.7. Note that additional location parameters of U and V can simply be incorporated in the preceding results by shifting them to the threshold.

The transformation of the univariate margins of a multivariate EVD to arbitrary univariate extreme value distributions yields again a multivariate EVD. A common approach in multivariate extreme value theory is, therefore, the transformation of a given EVD to an EVD with one's favorite univariate margins. This approach might, however, be misleading as the preceding results reveal that the marginal distributions of a multivariate EVD, actually, can matter.

Extreme value theory has become a standard toolkit within quantitative finance useful for describing non normal phenomena, see, e.g., Embrechts (2000, 2004), Klüppelberg (2004), Section 13 in Reiss and Thomas (2001). The above results now reveal surprising facts in particular about the expected shortfall, which is a popular risk measure of a linear portfolio. Interpreting a, b as *weights* and U, V as *risks*, the sum $aU + bV$ can be viewed as a (short) *linear portfolio*. Note that the limit excess df of $aU + bV$ *above* a high threshold can in case of reverse Weibull margins readily be turned into the limiting excess df of a linear portfolio *below* a small threshold approaching zero: A rv (U, V) follows a bivariate max-stable df with reverse (standard) Weibull margins if, and only if, the rv $(\tilde{U}, \tilde{V}) := (-U, -V)$ follows a bivariate min-stable df with Weibull margins.

The standard exponential df on $(0, \infty)$ is a particular example. The limit result (1.5) now becomes with arbitrary $a, b > 0$ and $t \in [0, 1]$

$$P\Big(a\tilde{U} + b\tilde{V} < tc \mid a\tilde{U} + b\tilde{V} < c\Big) \underset{c\downarrow 0}{\longrightarrow}$$
$$\underset{c\downarrow 0}{\longrightarrow} \begin{cases} t^{\alpha_1+\alpha_2} & \text{if } U, V \text{ are independent}\,, \\ t^{\max(\alpha_1,\alpha_2)} & \text{elsewhere}\,. \end{cases}$$

We see that in various cases, as the threshold increases or decreases, the limit excess distribution of $aU + bV$ does not depend on the parameters $a, b > 0$. A risk measure of a portfolio such as the *expected shortfall* (Acerbi and Sirtori (2001), Acerbi and Tasche (2001), Acerbi and Tasche (2002)), i.e., the expectation of $aU + bV$ given that the sum exceeds a high or a low threshold, is in this case asymptotically independent of the weights a, b. Such a risk measure of a linear portfolio has, consequently, to be taken with care, if the underlying joint df of the risks is assumed to be a max-stable or a min-stable df. For a linear portfolio $\sum_{i\le d} a_i U_i$ of arbitrary length d this was already observed in Macke (2005) in the case where $(U_1, ..., U_d)$ follows a d-dimensional EVD G with reverse exponential margins.

We remark that corresponding results might be established in higher dimensions as well, see, for a special case, Macke (2005). But the case of a dimension higher than two requires additional conditions such as very smooth dependence functions; it does not, however, provide essential new insight into the limit behavior of the corresponding excess distributions. In the two-dimensional case our mathematical tools are, on the other hand, so refined that we can establish our results under most general conditions. That is why we restrict ourselves in this paper to sums $aU + bV$ of length two.

It would, of course, be desirable to extend the preceding results (1.5)–(1.7) to rvs (U, V), whose distribution lies *in the domain of attraction* of a multivariate EVD. But this is not possible without further assumption. Take, for example, a rv (U, V), which follows a bivariate normal distribution $N(\mathbf{0}, \mathbf{\Sigma})$ with mean vector $\mathbf{0}$, variances 1 and and covariance $\rho \in (-1, 1)$. Then $N(\mathbf{0}, \mathbf{\Sigma})$ is in the domain of attraction of the EVD $G(x, y) = \exp(-e^{-x} - e^{-y})$, $x, y \in \mathbb{R}$, with independent Gumbel margins, i.e., there exist constants $a_n, c_n > 0$, $b_n, d_n \in \mathbb{R}$ such that

$$P\Big(\max_{1\le i\le n} U_i \le b_n + a_n x,\ \max_{1\le i\le n} V_i \le d_n + c_n y\Big) \underset{n\to\infty}{\longrightarrow} G(x, y)\,, \qquad x, y \in \mathbb{R}\,,$$

where $(U_1, V_1), (U_2, V_2), ...$ are independent copies of (U, V), see, e.g., equation (9.7) in Reiss and Thomas (2001). According to equation (1.7) one, therefore, should expect in this case that the limit of $P\big(aU + bV > c + t \mid aU + bV > c\big)$ is $\exp(-t/\max(a, b))$ as c converges to infinity. Standard arguments, however, yield that

$$P\Big(aU + bV > c + t \mid aU + bV > c\Big) \underset{c\to\infty}{\longrightarrow} 0$$

for arbitrary $a, b, t > 0$.

The results in this paper are related to results by Wüthrich (2003), Alink *et al.* (2004, 2005a, 2005b) and Barbe *et al.* (2006), who establish $P\big(\sum_{i\le d} X_i > t\big) \sim \Delta P(X_1 > t)$ as $t \to \infty$ with some diversification constant $\Delta > 0$. This is achieved under various conditions on the joint distribution of $(X_1, ..., X_d)$, thus extending the well known result with $\Delta = d$ in case of iid regularly varying X_i (Feller (1971, p. 279)) to dependent rvs. The above authors work, however, with identically distributed X_i so that the results stated here are not included in these papers.

This paper is organized as follows. As the derivation of our results is highly technical, we compile in Section 2 in a preparatory step various auxiliary results and tools. The main results are established in Section 3.

2. AUXILIARY RESULTS AND TOOLS

In a preparatory step we provide in this section several auxiliary results and mathematical tools, which might be of interest of their own.

A bivariate and nondegenerate EVD H has the characteristic property of max-stability, i.e., for each $n \in \mathbb{N}$ there are constants $a_{in} > 0$, $b_{i,n} \in \mathbb{R}$, $i = 1, 2$, such that

$$H^n\big(a_{1n}x + b_{1n},\, a_{2n}y + b_{2n}\big) \;=\; H(x,y)\;, \qquad x, y \in \mathbb{R}\;.$$

The margins of H are, consequently, univariate EVDs. The family of nondegenerate univariate EVDs is, with $\alpha > 0$, up to a scale and location shift given by

$$\begin{aligned} F_\alpha(x) \;&:=\; \begin{cases} \exp\big(-(-x)^\alpha\big), & x \le 0\,,\\ 1, & x > 0\;, \end{cases} \\ F_{-\alpha}(x) \;&:=\; \begin{cases} 0, & x \le 0\,,\\ \exp(-x^{-\alpha}), & x > 0\;, \end{cases} \qquad (2.1) \\ F_0(x) \;&:=\; \exp(-e^{-x})\,, \qquad x \in \mathbb{R}\;, \end{aligned}$$

being the family of (reverse) Weibull, Fréchet dfs and the Gumbel df; see, e.g., Section 2.2 in Falk *et al.* (2004). Note that F_1 is the standard reverse exponential df.

Let now (U, V) be a rv, which follows a bivariate EVD H with standard univariate extreme value margins as in (2.1). It is well-known that the df H of (U, V) equals that of $\big(H_1^{-1}(\exp(X)), H_2^{-1}(\exp(Y))\big)$, where (X, Y) follows an EVD G with reverse exponential margins F_1. By $F^{-1}(q) := \inf\{t \in \mathbb{R} \colon F(t) \ge q\}$, $q \in (0, 1)$, we denote the generalized inverse of a univariate df F; see, for example, Lemma 5.4.7 in Falk *et al.* (2004). In different notation we have, thus,

$$H(x,y) \;=\; G\Big(\log\big(H_1(x)\big),\log\big(H_2(y)\big)\Big) \;=\; G\big(\psi_1(x),\psi_2(y)\big)\;,$$

where $\psi_i(x)=\log(H_i(x))$, $i=1,2$, is each one of the three functions defined as follows:

$$\psi(x) := \begin{cases} -(-x)^\alpha, & x\le 0\,,\\ -x^{-\alpha}, & x>0\,,\\ -e^{-x}, & x\in\mathbb{R}\,. \end{cases}$$

We have, consequently,

$$(U,V) =_D \big(\psi_1^{-1}(X),\psi_2^{-1}(Y)\big)\;, \tag{2.2}$$

where $=_D$ denotes equality in distribution, and, we have by equation (1.1) for x,y with $0<H_1(x),H_2(x)<1$

$$H(x,y) \;=\; \exp\left(\Big(\psi_1(x)+\psi_2(y)\Big)\,D\bigg(\frac{\psi_1(x)}{\psi_1(x)+\psi_2(y)}\bigg)\right), \tag{2.3}$$

where D is a Pickands dependence function as defined by (1.2) and (1.3).

