TWO-FACTOR EXPERIMENTS WITH SPLIT UNITS CONSTRUCTED BY CYCLIC DESIGNS AND SQUARE LATTICE DESIGNS

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Abstract:

- We consider nested row-column designs with split units for a two-factor experiment. The most optimal design in this case is that of using for the whole plots a Latin square while for the subplot treatments with a completely randomized design for each whole plot. Such a design, in fact optimal, utilizes many experimental units and quite a large space. Hence to construct new designs of reduced size of the experiment we use a cyclic design for the whole plot treatments and a square lattice design for the subplot treatments. The proposed designs are generally balanced and they allow for giving the stratum efficiency factors, especially useful to design of experiments.

Key-Words:

- *Cyclic design; general balance; square lattice design; stratum efficiency factor.*

AMS Subject Classification:

- 62K15, 62K10, 05B05.

1. INTRODUCTION

In many biological and agricultural (field) experiments, a nested row-column design with split units is often used. The design is for a two-factor experiment of split-plot type with b blocks. The first factor A has v_1 levels $A_1, A_2, \ldots, A_{v_1}$ and the second factor B has v_2 levels $B_1, B_2, \ldots, B_{v_2}$. Each block is divided into k_1 rows and k_2 columns and these k_1k_2 units are treated as whole plots. Moreover, each whole plot is divided into k_3 subplots. The levels of A and B are applied to the whole plots (called whole plot treatments) and the subplots (called subplot treatments), respectively. Such a design is called a nested row-column design with split units.

Kachlicka and Mejza (1996) considered a mixed linear model with fixed treatment effects and random block, row, column, whole plot and subplot effects for the nested row-column design with split units. The hth factorial treatment combination effect τ_h is defined by

$$\tau_h = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}$$

for $h = (i-1)v_2 + j$, $i = 1, 2, \ldots, v_1$ and $j = 1, 2, \ldots, v_2$, where μ is the general mean, α_i denotes the main effect of the ith whole plot treatment A_i, β_j denotes the main effect of the jth subplot treatment B_j and $(\alpha\beta)_{ij}$ denotes the interaction effect of A_i and B_j. Here $\sum_{i=1}^{v_1} \alpha_i = 0$, $\sum_{j=1}^{v_2} \beta_j = 0$, $\sum_{i=1}^{v_1} (\alpha\beta)_{ij} = 0$ for $j = 1, 2, \ldots, v_2$ and $\sum_{j=1}^{v_2} (\alpha\beta)_{ij} = 0$ for $i = 1, 2, \ldots, v_1$. The mixed linear model results from a four-step randomization, i.e., the randomization of blocks, the randomization of rows within each block, the randomization of columns within each block and the randomization of subplots within each whole plot. This kind of randomization leads us to an experiment with orthogonal block structure as defined by Nelder (1965a, 1965b) and the multistratum analysis proposed by Nelder (1965a, 1965b) and Houtman and Speed (1983) can be applied to the analysis of data in the experiment. In this case, we have five strata, except zero stratum connected with the general mean only, (I) inter-block stratum, (II) inter-row stratum, (III) inter-column stratum, (IV) inter-whole plot stratum and (V) inter-subplot stratum. The statistical properties of the nested row-column design with split units are strictly connected with the eigenvalues and the eigenvectors of the stratum information matrices for the treatment combinations. The stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ are given by

$$\mathbf{A}_1 = \frac{1}{k_1k_2k_3}\mathbf{N}_0\mathbf{N}_0' - \frac{r}{v}\mathbf{J}_v, \quad \mathbf{A}_2 = \frac{1}{k_2k_3}\mathbf{N}_1\mathbf{N}_1' - \frac{1}{k_1k_2k_3}\mathbf{N}_0\mathbf{N}_0', \tag{1.1}$$

$$\mathbf{A}_3 = \frac{1}{k_1k_3}\mathbf{N}_2\mathbf{N}_2' - \frac{1}{k_1k_2k_3}\mathbf{N}_0\mathbf{N}_0', \tag{1.2}$$

$$\mathbf{A}_4 = \frac{1}{k_3}\mathbf{N}_3\mathbf{N}_3' - \frac{1}{k_1k_3}\mathbf{N}_2\mathbf{N}_2' - \frac{1}{k_2k_3}\mathbf{N}_1\mathbf{N}_1' + \frac{1}{k_1k_2k_3}\mathbf{N}_0\mathbf{N}_0' \tag{1.3}$$