Note that $\big(\psi_1^{-1}(X),\psi_2^{-1}(Y)\big)$ follows for an arbitrary choice of an EVD G with reverse exponential margins an EVD H with margins H_1,H_2 and, thus, representation (2.3) characterizes up to a scale and location parameter the complete class of bivariate EVDs with arbitrary margins.

The following auxiliary result provides a representation of an arbitrary Pickands dependence function D, which will be crucial for the derivation of our subsequent results. It implies in particular that any D is absolutely continuous and provides its derivative D'. For a proof of this result we refer to Lemma 6.2.1 in Falk *et al.* (2004).

Lemma 2.1. *An arbitrary Pickands dependence function D can be represented as*

$$D(z) \;=\; 1+\int_0^z M(x)-1\,dx \;=\; 1-\int_z^1 M(x)-1\,dx\;,$$

where $M\colon[0,1]\to[0,2]$ is a measure generating function with $M(1)=2$, $\int_0^1 M(x)\,dx=1$. The dependence function D is, consequently, absolutely continuous with derivative

$$D'(z) \;:=\; M(z)-1 \;\in\; [-1,1]\;.$$

It is easy to see that the converse of the preceding result is also true: any function $D\colon[0,1]\to[0,\infty)$ that can be represented as $D(z)=1+\int_0^z M(x)-1\,dx$, with $M\colon[0,1]\to[0,2]$ as in Lemma 2.1, satisfies condition (1.2) and (1.3) and is, consequently, a Pickands dependence function.

We will make extensive use of the conditional df $P(Y \le v \mid X = u)$, where (X, Y) follows a bivariate EVD with reverse exponential margins. This conditional df is provided in the next lemma. For a proof we refer to Lemma 2.1 in Falk and Michel (2006); the arguments are taken from Ghoudi *et al.* (1998).

Lemma 2.2. *Suppose that the rv (X, Y) follows an EVD G with reverse exponential margins and Pickands dependence function D. Then we have for $u < 0$*

$$P\Big(Y \le v \mid X = u\Big) =$$
$$= \begin{cases} \exp\Big\{u\Big(D\big(\tfrac{u}{u+v}\big) - 1\Big) + v\,D\big(\tfrac{u}{u+v}\big)\Big\}\,\Big(D\big(\tfrac{u}{u+v}\big) + D'\big(\tfrac{u}{u+v}\big)\big(1 - \tfrac{u}{u+v}\big)\Big) & \text{if } v < 0\,, \\ 1 & \text{if } v \ge 0\,. \end{cases}$$

3. MAIN RESULTS

In this section we compute the limiting excess df of the sum $aU + bV$, where (U, V) follows an arbitrary bivariate EVD. Without loss of generality (wlog) we suppose that the marginal univariate dfs have scale parameter 1. We begin with the case of reverse Weibull margins.

Theorem 3.1 (Reverse Weibull Margins)**.** *Suppose that (U,V) follows a bivariate EVD with reverse Weibull margins: $P(U \le x) = \exp\big(-(-x)^{\alpha_1}\big)$, $P(V \le x) = \exp\big(-(-x)^{\alpha_2}\big)$, $x \le 0$, $\alpha_1, \alpha_2 > 0$. If U, V are not independent, then we have for $a, b > 0$ and $0 \le t \le 1$*

$$P\Big(aU + bV > tc \mid aU + bV > c\Big) \underset{c\uparrow 0}{\longrightarrow} t^{\max(\alpha_1, \alpha_2)}\,.$$

If U, V are independent, then we have

$$P\Big(aU + bV > tc \mid aU + bV > c\Big) \underset{c\uparrow 0}{\longrightarrow} t^{\alpha_1 + \alpha_2}\,.$$

Proof: Wlog we assume $\alpha_1 \ge \alpha_2$. The assertion is an immediate consequence of

$$(3.1) \qquad (-c)^{-\alpha_1}\, P\big(aU + bV > c\big) \underset{c\uparrow 0}{\longrightarrow} K(a, b) > 0$$

if U, V are not independent, and of

$$(3.2) \qquad (-c)^{-(\alpha_1+\alpha_2)}\, P\big(aU + bV > c\big) \underset{c\uparrow 0}{\longrightarrow} a^{-\alpha_1}\, b^{-\alpha_2}\, \alpha_1 \int_0^1 (1-u)^{\alpha_2}\, u^{\alpha_1 - 1}\, du$$

if U, V are independent. This will be established in the following.

Wlog we can by (2.2) assume that $(U,V) = \big(-(-X)^{1/\alpha_1}, -(-Y)^{1/\alpha_2}\big)$, where (X,Y) follows a bivariate EVD $G(x,y) = \exp\big((x+y)\,D(x/(x+y))\big)$, $x,y \le 0$, with reverse exponential margins and Pickands dependence function D.

By conditioning on $X = u$, we obtain from Lemma 2.2 the representation

$$P\big(aU + bV > c\big) =$$

$$= \int_{-\infty}^{0} P\Big(-(-Y)^{1/\alpha_2} > \frac{c + a(-u)^{1/\alpha_1}}{b} \;\mid\; X = u\Big) \exp(u)\; du$$

$$= \int_{-(\frac{-c}{a})^{\alpha_1}}^{0} \Bigg(1 - P\bigg(Y \le -\Big(-\frac{c + a(-u)^{1/\alpha_1}}{b}\Big)^{\alpha_2} \;\mid\; X = u\bigg)\Bigg) \exp(u)\; du \tag{3.3}$$

$$= 1 - \exp\Big(-\Big(\frac{-c}{a}\Big)^{\alpha_1}\Big) \tag{3.4}$$

$$- \int_{-(\frac{-c}{a})^{\alpha_1}}^{0} \exp\Bigg(u\big(D(\tilde u) - 1\big) - \Big(-\frac{c + a(-u)^{1/\alpha_1}}{b}\Big)^{\alpha_2} D(\tilde u)\Bigg)$$

$$\times \Big(D(\tilde u) + D'(\tilde u)\,(1 - \tilde u)\Big) \exp(u)\; du \;,$$

where for $u \in \big(-(-c/a)^{\alpha_1}, 0\big]$

$$\tilde u := \frac{u}{u - \Big(-\frac{c + a(-u)^{1/\alpha_1}}{b}\Big)^{\alpha_2}} \;\in\; [0,1] \;.$$

In case of independence, i.e., $D = 1$, we obtain from equation (3.3) by using Taylor expansion of exp at 0 and substituting $u \mapsto -(-cu/a)^{\alpha_1}$

$$P\big(aU + bV > c\big) =$$

$$= \int_{-(\frac{-c}{a})^{\alpha_1}}^{0} \Bigg(1 - \exp\bigg(-\Big(-\frac{c + a(-u)^{1/\alpha_1}}{b}\Big)^{\alpha_2}\bigg)\Bigg) \exp(u)\; du$$

$$= -\Big(\frac{-c}{a}\Big)^{\alpha_1} \int_0^1 \bigg(1 - \exp\Big(-\Big(\frac{-c}{b}\Big)^{\alpha_2} (1-u)^{\alpha_2}\Big)\bigg)$$

$$\times \exp\Big(-\Big(\frac{-c}{a}\Big)^{\alpha_1} u^{\alpha_1}\Big)\, \alpha_1\, u^{\alpha_1 - 1}\; du$$

$$= \frac{(-c)^{\alpha_1+\alpha_2}}{a^{\alpha_1}\, b^{\alpha_2}}\; \alpha_1 \int_0^1 (1-u)^{\alpha_2}\, u^{\alpha_1-1} \big(1 + o(1)\big)\; du \;,$$

which implies equation (3.2).

It remains to establish equation (3.1). From equation (3.4) we obtain with the substitution $u \mapsto -(-cu/a)^{\alpha_1}$

$$
\begin{aligned}
&P\big(aU+bV>c\big) \;= \\
&\quad = 1-\exp\Big(-\Big(\frac{-c}{a}\Big)^{\alpha_1}\Big) \\
&\quad -\Big(\frac{-c}{a}\Big)^{\alpha_1}\int_0^1 \exp\Big(-\Big(\frac{-c}{a}\Big)^{\alpha_1} u_1^{\alpha}\big(D(u_c)-1\big)-\Big(\frac{-c}{b}\Big)^{\alpha_2}(1-u)^{\alpha_2}\,D(u_c)\Big) \\
&\qquad\qquad \times\Big(D(u_c)+D'(u_c)\,(1-u_c)\Big)\,\exp\Big(-\Big(\frac{-c}{a}\Big)^{\alpha_1}u^{\alpha_1}\Big)\,\alpha_1\,u^{\alpha_1-1}\,du\;,
\end{aligned}
\tag{3.5}
$$

where for $u \in (0,1)$

$$
u_c \;:=\; \frac{u^{\alpha_1}}{u^{\alpha_1}+(-c)^{\alpha_2-\alpha_1}\,\frac{a^{\alpha_1}}{b^{\alpha_2}}\,(1-u)^{\alpha_2}} \;\underset{c\uparrow 0}{\downarrow}\; 0 \qquad \text{if } \alpha_1>\alpha_2\;.
$$

Hence we obtain in the case $\alpha_1 > \alpha_2$

$$
\Big(\frac{-c}{a}\Big)^{-\alpha_1} P\big(aU+bV>c\big) \;\underset{c\uparrow 0}{\longrightarrow}\; -D'(0) \;=\; 1-M(0) \;>\; 0\;.
$$

The fact that $M(0)<1$ can be seen as follows: Suppose that $M(0)\ge 1$. Then we obtain from Lemma 2.1 that $D(z)=1+\int_0^z M(x)-1\,dx \ge 1$, $0\le z\le 1$, and, thus, D is the constant function 1. But this case was excluded. Thus we have established equation (3.1) in the case $\alpha_1>\alpha_2$. It remains to prove (3.1) also in the case $\alpha_1=\alpha_2$.

Suppose that $\alpha_1=\alpha_2$. Equation (3.5) implies

$$
\begin{aligned}
&\Big(\frac{-c}{a}\Big)^{-\alpha_1} P\big(aU+bV>c\big) \;\underset{c\uparrow 0}{\longrightarrow} \\
&\qquad \underset{c\uparrow 0}{\longrightarrow}\; \int_0^1 \Big(1-D(u^*)-D'(u^*)\,(1-u^*)\Big)\,\alpha_1\,u^{\alpha_1-1}\,du \;>\; 0
\end{aligned}
\tag{3.6}
$$

where for $u\in[0,1]$

$$
u^* \;:=\; \frac{u^{\alpha_1}}{u^{\alpha_1}+(1-u)^{\alpha_1}\,(\frac{a}{b})^{\alpha_1}} \;\in\; [0,1]\;.
$$

We show in the following that the limit integral in (3.6) is strictly positive. Note that we have by Lemma 2.1 for $u\in[0,1]$

$$
1-D(u)-D'(u)\,(1-u) \;=\; \int_u^1 M(x)-M(u)\;dx \;\ge\; 0\;,
$$

where the integral on the right hand side above is a function in u, which is continuous from the right. Suppose that the integral in equation (3.6) is zero. This implies $\int_u^1 M(x)-M(u)\,dx=0$ for $u\in[0,1)$. Then we have in particular $\int_0^1 M(x)-M(0)\,dx=0$, which implies $M(x)=M(0)$, $x\in[0,1)$, and, thus, $D(z)= 1+\int_0^z M(0)-1\,dx = 1+z\big(M(0)-1\big)$, $z\in[0,1]$. From the fact that $D(1)=1$ we obtain $M(0)=1$ and, hence, that $D(z)=1$, $z\in[0,1]$. But this case was excluded. The limit integral in (3.6) is, therefore, strictly positive. This completes the proof of equation (3.1) and, thus, the proof of Theorem 3.1. □

The case of Fréchet margins requires completely different proofs for identical and nonidentical margins. The two cases are, therefore, stated separately in Theorem 3.3 and in Theorem 3.2. We begin with the case of different margins, since this case is an immediate consequence of the following result for regularly varying rvs. For a proof of this result we refer to Lemma 2 in Klüppelberg *et al.* (2006) [17].

Lemma 3.1. *Let Y and Z be rvs on a common probability space such that Y has regularly varying right tail with index $-\kappa < 0$. Let $d > \kappa$ and suppose that $E(|Z|^d) < \infty$. Then*

$$\lim_{x\to\infty} \frac{P\big(Y + Z > x\big)}{P\big(Y > x\big)} = 1 \ .$$

Theorem 3.2 (Different Fréchet Margins)**.** *Suppose that (U,V) follows a bivariate EVD with different standard Fréchet margins: $P(U \le x) = \exp(-x^{-\alpha_1})$, $P(V \le x) = \exp(-x^{-\alpha_2})$, $x > 0$, $\alpha_1 \ne \alpha_2$. Then we have for $a,b > 0$ and $t \ge 1$*

$$P\Big(aU + bV > ct \mid aU + bV > c\Big) \underset{c\to\infty}{\longrightarrow} t^{-\min(\alpha_1,\alpha_2)} \ .$$

Note that the case of identical Fréchet margins $\alpha_1 = \alpha_2 =: \alpha$ is not covered by Lemma 3.1, as in this case $E(|U|^d) = E(|V|^d) = \infty$ for any $d > \alpha$.

Theorem 3.3 (Identical Fréchet Margins)**.** *Suppose that (U,V) follows a bivariate EVD with identical Fréchet margins: $P(U \le x) = P(V \le x) = \exp(-x^{-\alpha})$, $x > 0$, for some $\alpha > 0$. Then we obtain for $a,b > 0$ and $t \ge 1$*

$$P\Big(aU + bV > ct \mid aU + bV > c\Big) \underset{c\to\infty}{\longrightarrow} t^{-\alpha} \tag{3.7}$$

if U,V are independent. If U,V are not independent, this result remains true if we require in addition that the underlying dependence function D satisfies for some $\delta > 1$ the expansion

$$1 - D(z) - D'(z)\,(1 - z) = O\big((1 - z)^\delta\big) \ . \tag{3.8}$$

Condition (3.8) is, for example, satisfied by the dependence function $D(z) = \big(z^\lambda + (1 - z)^\lambda\big)^{1/\lambda}$, $1 \le \lambda \le \infty$, which corresponds to the Gumbel type B EVD. It is also obviously satisfied by the dependence function $D(z) = 1 - \lambda \min(z, 1 - z)$, $\lambda \in [0,1]$, which corresponds to the Marshall–Olkin EVD. We conjecture that it is satisfied by an arbitrary dependence function, but this is an open question.

Proof: Wlog we can assume $(U,V) = \big((-X)^{-1/\alpha}, (-Y)^{-1/\alpha}\big)$, where (X,Y) follows a bivariate EVD G with reverse exponential margins and dependence function D.