and

$$\mathbf{A}_5 = r\mathbf{I}_v - \frac{1}{k_3}\mathbf{N}_3\mathbf{N}_3', \tag{1.4}$$

where $v = v_1v_2$, $\mathbf{N}_0$, $\mathbf{N}_1$, $\mathbf{N}_2$ and $\mathbf{N}_3$ are the incidence matrices for the treatment combinations vs. blocks, rows, columns and whole plots, respectively, $\mathbf{I}_v$ is the identity matrix of order v and $\mathbf{J}_v$ is the $v \times v$ matrix with every element unity. Here we assume that every treatment combination A_iB_j $(i = 1, 2, \ldots, v_1, j = 1, 2, \ldots, v_2)$ occurs in precisely r blocks and the treatment combinations are ordered lexicographically.

A generally balanced design was firstly introduced by Nelder (1965a, 1965b), for which the stratum information matrices are spanned by a common set of eigenvectors. Let $\boldsymbol{s}_0, \boldsymbol{s}_1, \ldots, \boldsymbol{s}_{v-1}$ be the mutually orthonormal common eigenvectors of the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$. Since $\mathbf{A}_f\mathbf{1}_v = \mathbf{0}$ for $f = 1, 2, 3, 4, 5$, $\frac{1}{\sqrt{v}}\mathbf{1}_v$ may be chosen as the first eigenvector $\boldsymbol{s}_0$, where $\mathbf{1}_v$ is the $v \times 1$ vector of unit elements. Let ξ_{fh} be an eigenvalue of a matrix $\mathbf{A}_f^* = r^{-1}\mathbf{A}_f$ corresponding to the eigenvector $\boldsymbol{s}_h$ for $f = 1, 2, 3, 4, 5$ and $h = 1, 2, \ldots, v-1$. Then, a basic contrast of the treatment effects (see Pearce et al. (1974)) is defined by $\boldsymbol{s}_h'\boldsymbol{\tau}$ for $h = 1, 2, \ldots, v-1$, where $\boldsymbol{\tau}$ is the $v \times 1$ vector of the treatment effects. The eigenvalue ξ_{fh} can be identified as a stratum efficiency factor of the design concerning estimation of the hth basic contrast in the fth stratum for $f = 1, 2, 3, 4, 5$ and $h = 1, 2, \ldots, v-1$ (see, Houtman and Speed (1983)).

Many experiments require a long time or a large space (units) often making it impossible to carry out a conventional, complete (orthogonal) design of the considered type. For example, in agricultural field experiments, because of soil fertility it is difficult to find units (plots) fulfilling restrictions concerning the homogeneity of blocks, rows, columns, whole plots or subplots. Then, to satisfy the main experimental principles it is necessary to design the experiment as an incomplete (non-orthogonal) one. Such an experiment usually utilizes smaller units, with respect to size and also utilizes smaller number of units (the experiment is cheaper). The problem is to find an incomplete design proper to experimental material structure and optimal with respect to statistical properties of the design.

Kuriki et al. (2009), Mejza et al. (2009) and Mejza and Kuriki (2013) constructed nested row-column designs with split units by the Kronecker product of the incidence matrices of two designs. They used a Youden square for the whole plot treatments and various proper designs for the subplot treatments. Mejza et al. (2014) have used a balanced incomplete block design with nested rows and columns instead of the Youden square to construct a nested row-column design with split units. The designs obtained by this way need usually a large number of units. In this paper, we construct a nested row-column design with split units by a modified Kronecker product (called a semi-Kronecker product) of the incidence matrices of two designs. We use a cyclic design for the whole plot treatments and a square lattice design for the subplot treatments. We give the stratum efficiency factors for such a nested row-column design with split units, which has the general balance property.

These designs have smaller numbers of blocks than the conventional experiments. Therefore, they would be useful in practice, for example, the reduction of the experimental expenses and effort, and the easier implementation of the experiments by using the well-known cyclic designs and square lattice designs in the literature (see, John (1987), John and Williams (1995) and Raghavarao (1971), etc.).