First we consider the case $D(z)=1$, $z\in[0,1]$, i.e., the case of independence of X,Y or, equivalently, of U,V. We claim that in this case

$$c^\alpha P\big(aU+bV>c\big) \underset{c\to\infty}{\longrightarrow} a^\alpha+b^\alpha\ , \tag{3.9}$$

from which equation (3.7) follows immediately. Equation (3.9) can be seen as follows. Note that $P\big(aU+bV>c \mid X=u\big)=1$ if $u>-(a/c)^\alpha$ and, thus,

$$\begin{aligned}
&P\big(aU+bV>c\big) \;=\\
&= \int_{-\infty}^0 P\Big(aU+bV>c \mid X=u\Big)\exp(u)\;du\\
&= \int_{-\left(\frac{a}{c}\right)^\alpha}^0 \exp(u)\;du + \int_{-\infty}^{-\left(\frac{a}{c}\right)^\alpha} P\left(Y>-\left(\frac{b}{c-a(-u)^{-1/\alpha}}\right)^\alpha \mid X=u\right)\exp(u)\;du\\
&= 1-\exp\left(-\left(\frac{a}{c}\right)^\alpha\right) + \int_{-\infty}^{-\left(\frac{a}{c}\right)^\alpha}\left(1-\exp\left(-\left(\frac{b}{c-a(-u)^{-1/\alpha}}\right)^\alpha\right)\right)\exp(u)\;du\ .
\end{aligned}$$

Since $1-\exp\big(-(a/c)^\alpha\big)=(a/c)^\alpha\big(1+o(1)\big)$, it suffices to show that the integral on the right hand side above equals $(b/c)^\alpha\big(1+o(1)\big)$. Split the integral into the sum of the subintegrals $\int_{-\infty}^{-2(a/c)^\alpha}\cdots\,du+\int_{-2(a/c)^\alpha}^{-(a/c)^\alpha}\cdots\,du$. By the substitution $u\mapsto-(a/c)^\alpha u$ the second subintegral equals

$$\left(\frac{a}{c}\right)^\alpha \int_1^2\left(1-\exp\left(-\left(\frac{b}{c}\right)^\alpha\big(1-u^{-1/\alpha}\big)^{-\alpha}\right)\right)\exp\left(-\left(\frac{a}{c}\right)^\alpha u\right)\,du \;=\; o(c^{-\alpha})$$

by the dominated convergence theorem. From the Taylor expansion $\exp(-x)=1-x+\exp(-\vartheta_x\,x)\,x^2/2$ with $0<\vartheta_x<1$ and the fact that $0<\exp(-\vartheta_x\,x)<1$ for $x>0$ we obtain that the first subintegral equals

$$\int_{-\infty}^{-2\left(\frac{a}{c}\right)^\alpha}\left(\frac{b}{c}\right)^\alpha\left(1-\frac{a}{c}(-u)^{-1/\alpha}\right)^{-\alpha}\exp(u)\;du+O(c^{-2\alpha}) \;=\; \left(\frac{b}{c}\right)^\alpha\big(1+o(1)\big)\ .$$

Thus we have shown (3.9).

If D is not the constant function 1, we have

$$\begin{aligned}
&c^\alpha P\big(aU+bV>c\big) \underset{c\to\infty}{\longrightarrow}\\
&\underset{c\to\infty}{\longrightarrow}\; b\int_0^1\Big(1-D(z)-D'(z)(1-z)\Big)\,z^{1/\alpha-1}\,\frac{\Big(b\,z^{1/\alpha}+a(1-z)^{1/\alpha}\Big)^{\alpha-1}}{(1-z)^2}\,dz\\
&\quad +b^\alpha\Big(2-M(1-0)\Big)\;>\;0\ ,
\end{aligned} \tag{3.10}$$

where M is the measure generating function in the representation $D(z)=1+\int_0^z M(x)-1\,dx$ and $M(1-0):=\lim_{\varepsilon\downarrow 0}M(1-\varepsilon)$ is the limit from the left of M at 1.

This is established in the following. Repeating previous arguments we obtain

$$P\big(aU+bV>c\big) = \\ = 1-\exp\Big(-\Big(\frac{a}{c}\Big)^\alpha\Big) \\ + \int_{-\infty}^{-\left(\frac{a}{c}\right)^\alpha}\left(1-P\left(Y\le -\Big(\frac{b}{c-a(-u)^{-1/\alpha}}\Big)^\alpha \mid X=u\right)\right)\exp(u)\;du\;.$$

The integral equals, by Lemma 2.2,

$$\int_{-\infty}^{-\left(\frac{a}{c}\right)^\alpha}\left\{1-\exp\left(\left(u-\Big(\frac{b}{c-a(-u)^{-1/\alpha}}\Big)^\alpha\right)D(\tilde u)\right)\exp(-u) \\ \times\Big(D(\tilde u)+D'(\tilde u)\,(1-\tilde u)\Big)\right\}\exp(u)\;du \;=$$

$$= \int_{-\infty}^{-\left(\frac{a}{c}\right)^\alpha}\Big(1-D(\tilde u)-D'(\tilde u)\,(1-\tilde u)\Big)\exp(u)\;du \\ + \int_{-\infty}^{-\left(\frac{a}{c}\right)^\alpha}\left\{1-\exp\left(u\big(D(\tilde u)-1\big)-\Big(\frac{b}{c-a(-u)^{-1/\alpha}}\Big)^\alpha D(\tilde u)\right)\right\} \\ \times\Big(D(\tilde u)+D'(\tilde u)(1-\tilde u)\Big)\exp(u)\;du\;, \tag{3.11}$$

where for $u<-(a/c)^\alpha$

$$\tilde u := \frac{1}{1+\frac{b^\alpha}{\left(c(-u)^{1/\alpha}-a\right)^\alpha}} \;\in\; (0,1)$$

converges to 1 as $c\to\infty$. Putting for $z\in(0,1)$

$$g(z) := -\frac{1}{c^\alpha}\left(b\Big(\frac{z}{1-z}\Big)^{1/\alpha}+a\right)^\alpha .$$

and substituting $u\mapsto g(z)$, we obtain that the first integral in equation (3.11) equals

$$-\int_0^1\Big(1-D(z)-D'(z)\,(1-z)\Big)\exp\big(g(z)\big)\,g'(z)\;dz \;= \\ = \frac{b}{c^\alpha}\int_0^1\Big(1-D(z)-D'(z)\,(1-z)\Big)\exp\big(g(z)\big) \\ \times\Big(b\,z^{1/\alpha}+a(1-z)^{1/\alpha}\Big)^{\alpha-1}(1-z)^{-2}\,z^{1/\alpha-1}\;dz\;.$$

From condition (3.8) and the dominated convergence theorem we, therefore, obtain

$$c^\alpha\int_{-\infty}^{-\left(\frac{a}{c}\right)^\alpha}\Big(1-D(\tilde u)-D'(\tilde u)\,(1-\tilde u)\Big)\exp(u)\;du \underset{c\to\infty}{\longrightarrow} \\ \underset{c\to\infty}{\longrightarrow}\; b\int_0^1\Big(1-D(z)-D'(z)\,(1-z)\Big)\,z^{1/\alpha-1} \\ \times\Big(b\,z^{1/\alpha}+a(1-z)^{1/\alpha}\Big)^{\alpha-1}(1-z)^{-2}\;dz \;\in\;(0,\infty)\;. \tag{3.12}$$

The second integral in equation (3.11) is split into the sum of the subintervals

$$\int_{-\infty}^{-2\left(\frac{a}{c}\right)^\alpha} \cdots\, du + \int_{-2\left(\frac{a}{c}\right)^\alpha}^{-\left(\frac{a}{c}\right)^\alpha} \cdots\, du \;=:\; I(c) + II(c)\;.$$

Substituting $u \mapsto -(a/c)^\alpha u$ and putting for $u \in (1,2)$

$$\bar{u} := \frac{1}{1 + (\frac{b}{a})^\alpha\, (u^{1/\alpha} - 1)^{-\alpha}} \;\in\; (0,1)\;,$$

the second subintegral above equals

$$\begin{aligned} II(c) \;&=\; \left(\frac{a}{c}\right)^\alpha \int_1^2 \left\{1 - \exp\left(-\left(\frac{a}{c}\right)^\alpha u\big(D(\bar{u}) - 1\big) - \left(\frac{b}{c}\right)^\alpha \big(1 - u^{-1/\alpha}\big)^{-\alpha} D(\bar{u})\right)\right\} \\ &\qquad \times \Big(D(\bar{u}) + D'(\bar{u})\,(1 - \bar{u})\Big) \exp\left(-\left(\frac{a}{c}\right)^\alpha u\right)\, du \\ &= o\big(c^{-\alpha}\big) \end{aligned}$$

by the dominated convergence theorem.

Taylor expansion of exp at zero yields that the first subintegral equals

$$\begin{aligned} I(c) \;&=\; \int_{-\infty}^{-2\left(\frac{a}{c}\right)^\alpha} \left(\big(1 - D(\tilde{u})\big)\, u + \left(\frac{b}{c - a(-u)^{-1/\alpha}}\right)^\alpha D(\tilde{u})\right) \\ &\qquad \times \exp\left\{\vartheta_u\, u\big(D(\tilde{u}) - 1\big) - \vartheta_u \left(\frac{b}{c - a(-u)^{-1/\alpha}}\right)^\alpha D(\tilde{u})\right\} \\ &\qquad \times \Big(D(\tilde{u}) + D'(\tilde{u})\,(1 - \tilde{u})\Big) \exp(u)\; du\;, \end{aligned}$$

where $0 < \vartheta_u < 1$. Recall that $1 - D(\tilde{u}) \in [0, 1/2]$, $D'(\tilde{u})(1 - \tilde{u}) \in [-1, 1]$ and note that for $u \le -2(a/c)^\alpha$

$$\left(\frac{b}{c - a(-u)^{-1/\alpha}}\right)^\alpha \;=\; \left(\frac{b}{c}\right)^\alpha \left(\frac{1}{1 - \frac{1}{\left(-\left(\frac{c}{a}\right)^\alpha u\right)^{1/\alpha}}}\right)^\alpha \;\le\; \frac{1}{(1 - 2^{-1/\alpha})^\alpha} \left(\frac{b}{c}\right)^\alpha.$$

We have, further, by Lemma 2.1

$$1 - D(\tilde{u}) \;=\; \int_{\tilde{u}}^1 M(x) - 1\, dx \;=\; \big(M(1-0) - 1\big)\,(1 - \tilde{u})\,\big(1 + r(\tilde{u})\big)\;,$$

where

$$0 \;\geq\; r(\tilde u) \;:=\; \frac{\displaystyle\int_{\tilde u}^1 M(x)-1\,dx}{\bigl(M(1-0)-1\bigr)\,(1-\tilde u)} - 1$$

$$= \frac{\displaystyle\int_{\tilde u}^1 M(x)-M(1-0)\,dx}{\bigl(M(1-0)-1\bigr)\,(1-\tilde u)}$$

$$\geq \frac{M(\tilde u)-M(1-0)}{M(1-0)-1}$$

$$\geq -\frac{M(1-0)-M(0)}{M(1-0)-1}$$

is bounded and converges to 0 as $c\to\infty$. We have, further, for $u \leq -2(a/c)^\alpha$

$$(1-\tilde u)\,u \;=\; \frac{\frac{b^\alpha}{\bigl(c(-u)^{1/\alpha}-a\bigr)^\alpha}}{1+\frac{b^\alpha}{\bigl(c(-u)^{1/\alpha}-a\bigr)^\alpha}}\; u$$

$$= \frac{b^\alpha}{c^\alpha}\;\frac{u}{\bigl((-u)^{1/\alpha}-\frac{a}{c}\bigr)^\alpha}\;\frac{1}{1+\frac{b^\alpha}{\bigl(c(-u)^{1/\alpha}-a\bigr)^\alpha}}$$

$$= -\frac{b^\alpha}{c^\alpha}\bigl(1+s_c(u)\bigr)\;,$$

where s_c is bounded and $s_c(u) \underset{c\to\infty}{\longrightarrow} 0$.

We obtain, consequently, from the dominated convergence theorem

$$\text{(3.13)}\qquad c^\alpha I(c) \underset{c\to\infty}{\longrightarrow} \int_{-\infty}^0 b^\alpha\bigl(2-M(1-0)\bigr)\exp(u)\,du \;=\; \bigl(2-M(1-0)\bigr)\,b^\alpha \;\geq\; 0\,.$$

Equation (3.10) now follows from (3.11), (3.12) and (3.13). □

Theorem 3.4 (Gumbel Margins)**.** *Suppose that the rv (U,V) follows a bivariate EVD with identical Gumbel margins: $P(U\leq x) = P(V\leq x) = \exp(-e^{-x})$, $x\in\mathbb{R}$. Then we obtain for $a,b>0$ and $t\geq 0$*

$$P\Bigl(aU+bV>c+t \mid aU+bV>c\Bigr) \underset{c\to\infty}{\longrightarrow}$$

$$\underset{c\to\infty}{\longrightarrow} \begin{cases} \exp\Bigl(-\dfrac{t}{\max(a,b)}\Bigr) & \textit{if } U,V \textit{ are independent}\,,\\[2ex] \exp\Bigl(-\dfrac{t}{a+b}\Bigr) & \textit{elsewhere}\,. \end{cases}$$

Proof: We consider first the case, where U, V are independent. Wlog we assume $a > b$. The case $a = b$ requires a different approach, see below. The assertion is immediate from

$$e^{\frac{c}{a}}\, P\big(aU + bV > c\big) \underset{c\to\infty}{\longrightarrow} \frac{b}{a}\int_{-\infty}^{\infty}\Big(1 - \exp(-e^{-u})\Big)\, e^{\frac{b}{a}u}\, du \;\in\; (0,\infty)\;, \tag{3.14}$$

which we establish in the sequel.

Put $F(u) := \exp(-e^{-u})$, $u \in \mathbb{R}$. We have

$$\begin{aligned} P\big(aU + bV > c\big) &= \int_{-\infty}^{\infty} P\Big(aU + bV > c \mid U = u\Big)\, F'(u)\, du \\ &= \int_{-\infty}^{\infty}\Big(1 - F\Big(\frac{c - au}{b}\Big)\Big)\, F'(u)\, du\;. \end{aligned}$$

With the substitution $u \mapsto (c - bu)/a$, the preceding integral equals

$$\begin{aligned} &\frac{b}{a}\int_{-\infty}^{\infty}\Big(1 - F(u)\Big)\, F'\Big(\frac{c - bu}{a}\Big)\, du \;= \\ &\qquad = \frac{b}{a}\int_{-\infty}^{\infty}\Big(1 - \exp(-e^{-u})\Big)\, e^{\frac{bu-c}{a}}\, \exp\big(-e^{\frac{bu-c}{a}}\big)\, du \\ &\qquad = e^{-\frac{c}{a}}\,\frac{b}{a}\int_{-\infty}^{\infty}\Big(1 - \exp(-e^{-u})\Big)\, e^{\frac{b}{a}u}\, \exp\big(-e^{\frac{bu-c}{a}}\big)\, du \end{aligned}$$

and, thus, equation (3.14) follows from the dominated convergence theorem; recall that $a > b$.