Other variants of incomplete split plot designs are given, for example, by Mejza and Mejza (1996), Ozawa et al. (2004), Aastveit et al. (2009), Mejza et al. (2012) and Kuriki et al. (2012).

2. A CONSTRUCTION BY A CYCLIC DESIGN AND A SQUARE LATTICE DESIGN

Firstly, we need the semi-Kronecker product (see, Khatri and Rao (1968) and Mejza, Kuriki and Mejza (2001)) of two matrices that will be used to construct nested row-column designs with split units. Suppose that two matrices $\mathbf{E}$ and $\mathbf{F}$ are divided into the same number of submatrices as follows:

$$\mathbf{E} = (\mathbf{E}_1 : \mathbf{E}_2 : \cdots : \mathbf{E}_m) \quad \text{and} \quad \mathbf{F} = (\mathbf{F}_1 : \mathbf{F}_2 : \cdots : \mathbf{F}_m).$$

Then, the semi-Kronecker product $\mathbf{E} \tilde{\otimes} \mathbf{F}$ is defined by

$$\mathbf{E} \tilde{\otimes} \mathbf{F} = (\mathbf{E}_1 \otimes \mathbf{F}_1 : \mathbf{E}_2 \otimes \mathbf{F}_2 : \cdots : \mathbf{E}_m \otimes \mathbf{F}_m),$$

where $\otimes$ denotes the usual Kronecker product.

Next, we need a cyclic design and a square lattice design. Let V be a set of v treatments and let $\mathcal{B}$ be a collection of subsets (called blocks) of V. A design $(V, \mathcal{B})$ is denoted by $\mathrm{D}(v, r, k)$ if every treatment occurs in precisely r blocks and each block contains k treatments. Let Z_v be the additive group of integers modulo v and let $(V, \mathcal{B})$ be a $\mathrm{D}(v, r, k)$ with $V = Z_v$ for which if $\{a_1, a_2, \ldots, a_k\}$ is a block, then $\{a_1 + 1, a_2 + 1, \ldots, a_k + 1\}$ is also a block. A set of blocks $\{\{a_1 + i, a_2 + i, \ldots, a_k + i\} \,|\, i \in Z_v\}$ is called a cyclic class and a block taken arbitrarily from each cyclic class is called an initial block. If the collection $\mathcal{B}$ of blocks is divided into some cyclic classes, then $(V, \mathcal{B})$ is said to be cyclic and it is denoted by $\mathrm{CD}(v, r, k)$. Here we consider only a case where the number of blocks in each cyclic class is v.

Let $(V, \mathcal{B})$ be a $\mathrm{D}(v, r, k)$. If the collection $\mathcal{B}$ of blocks can be grouped in such a way that every treatment occurs precisely once in every group (called a resolution class), then $(V, \mathcal{B})$ is said to be resolvable. A resolvable $\mathrm{D}(v, r, k)$ $(V, \mathcal{B})$ such that $v = s^2$, $r \leq s + 1$ and $k = s$ for a positive integer s is called a square lattice design if any two blocks from different resolution classes contain just one common treatment, and it is denoted by $\mathrm{SLD}(s^2, r, s)$. If $r = s + 1$, it is called a balanced square lattice design and it is well known that there exists a balanced square lattice design if s is a prime or a prime power (see, Raghavarao (1971)).

Now we construct a nested row-column design with split units. Let $(V_A, \mathcal{B}_A)$ be a $\mathrm{CD}(v_A, r_A, k_A)$ with $m = r_A/k_A$ initial blocks. Each cyclic class of $(V_A, \mathcal{B}_A)$ is treated as a block with k_A rows and v_A columns such that the columns are blocks of $\mathcal{B}_A$ and that every treatment of V_A occurs precisely once in each row. Such a design is denoted by $\mathcal{D}_A$. An $\mathrm{SLD}(s^2, m, s)$ is denoted by $\mathcal{D}_B$. The whole plot treatments occur in $\mathcal{D}_A$ and the subplot treatments occur in $\mathcal{D}_B$. We construct a nested row-column design, say $\mathcal{D}$, with split units embedding each block of the ith resolution class of $\mathcal{D}_B$ in every whole plot of the ith block of $\mathcal{D}_A$ for $i = 1, 2, \ldots, m$. The parameters of $\mathcal{D}$ are $v_1 = v_A$, $v_2 = s^2$, $b = ms$, $r = mk_A = r_A$, $k_1 = k_A$, $k_2 = v_A$ and $k_3 = s$.

Example 2.1. We use an A-efficient cyclic design CD(6, 6, 3) with initial blocks $\{0, 1, 2\}$ and $\{0, 1, 3\}$ given by John (1987). From two cyclic classes of this design, we have the following two blocks with 3 rows and 6 columns of $\mathcal{D}_A$:

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1

and

0	1	2	3	4	5
1	2	3	4	5	0
3	4	5	0	1	2

.

We also use a square lattice design SLD(9, 2, 3) $\mathcal{D}_B = (V_B, \mathcal{B}_B)$ with $V_B = \{1, 2, \ldots, 9\}$. The following columns are 6 blocks of $\mathcal{D}_B$:

1	4	7
2	5	8
3	6	9

and

1	2	3
4	5	6
7	8	9

,

where the first resolution class is constituted by the first 3 blocks and the second one is constituted by the remaining blocks. We construct a nested row-column design $\mathcal{D}$ with split units embedding each block of the first (second) resolution class of $\mathcal{D}_B$ in every whole plot of the first (second) block of $\mathcal{D}_A$, replacing the treatments $0, 1, 2, 3, 4, 5$ of $\mathcal{D}_A$ with $A_1, A_2, A_3, A_4, A_5, A_6$ and the treatments $1, 2, \ldots, 9$ of $\mathcal{D}_B$ with $B_1, B_2, \ldots, B_9$. The design $\mathcal{D}$ has 6 blocks as follows:

<table>
<tr><td colspan="3">A_1</td><td colspan="3">A_2</td><td colspan="3">A_3</td><td colspan="3">A_4</td><td colspan="3">A_5</td><td colspan="3">A_6</td></tr>
<tr><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td></tr>
<tr><td colspan="3">A_2</td><td colspan="3">A_3</td><td colspan="3">A_4</td><td colspan="3">A_5</td><td colspan="3">A_6</td><td colspan="3">A_1</td></tr>
<tr><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td></tr>
<tr><td colspan="3">A_3</td><td colspan="3">A_4</td><td colspan="3">A_5</td><td colspan="3">A_6</td><td colspan="3">A_1</td><td colspan="3">A_2</td></tr>
<tr><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td><td>B_1</td><td>B_2</td><td>B_3</td></tr>
</table>

<table>
<tr><td colspan="3">A_1</td><td colspan="3">A_2</td><td colspan="3">A_3</td><td colspan="3">A_4</td><td colspan="3">A_5</td><td colspan="3">A_6</td></tr>
<tr><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td></tr>
<tr><td colspan="3">A_2</td><td colspan="3">A_3</td><td colspan="3">A_4</td><td colspan="3">A_5</td><td colspan="3">A_6</td><td colspan="3">A_1</td></tr>
<tr><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td></tr>
<tr><td colspan="3">A_3</td><td colspan="3">A_4</td><td colspan="3">A_5</td><td colspan="3">A_6</td><td colspan="3">A_1</td><td colspan="3">A_2</td></tr>
<tr><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td><td>B_4</td><td>B_5</td><td>B_6</td></tr>
</table>

A_1			A_2			A_3			A_4			A_5			A_6		
B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9
A_2			A_3			A_4			A_5			A_6			A_1		
B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9
A_3			A_4			A_5			A_6			A_1			A_2		
B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9	B_7	B_8	B_9

A_1			A_2			A_3			A_4			A_5			A_6		
B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7
A_2			A_3			A_4			A_5			A_6			A_1		
B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7
A_4			A_5			A_6			A_1			A_2			A_3		
B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7	B_1	B_4	B_7

A_1			A_2			A_3			A_4			A_5			A_6		
B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8
A_2			A_3			A_4			A_5			A_6			A_1		
B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8
A_4			A_5			A_6			A_1			A_2			A_3		
B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8	B_2	B_5	B_8

A_1			A_2			A_3			A_4			A_5			A_6		
B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9
A_2			A_3			A_4			A_5			A_6			A_1		
B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9
A_4			A_5			A_6			A_1			A_2			A_3		
B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9	B_3	B_6	B_9

□

We note that if the nested row-column design with split units is constructed by the usual Kronecker product of the incidence matrices (see, Mejza et al. (2014)), then the number of blocks becomes $m^2 s = 12$. Generally, the number of blocks of a nested row-column design with split units by the Kronecker product is m times larger than those of a nested row-column design with split units by the semi-Kronecker product.