Next we consider the case, where U, V are independent and $a = b$. The assertion is a consequence of

$$\frac{e^{c/a}}{c/a}\, P\big(aU + aV > c\big) \underset{c\to\infty}{\longrightarrow} 1\;, \tag{3.15}$$

which we establish in the following. Wlog we assume $a = 1$. Repeating the arguments in the derivation of (3.14) we obtain

$$\begin{aligned} P\big(aU + bV > c\big) &= e^{-c}\Big\{\int_{-\infty}^{0}\Big(1 - \exp(-e^{-u})\Big)\, e^{u}\, \exp(-e^{u-c})\, du \\ &\qquad + \int_{0}^{\infty}\Big(1 - \exp(-e^{-u})\Big)\, e^{u}\, \exp(-e^{u-c})\, du\Big\} \\ &=:\; e^{-c}\Big\{I(c) + II(c)\Big\}\;. \end{aligned}$$

The dominated convergence theorem implies that

$$I(c) \underset{c\to\infty}{\longrightarrow} \int_{-\infty}^{0}\Big(1 - \exp(-e^{-u})\Big)\, e^{u}\, du \;\in\; (0,1)\;.$$

Taylor expansion of exp at 0 and the substitution $u \mapsto u + c$ yields

$$\begin{aligned} I\!I(c) &= \int_0^\infty \Big(e^{-u} + O(e^{-2u})\Big)\, e^u \exp(-e^{u-c})\ du \\ &= \int_0^\infty \exp(-e^{u-c})\ du\ +\ O(1) \\ &= \int_{-c}^0 \exp(-e^u)\ du\ + \int_0^\infty \exp(-e^u)\ du\ +\ O(1) \\ &= \int_{-c}^0 \exp(-e^u)\ du\ +\ O(1)\ . \end{aligned}$$

In order to establish equation (3.15) it suffices, therefore, to show that

$$c^{-1} \int_{-c}^0 \exp(-e^u)\ du \underset{c\to\infty}{\longrightarrow} 1\ .$$

But this follows from straightforward computations.

Finally we consider the case, where U, V are not independent. Wlog we assume that $a \geq b$ and that $(U,V) = \big(-\log(-X), -\log(-Y)\big)$, where (X,Y) follows a bivariate EVD with reverse exponential margins and dependence function D, which is not the constant function 1. The assertion is a consequence of the fact

$$\text{(3.16)} \qquad \exp\Big(\frac{c}{a+b}\Big)\, P\big(aU + bV > c\big) \underset{c\to\infty}{\longrightarrow}$$
$$\underset{c\to\infty}{\longrightarrow} \frac{b}{a+b} \int_0^1 \Big(1 - D(z) - D'(z)\,(1-z)\Big)\, z^{-\frac{a}{a+b}}\, (1-z)^{\frac{a}{a+b}-2}\ dz\ \in\ (0,\infty)\ ,$$

which we establish in the following.

Put for $u < 0$

$$\tilde u := \frac{1}{1 + \exp(-\frac{c}{b})\,(-u)^{-(a+b)/b}} \ \in\ (0,1)\ .$$

Then we have by Lemma 2.2

$$\begin{aligned} P\big(aU + bV > c\big) &= \int_{-\infty}^0 \left(1 - P\Big(Y \leq -\exp\Big(-\frac{c}{b}\Big)(-u)^{-a/b} \mid X = u\Big)\right) \exp(u)\ du \\ &= \int_{-\infty}^0 \Bigg(1 - \exp\Big\{u\big(D(\tilde u) - 1\big) - \exp\Big(-\frac{c}{b}\Big)\,(-u)^{-a/b} D(\tilde u)\Big\} \\ &\qquad\qquad \times \Big(D(\tilde u) + D'(\tilde u)\,(1-\tilde u)\Big)\Bigg) \exp(u)\ du \\ &= \int_{-\infty}^0 \Big(1 - D(\tilde u) - D'(\tilde u)\,(1-\tilde u)\Big) \exp(u)\ du \\ &\quad + \int_{-\infty}^0 \Bigg(1 - \exp\Big\{u\big(D(\tilde u) - 1\big) - \exp\Big(-\frac{c}{b}\Big)\,(-u)^{-a/b} D(\tilde u)\Big\}\Bigg) \\ &\qquad\qquad \times \Big(D(\tilde u) + D'(\tilde u)\,(1-\tilde u)\Big) \exp(u)\ du \\ &=:\ \tilde I(c) + \widetilde{I\!I}(c)\ . \end{aligned}$$

Put for $z \in (0,1)$

$$g(z) := -\exp\Big(-\frac{c}{a+b}\Big)\Big(\frac{z}{1-z}\Big)^{\frac{b}{a+b}} .$$

Then we have $\widetilde{g(z)} = z$ and, thus, with

$$g'(z) = -\exp\Big(-\frac{c}{a+b}\Big)\frac{b}{a+b}\, z^{-\frac{a}{a+b}}(1-z)^{\frac{a}{a+b}-2} ,$$

the substitution $u \mapsto g(z)$ yields

$$\begin{aligned}\tilde{I}(c) &= -\int_0^1 \Big(1 - D(z) - D'(z)(1-z)\Big)\exp\big(g(z)\big)\, g'(z)\, dz \\ &= \exp\Big(-\frac{c}{a+b}\Big)\frac{b}{a+b} \\ &\quad \times \int_0^1 \Big(1 - D(z) - D'(z)(1-z)\Big)\exp\big(g(z)\big)\, z^{-\frac{a}{a+b}}(1-z)^{\frac{a}{a+b}-2}\, dz .\end{aligned}$$

Note that the function g depends on the threshold c with $g(z) \underset{c\to\infty}{\longrightarrow} 0$ and that $g(z) < 0$, $z \in (0,1)$. The dominated convergence theorem implies, therefore, that

$$\begin{aligned}&\exp\Big(\frac{c}{a+b}\Big)\tilde{I}(c) \underset{c\to\infty}{\longrightarrow} \\ &\underset{c\to\infty}{\longrightarrow} \frac{b}{a+b}\int_0^1 \Big(1 - D(z) - D'(z)(1-z)\Big)\, z^{-\frac{a}{a+b}}(1-z)^{\frac{a}{a+b}-2}\, dz \in (0,\infty) .\end{aligned}$$

The integral on the right hand side is finite since $1 - D(z) \le 1 - z$ and $D'(z) \in [-1,1]$. It is positive by the arguments at the end of the proof of Theorem 3.1.

In order to establish (3.16) it suffices, therefore, to show that

$$\exp\Big(\frac{c}{a+b}\Big)\widetilde{II}(c) \underset{c\to\infty}{\longrightarrow} 0 . \tag{3.17}$$

This can be seen as follows. Choose $z_c \in (0,1)$ with $g(z_c) = -c/b$, i.e.,

$$z_c = \frac{1}{1 + \big(\frac{b}{c}\big)^{(a+b)/b}\exp\big(-\frac{c}{b}\big)} .$$

Split the integral $\widetilde{II}(c)$ into the sum of the subintervals

$$\widetilde{II}(c) = \int_{-\infty}^{g(z_c)} \cdots\, du + \int_{g(z_c)}^{0} \cdots\, du .$$

The first integral is of order $O\big(\exp(-2c/(3b))\big) = o\big(\exp(-c/(a+b))\big)$; recall that we assume $a \ge b$ and that $1 - D(\tilde{u}) < 1/3$ for $u \le -c/b$ if c is large. By using again the substitution $u \mapsto g(z)$ and Taylor expansion of exp at 0, the second integral on the right hand side above equals

$$\begin{aligned}
&\exp\Big(-\frac{c}{a+b}\Big)\,\frac{b}{a+b}\\
&\times\int_0^{z_c}\Bigg(1-\exp\Big\{g(z)\,\big(D(z)-1\big)-\exp\Big(-\frac{c}{a+b}\Big)\,\Big(\frac{1-z}{z}\Big)^{\frac{a}{a+b}}\,D(z)\Big\}\Bigg)\\
&\qquad\times\Big(D(z)+D'(z)(1-z)\Big)\,\exp\big(g(z)\big)\,z^{-\frac{a}{a+b}}\,(1-z)^{\frac{a}{a+b}-2}\,dz\;=
\end{aligned}$$

$$\begin{aligned}
&=\exp\Big(-\frac{c}{a+b}\Big)\,\frac{b}{a+b}\\
&\times\int_{1/2}^{z_c}\Bigg(1-\exp\Big\{g(z)\,\big(D(z)-1\big)-\exp\Big(-\frac{c}{a+b}\Big)\,\Big(\frac{1-z}{z}\Big)^{\frac{a}{a+b}}\,D(z)\Big\}\Bigg)\\
&\qquad\times\Big(D(z)+D'(z)(1-z)\Big)\,\exp\big(g(z)\big)\,z^{-\frac{a}{a+b}}\,(1-z)^{\frac{a}{a+b}-2}\,dz\\
&+\;o\Bigg(\exp\Big(-\frac{c}{a+b}\Big)\Bigg)
\end{aligned}$$

$$\begin{aligned}
&=\exp\Big(-\frac{c}{a+b}\Big)\,\frac{b}{a+b}\\
&\times\int_{1/2}^{z_c}\Bigg(-g(z)\,\big(D(z)-1\big)+\exp\Big(-\frac{c}{a+b}\Big)\,\Big(\frac{1-z}{z}\Big)^{\frac{a}{a+b}}\,D(z)\Bigg)\,(1-z)^{\frac{a}{a+b}-2}\,O(1)\;dz\\
&+\;o\Bigg(\exp\Big(-\frac{c}{a+b}\Big)\Bigg)
\end{aligned}$$

$$=o\Bigg(\exp\Big(-\frac{c}{a+b}\Big)\Bigg)\;,$$

which follows from elementary computations; recall that $g(z)\underset{c\to\infty}{\longrightarrow}0$ and that $1-D(z)\le 1-z$. We have, thus, established (3.17), which completes the proof of Theorem 3.4. □