Let

$$\mathbf{N}_A = (\mathbf{N}_{A1} : \mathbf{N}_{A2} : \cdots : \mathbf{N}_{Am}) \quad \text{and} \quad \mathbf{N}_B = (\mathbf{N}_{B1} : \mathbf{N}_{B2} : \cdots : \mathbf{N}_{Bm})$$

be the incidence matrices of the cyclic design $\mathrm{CD}(v_A, r_A, k_A)$ and the square lattice design $\mathrm{SLD}(s^2, m, s)$, where $\mathbf{N}_{Ai}$ and $\mathbf{N}_{Bi}$ correspond to the ith cyclic and resolution classes, respectively. By the definition of the square lattice design,

$$\mathbf{N}'_{Bi}\mathbf{N}_{Bi} = s\mathbf{I}_s \quad \text{and} \quad \mathbf{N}'_{Bi}\mathbf{N}_{Bj} = \mathbf{J}_s \tag{2.1}$$

hold for $i, j = 1, 2, \ldots, m$, $i \neq j$. Then, the incidence matrix $\mathbf{N}_2$ of the nested row-column design $\mathcal{D}$ with split units is given by the semi-Kronecker product of $\mathbf{N}_A$ and $\mathbf{N}_B$, i.e.,

$$\mathbf{N}_2 = \mathbf{N}_A \,\tilde{\otimes}\, \mathbf{N}_B = (\mathbf{N}_{A1} \otimes \mathbf{N}_{B1} : \mathbf{N}_{A2} \otimes \mathbf{N}_{B2} : \cdots : \mathbf{N}_{Am} \otimes \mathbf{N}_{Bm})$$

in a suitable order of columns of $\mathcal{D}$, and the concurrence matrices $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$ of $\mathcal{D}$ are given by

$$\mathbf{N}_0\mathbf{N}_0' = \sum_{i=1}^{m} \left(k_A^2\mathbf{J}_{v_A} \otimes \mathbf{N}_{Bi}\mathbf{N}_{Bi}'\right) = k_A^2\mathbf{J}_{v_A} \otimes \mathbf{N}_B\mathbf{N}_B', \tag{2.2}$$

$$\mathbf{N}_1\mathbf{N}_1' = \sum_{i=1}^{m} \left(k_A\mathbf{J}_{v_A} \otimes \mathbf{N}_{Bi}\mathbf{N}_{Bi}'\right) = k_A\mathbf{J}_{v_A} \otimes \mathbf{N}_B\mathbf{N}_B', \tag{2.3}$$

$$\mathbf{N}_2\mathbf{N}_2' = \sum_{i=1}^{m} \left(\mathbf{N}_{Ai}\mathbf{N}_{Ai}' \otimes \mathbf{N}_{Bi}\mathbf{N}_{Bi}'\right) \tag{2.4}$$

and

$$\mathbf{N}_3\mathbf{N}_3' = \sum_{i=1}^{m} \left(k_A\mathbf{I}_{v_A} \otimes \mathbf{N}_{Bi}\mathbf{N}_{Bi}'\right) = k_A\mathbf{I}_{v_A} \otimes \mathbf{N}_B\mathbf{N}_B'. \tag{2.5}$$

3. STRATUM EFFICIENCY FACTORS FOR $\mathcal{D}$

In this section, we give the stratum efficiency factors for the nested row-column design $\mathcal{D}$ with split units constructed in Section 2. To find the stratum efficiency factors, it is necessary to find the eigenvalues of the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ of $\mathcal{D}$. It is easy to find these eigenvalues if $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ have the common eigenvectors, i.e., if $\mathcal{D}$ is generally balanced. It follows, from (2.1), that

$$\mathbf{N}_{Bi}\mathbf{N}_{Bi}'\mathbf{N}_{Bj}\mathbf{N}_{Bj}' = \mathbf{J}_{s^2} \tag{3.1}$$

holds for $i, j = 1, 2, \cdots, m$, $i \neq j$. From (3.1), it is easily verified that the concurrence matrices $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$ given in (2.2)–(2.5) are mutually commutative. Thus, by use of (1.1)–(1.4), the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ are mutually commutative, which means that $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$ have the common eigenvectors. Therefore, $\mathcal{D}$ is generally balanced.