In the subsequent theorems we compile the limit excess distributions of $aU+bV$ for all combinations of different marginal univariate EVDs. Note that the df of (U,V) is a bivariate EVD if, and only if the df of (V,U) is a bivariate EVD. This implies that the order of the prescribed marginal dfs of (U,V) in the subsequent results does not matter.

Theorem 3.5 (Reverse Weibull and Gumbel Margins)**.** *Suppose that* (U,V) *follows a bivariate EVD and that* $P(U\le x)=\exp(-e^{-x})$, $x\in\mathbb{R}$, $P(V\le y)=\exp\big(-(-y)^{\alpha}\big)$, $y\le 0$, $\alpha>0$. *Then we have for* $a,b>0$ *and* $t\ge 0$

$$P\Big(aU+bV>c+t\mid aU+bV>c\Big)\underset{c\to\infty}{\longrightarrow}\exp(-t/a)\;.$$

The combination of a reverse Weibull and a Gumbel margin is, consequently, dominated by the Gumbel part. The corresponding scale parameter is preserved in the limit.

Proof: Wlog we can assume the representation $U = -\log(-X)$, $V = -(-Y)^{1/\alpha}$, where (X, Y) follows a bivariate EVD with reverse exponential margins and dependence function $D(z) = 1 + \int_0^z M(x) - 1\,dx$, see Lemma 2.1. We will establish in the following

$$(3.18) \qquad e^{c/a} P\big(aU + bV > c\big) \underset{c\to\infty}{\longrightarrow} \int_{-\infty}^{0} \exp\Big(-\frac{b}{a}(-u)^{1/\alpha}\Big) \exp(u)\,du \in (0,1)$$

if U and V are independent and

$$(3.19) \qquad e^{c/a} P\big(aU + bV > c\big) \underset{c\to\infty}{\longrightarrow} 1 - M(0) \int_0^1 \exp\Big(-\big(-\log(u^{a/b})\big)^{\alpha}\Big)\,du \in (0,1)$$

elsewhere. This implies the assertion.

First we establish (3.18). Conditioning on $Y = u$ we obtain

$$\begin{aligned}
P\big(aU + bV > c\big) &= \int_{-\infty}^{0} P\Big(-a\log(-X) - b(-Y)^{1/\alpha} > c \mid Y = u\Big) \exp(u)\,du \\
&= \int_{-\infty}^{0} \Bigg(1 - P\Bigg(X \le -\exp\Bigg(-\frac{c + b(-u)^{1/\alpha}}{a}\Bigg) \mid Y = u\Bigg)\Bigg) \exp(u)\,du \\
&= \int_{-\infty}^{0} \Big(1 - \exp\big(-e^{-c/a - b(-u)^{1/\alpha}/a}\big)\Big) \exp(u)\,du \\
&= \int_{-\infty}^{0} \Big(e^{-c/a - b(-u)^{1/\alpha}/a} + O\big(e^{-2c/a - 2b(-u)^{1/\alpha}/a}\big)\Big) \exp(u)\,du
\end{aligned}$$

and, thus,

$$e^{c/a} P\big(aU + bV > c\big) \underset{c\to\infty}{\longrightarrow} \int_{-\infty}^{0} \exp\Big(-\frac{b}{a}(-u)^{1/\alpha}\Big) \exp(u)\,du\,,$$

which is (3.18).

Next we establish (3.19). Conditioning on $X = u$ we obtain from Lemma 2.2

$$\begin{aligned}
P\big(aU + bV > c\big) &= \int_{-\infty}^{0} P\Big(-a\log(-u) - b(-Y)^{1/\alpha} > c \mid X = u\Big) \exp(u)\,du \\
&= \int_{-\exp(-c/a)}^{0} \Bigg(1 - P\Bigg(Y \le -\Bigg(\frac{-c - a\log(-u)}{b}\Bigg)^{\alpha} \mid X = u\Bigg)\Bigg) \exp(u)\,du \\
&= \int_{-\exp(-c/a)}^{0} \Bigg(1 - \exp\Bigg\{u\big(D(\tilde{u}) - 1\big) - \Bigg(\frac{-c - a\log(-u)}{b}\Bigg)^{\alpha} D(\tilde{u})\Bigg\} \\
&\qquad \times \Big(D(\tilde{u}) + D'(\tilde{u})(1 - \tilde{u})\Big)\Bigg) \exp(u)\,du\,,
\end{aligned}$$

where

$$\tilde{u} := \frac{u}{u - \Big(\frac{-c - a\log(-u)}{b}\Big)^{\alpha}} \in [0,1]\,.$$

With the substitution $u \mapsto -\exp(-c/a)\,u$, the above integral equals

$$\exp\Bigl(-\frac{c}{a}\Bigr)\int_0^1 \Biggl(1-\exp\Bigl\{-\exp\Bigl(-\frac{c}{a}\Bigr)\,u\bigl(D(\bar u)-1\bigr)-\bigl(-\log(u^{a/b})\bigr)^{\alpha}D(\bar u)\Bigr\}$$
$$\times\Bigl(D(\bar u)+D'(\bar u)\,(1-\bar u)\Bigr)\Biggr)\,\exp\Bigl(-\exp\Bigl(-\frac{c}{a}\Bigr)\,u\Bigr)\;du\;,$$

where for $u \in (0,1)$

$$\bar u := \bigl(\widetilde{-\exp(-c/a)}\,u\bigr) = \frac{u}{u+\exp(c/a)\bigl(-\log(u^{a/b})\bigr)^{\alpha}} \underset{c\to\infty}{\downarrow} 0\;.$$

We obtain, consequently,

$$\exp\Bigl(\frac{c}{a}\Bigr)\,P\bigl(aU+bV>c\bigr) =$$
$$= \int_0^1 \Bigl(1-D(\bar u)-D'(\bar u)\,(1-\bar u)\Bigr)\,\exp\Bigl(-\exp\Bigl(-\frac{c}{a}\Bigr)\,u\Bigr)\;du$$
$$+\int_0^1\Biggl\{1-\exp\Bigl(-\exp\Bigl(-\frac{c}{a}\Bigr)\,u\,\bigl(1-D(\bar u)\bigr)-\bigl(-\log(u^{a/b})\bigr)^{\alpha}D(\bar u)\Bigr)\Biggr\}$$
$$\times\Bigl(D(\bar u)+D'(\bar u)\,(1-\bar u)\Bigr)\,\exp\Bigl(-\exp\Bigl(-\frac{c}{a}\Bigr)\,u\Bigr)\;du$$
$$\underset{c\to\infty}{\longrightarrow}\; -D'(0)+\int_0^1\Bigl\{1-\exp\Bigl(-\bigl(-\log(u^{a/b})\bigr)^{\alpha}\Bigr)\Bigr\}\,\bigl(1+D'(0)\bigr)\;du$$
$$= 1-M(0)\int_0^1\exp\Bigl(-\bigl(-\log(u^{a/b})\bigr)^{\alpha}\Bigr)\;du \;\in\; (0,1)\;.$$

Note that necessarily $M(0)<1$. Otherwise we had $D(z)=1+\int_0^z M(x)-1\,dx \ge 1$ and, thus, D would be the constant function 1. But this case was excluded. Thus we have established (3.19), which completes the proof of Theorem 3.5. □

Theorem 3.6 (Reverse Weibull and Fréchet Margins)**.** *Suppose that* (U,V) *follows a bivariate EVD with* $P(U\le x)=\exp\bigl(-(-x)^{\alpha_1}\bigr)$, $x\le 0$, *and* $P(V\le y)=\exp(-y^{-\alpha_2})$, $y>0$, $\alpha_1,\alpha_2>0$. *Then we have for* $a,b>0$ *and* $t\ge 1$

$$P\Bigl(aU+bV>t\,c \mid aU+bV>c\Bigr) \underset{c\to\infty}{\longrightarrow} t^{-\alpha_2}\;.$$

The combination of a reverse Weibull and a Fréchet margin is, therefore, determined by the Fréchet part. The limit excess df is independent of the scale parameters.