In order to find the common eigenvectors of the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$, i.e., those of the concurrence matrices $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$, we consider the eigenvectors of $\mathbf{N}_{Ai}\mathbf{N}_{Ai}'$ for the ith cyclic class of the cyclic design $\mathrm{CD}(v_A, r_A, k_A)$ and those of $\mathbf{N}_{Bi}\mathbf{N}_{Bi}'$ for the ith resolution class of the square lattice design $\mathrm{SLD}(s^2, m, s)$ for $i = 1, 2, \ldots, m$. For the incidence matrix $\mathbf{N}_A$ of the $\mathrm{CD}(v_A, r_A, k_A)$, since $\mathbf{N}_{A1}\mathbf{N}_{A1}', \mathbf{N}_{A2}\mathbf{N}_{A2}', \ldots, \mathbf{N}_{Am}\mathbf{N}_{Am}'$ are symmetric circulant matrices, these matrices have the mutually orthonormal common eigenvectors, which are denoted by $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_{v_A-1}$ with $\boldsymbol{x}_0 = \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}$. The corresponding eigenvalues of $\mathbf{N}_{Ai}\mathbf{N}'_{Ai}$ are given by

$$\theta_j^{(i)} = \sum_{h=0}^{v_A-1} \lambda_h^{(i)} \cos\left(\frac{2\pi jh}{v_A}\right)$$

for $i = 1, 2, \ldots, m$ and $j = 0, 1, \ldots, v_A - 1$, where $\lambda_h^{(i)}$ $(h \neq 0)$ denotes the number of blocks containing two treatments 0 and h in the ith cyclic class of the CD(v_A, r_A, k_A) and $\lambda_0^{(i)} = k_A$. In particular, $\theta_0^{(i)} = k_A^2$ and the corresponding eigenvector is $\boldsymbol{x}_0 = \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}$ (see, John (1987) and John and Williams (1995)). These eigenvalues and common eigenvectors are summarized in the following table:

Table 1: Eigenvalues and common eigenvectors of $\mathbf{N}_{Ai}\mathbf{N}'_{Ai}$ in the CD(v_A, r_A, k_A).

Eigenvalues	Common eigenvectors
k_A^2	$\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}$
$\theta_j^{(i)}$	$\boldsymbol{x}_j$ $(j = 1, 2, \ldots, v_A - 1)$

Similarly, for the incidence matrix $\mathbf{N}_B$ of the SLD(s^2, m, s), from (2.1), $\mathbf{N}_{Bi}\mathbf{N}'_{Bi}$ has the eigenvalues s and 0 with multiplicities s and $s(s-1)$ for each $i = 1, 2, \ldots, m$. From (3.1), $\mathbf{N}_{B1}\mathbf{N}'_{B1}, \mathbf{N}_{B2}\mathbf{N}'_{B2}, \ldots, \mathbf{N}_{Bm}\mathbf{N}'_{Bm}$ are mutually commutative, so these matrices have the common eigenvectors. Let $\mathbf{Q} = (\boldsymbol{q}_0, \boldsymbol{q}_1, \ldots, \boldsymbol{q}_{s-1})$ be an orthogonal matrix of order s with $\boldsymbol{q}_0 = \frac{1}{\sqrt{s}}\mathbf{1}_s$. For each $i = 1, 2, \ldots, m$, from (2.1), the mutually orthonormal eigenvectors of $\mathbf{N}_{Bi}\mathbf{N}'_{Bi}$ corresponding to the eigenvalue s are given by