Proof: It is sufficient to show that for $n\in\mathbb{N}$

$$n^{-1/\alpha_2}\max_{1\le i\le n}\bigl(aU_i+bV_i\bigr) \underset{D}{\longrightarrow} \exp\bigl(-(y/b)^{-\alpha_2}\bigr)\;,\qquad y>0\;, \tag{3.20}$$

where $(U_1, V_1), (U_2, V_2), \dots$ are independent copies of (U, V). But (3.20) is immediate from the inequalities

$$a \min_{1\le i\le n} U_i + b \max_{1\le i\le n} V_i \;\le\; \max_{1\le i\le n}\bigl(aU_i + bV_i\bigr) \;\le\; b \max_{1\le i\le n} V_i$$

and the facts that

$$n^{-1/\alpha_2} \max_{1\le i\le n} V_i \underset{D}{=} \exp(-y^{-\alpha_2}) \;, \qquad y > 0 \;,$$
$$n^{-1/\alpha_2} \min_{1\le i\le n} U_i \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in probability} \;. \qquad \square$$

Theorem 3.7 (Fréchet and Gumbel Margins)**.** *Suppose that the rv* (U,V) *follows a bivariate EVD with* $P(U\le x) = \exp(-x^{-\alpha})$, $x>0$, *where* $\alpha>0$, *and* $P(V\le y) = \exp(-e^{-y})$, $y\in\mathbb{R}$. *Then we have for* $a, b > 0$ *and* $t \ge 1$

$$P\Bigl(aU + bV > tc \mid aU + bV > c\Bigr) \underset{c\to\infty}{\longrightarrow} t^{-\alpha} \;.$$

The combination of a Fréchet and a Gumbel margin is, consequently, determined by the Fréchet part. The limit excess df is independent of the scale parameters.

Proof: It is sufficient to show that for $n\in\mathbb{N}$

$$\text{(3.21)} \qquad n^{-1/\alpha} \max_{1\le i\le n}\bigl(aU_i + bV_i\bigr) \underset{D}{\longrightarrow} \exp\bigl(-(x/b)^{-\alpha}\bigr) \;, \qquad x > 0 \;,$$

where $(U_1, V_1), (U_2, V_2), \dots$ are independent copies of (U, V). But (3.21) is immediate from the inequalities

$$a \max_{1\le i\le n} U_i + b \min_{1\le i\le n} V_i \;\le\; \max_{1\le i\le n}\bigl(aU_i + bV_i\bigr) \;\le\; a \max_{1\le i\le n} U_i + b \max_{1\le i\le n} V_i$$

and the facts that

$$n^{-1/\alpha} \max_{1\le i\le n} U_i \underset{D}{=} \exp(-x^{-\alpha}) \;, \qquad x > 0 \;,$$
$$n^{-1/\alpha} \min_{1\le i\le n} V_i \underset{n\to\infty}{\longrightarrow} 0 \,, \quad n^{-1/\alpha} \max_{1\le i\le n} V_i \underset{n\to\infty}{\longrightarrow} 0 \qquad \text{in probability} \;. \qquad \square$$

ACKNOWLEDGMENTS

The author is indebted to the referee for bringing to his attention the papers by Alink *et al.* (2004, 2005a, 2005b) and Barbe *et al.* (2006).

REFERENCES

[1] Acerbi, C. and Sirtori, C. (2001). *Expected shortfall as a tool for financial risk management.* http://www.gloriamundi.org/picsresources/ncs.pdf.

[2] Acerbi, C. and Tasche, D. (2001). *Expected shortfall: a natural coherent alternative to value at risk.* http://www.gloriamundi.org/picsresources/expshortfall.pdf.

[3] Acerbi, C. and Tasche, D. (2002). *On the coherence of expected shortfall.* http://www.gloriamundi.org/picsresources/cadt.pdf.

[4] Alink, S.; Löwe, M. and Wüthrich, M.V. (2004). Diversification of aggregate dependent risks, *Insurance: Mathematics and Economics*, **35**, 77–95.

[5] Alink, S.; Löwe, M. and Wüthrich, M.V. (2005a). Analysis of the expected shortfall of aggregate dependent risks, *Astin Bulletin*, **35**, 25–43.

[6] Alink, S.; Löwe, M. and Wüthrich, M.V. (2005b). *Analysis of the diversification effect of aggregate dependent risks.* http://www.math.ethz.ch/∼wueth/papers2.html.

[7] Barbe, Ph.; Fougères, A.-L. and Genest, C. (2006). On the tail behaviour of dependent risks, *Astin Bulletin*, **36**, 361–373.

[8] Dupuis, D.J. and Tawn, J.A. (2001). Effects of mis-specification in bivariate extreme value problems, *Extremes*, **2**, 339–365.

[9] Embrechts, P. (2000). *Extremes and Integrated Risk Management*, Risk Books, London.

[10] Embrechts, P. (2004). *Extremes in economics and the economics of extremes.* In "Extreme Values in Finance, Telecommunications, and the Environment" (B. Finkenstädt and H. Rootzén, Eds.), Chapman & Hall/CRC, Boca Raton, 169–183.

[11] Falk, M. (2006). A representation of bivariate extreme value distributions via norms on $\mathbb{R}^2$, *Extremes*, **9**, 63–68.

[12] Falk, M. and Michel, R. (2006). Testing for tail independence in extreme value models, *Ann. Inst. Statist. Math.*, **58**, 261–290.

[13] Falk, M.; Hüsler, J. and Reiss, R.-D. (2004). *Laws of Small Numbers: Extremes and Rare Events*, 2nd ed., Birkhäuser, Basel.

[14] Feller, W. (1971). *An Introduction to Probability Theory and its Applications*, Volume 2, 2nd ed., Wiley, New York.

[15] Ghoudi, K.; Khoudraji, A. and Rivest, L.P. (1998). Statistical properties of couples of bivariate extreme–value copulas, *Canad. J. Statist.*, **26**, 187–197.

[16] Klüppelberg, C. (2004). *Risk management with extreme value theory.* In "Extreme Values in Finance, Telecommunications, and the Environment" (B. Finkenstädt and H. Rootzén, Eds.), Chapman & Hall/CRC, Boca Raton, 101–168.

[17] Klüppelberg, C.; Lindner, A. and Maller, R. (2006). *Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models.* In "From Stochastic Calculus to Mathematical Finance. The Shiryaev Festschrift" (Yu. Kabanov, R. Lipster and J. Stoyanov, Eds.), Springer, Berlin, 393–419.

[18] Macke, M. (2005). *On the distribution of linear combinations of random variables with underlying multivariate extreme value and generalized Pareto distributions and an application to the expected shortfall of portfolios*, Diploma thesis (in German), University of Würzburg.

[19] Reiss, R.-D. and Thomas, M. (2001). *Statistical Analysis of Extreme Values*, 2nd ed., Birkhäuser, Basel.

[20] Wüthrich, M.V. (2003). Asymptotic value-at-risk estimates for sums of dependent random variables, *Astin Bulletin*, **33**, 75–92.