$$\boldsymbol{z}_{ip} = \frac{1}{\sqrt{s}}\mathbf{N}_{Bi}\boldsymbol{q}_p$$

for $p = 0, 1, \ldots, s - 1$. In particular, $\boldsymbol{z}_{i0} = \frac{1}{s}\mathbf{1}_{s^2}$. The eigenvectors $\boldsymbol{z}_{ip}$ are also the eigenvectors of $\mathbf{N}_{Bh}\mathbf{N}'_{Bh}$ $(h \neq i)$ for any other resolution class, and the eigenvalues of $\mathbf{N}_{Bh}\mathbf{N}'_{Bh}$ corresponding to $\boldsymbol{z}_{i0}$ and $\boldsymbol{z}_{ip}$ $(p \neq 0)$ are s and 0, respectively. Furthermore, the mutually orthonormal common eigenvectors of $\mathbf{N}_{B1}\mathbf{N}'_{B1}, \mathbf{N}_{B2}\mathbf{N}'_{B2}, \ldots, \mathbf{N}_{Bm}\mathbf{N}'_{Bm}$ corresponding to the eigenvalue 0 are denoted by $\boldsymbol{z}_q^*$ for $q = 1, 2, \ldots, s^2 - m(s-1) - 1$. These eigenvalues and common eigenvectors are summarized in Table 2.

Combining the eigenvectors of Table 1 and Table 2, we consider the following 6 sets of vectors:

$$(1)\ \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A} \otimes \frac{1}{s}\mathbf{1}_{s^2}, \quad (2)\ \boldsymbol{x}_j \otimes \frac{1}{s}\mathbf{1}_{s^2}, \quad (3)\ \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A} \otimes \boldsymbol{z}_{ip},$$

$$(4)\ \frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A} \otimes \boldsymbol{z}_q^*, \quad (5)\ \boldsymbol{x}_j \otimes \boldsymbol{z}_{ip}, \quad (6)\ \boldsymbol{x}_j \otimes \boldsymbol{z}_q^*$$

Table 2: Eigenvalues and common eigenvectors of $\mathbf{N}_{Bi}\mathbf{N}'_{Bi}$ in the SLD(s^2, m, s).

Eigenvalues				Common eigenvectors
$\mathbf{N}_{B1}\mathbf{N}'_{B1}$	$\mathbf{N}_{B2}\mathbf{N}'_{B2}$	$\cdots$	$\mathbf{N}_{Bm}\mathbf{N}'_{Bm}$	
s	s	$\cdots$	s	$\frac{1}{s}\mathbf{1}_{s^2}$
s	0	$\cdots$	0	$\boldsymbol{z}_{1p}$ $(p = 1, 2, \ldots, s-1)$
0	s	$\cdots$	0	$\boldsymbol{z}_{2p}$ $(p = 1, 2, \ldots, s-1)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	0	$\cdots$	s	$\boldsymbol{z}_{mp}$ $(p = 1, 2, \ldots, s-1)$
0	0	$\cdots$	0	$\boldsymbol{z}^*_q$ $(q = 1, 2, \ldots, s^2 - m(s-1) - 1)$

for $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, v_A - 1,\ p = 1, 2, \ldots, s-1$ and $q = 1, 2, \ldots, s^2 - m(s-1) - 1$. The vectors of (1)–(6) are mutually orthonormal and the total number of the vectors is $v_A s^2$. We show that the vectors of (1)–(6) are the common eigenvectors of $\mathbf{N}_0\mathbf{N}'_0$, $\mathbf{N}_1\mathbf{N}'_1$, $\mathbf{N}_2\mathbf{N}'_2$ and $\mathbf{N}_3\mathbf{N}'_3$, and we find the corresponding eigenvalues of $\mathbf{N}_0\mathbf{N}'_0$, $\mathbf{N}_1\mathbf{N}'_1$, $\mathbf{N}_2\mathbf{N}'_2$ and $\mathbf{N}_3\mathbf{N}'_3$.

Firstly, we take into account the matrix $\mathbf{N}_0\mathbf{N}'_0$. For (1), we have, from (2.2), Table 1 and Table 2,

$$\begin{aligned}
&\mathbf{N}_0\mathbf{N}'_0\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\otimes\frac{1}{s}\mathbf{1}_{s^2}\right) = \left(k_A^2\mathbf{J}_{v_A}\otimes\mathbf{N}_B\mathbf{N}'_B\right)\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\otimes\frac{1}{s}\mathbf{1}_{s^2}\right)\\
&= \left(k_A^2\mathbf{J}_{v_A}\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\right)\otimes\left(\mathbf{N}_B\mathbf{N}'_B\frac{1}{s}\mathbf{1}_{s^2}\right) = \left(v_A k_A^2\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\right)\otimes\left(ms\frac{1}{s}\mathbf{1}_{s^2}\right)\\
&= m v_A k_A^2 s\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\otimes\frac{1}{s}\mathbf{1}_{s^2}\right).
\end{aligned}$$

The corresponding eigenvalue is $m v_A k_A^2 s$.

For (2), we have

$$\mathbf{N}_0\mathbf{N}'_0\left(\boldsymbol{x}_j\otimes\frac{1}{s}\mathbf{1}_{s^2}\right) = \left(k_A^2\mathbf{J}_{v_A}\boldsymbol{x}_j\right)\otimes\left(\mathbf{N}_B\mathbf{N}'_B\frac{1}{s}\mathbf{1}_{s^2}\right) = \mathbf{0}.$$

The corresponding eigenvalue is zero for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, v_A - 1$.

For (3), we have

$$\begin{aligned}
&\mathbf{N}_0\mathbf{N}'_0\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\otimes\boldsymbol{z}_{ip}\right) = \left(k_A^2\mathbf{J}_{v_A}\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\right)\otimes\left(\mathbf{N}_B\mathbf{N}'_B\boldsymbol{z}_{ip}\right)\\
&= \left(v_A k_A^2\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\right)\otimes\left(\sum_{h=1}^{m}\mathbf{N}_{Bh}\mathbf{N}'_{Bh}\boldsymbol{z}_{ip}\right) = \left(v_A k_A^2\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\right)\otimes\left(s\boldsymbol{z}_{ip}\right)\\
&= v_A k_A^2 s\left(\frac{1}{\sqrt{v_A}}\mathbf{1}_{v_A}\otimes\boldsymbol{z}_{ip}\right).
\end{aligned}$$

The corresponding eigenvalue is $v_A k_A^2 s$ for each $i = 1, 2, \ldots, m$ and $p = 1, 2, \ldots, s-1$.

For (4), we have

$$\mathbf{N}_0\mathbf{N}_0' \left(\frac{1}{\sqrt{v_A}} \mathbf{1}_{v_A} \otimes \boldsymbol{z}_q^* \right) = \left(k_A^2 \mathbf{J}_{v_A} \frac{1}{\sqrt{v_A}} \mathbf{1}_{v_A} \right) \otimes \left(\mathbf{N}_B \mathbf{N}_B' \boldsymbol{z}_q^* \right)$$

$$= \left(v_A k_A^2 \frac{1}{\sqrt{v_A}} \mathbf{1}_{v_A} \right) \otimes \left(\sum_{i=1}^{m} \mathbf{N}_{Bi} \mathbf{N}_{Bi}' \boldsymbol{z}_q^* \right) = \mathbf{0}.$$

The corresponding eigenvalue is zero for $q = 1, 2, \ldots, s^2 - m(s-1) - 1$. Moreover, for (5) and (6), the eigenvalue is also zero.

Similarly, from (2.3)–(2.5), we can show that the vectors of (1)–(6) are also the eigenvectors of $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$. The corresponding eigenvalues of $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$ are summarized in the table below:

Table 3: Eigenvalues and common eigenvectors of $\mathbf{N}_0\mathbf{N}_0'$, $\mathbf{N}_1\mathbf{N}_1'$, $\mathbf{N}_2\mathbf{N}_2'$ and $\mathbf{N}_3\mathbf{N}_3'$.

Eigenvalues				Common eigenvectors
$\mathbf{N}_0\mathbf{N}_0'$	$\mathbf{N}_1\mathbf{N}_1'$	$\mathbf{N}_2\mathbf{N}_2'$	$\mathbf{N}_3\mathbf{N}_3'$	
$mv_A k_A^2 s$	$mv_A k_A s$	$mk_A^2 s$	$mk_A s$	(1)
0	0	$\sum_{i=1}^m \theta_j^{(i)} s$	$mk_A s$	(2)
$v_A k_A^2 s$	$v_A k_A s$	$k_A^2 s$	$k_A s$	(3)
0	0	$\theta_j^{(i)} s$	$k_A s$	(5)
0	0	0	0	(4), (6)

Here $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, v_A - 1$.

The vectors (1)–(6) are also the common eigenvectors of the stratum information matrices $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, $\mathbf{A}_4$ and $\mathbf{A}_5$. By use of (1.1)–(1.4) and Table 3, the stratum efficiency factors for $\mathcal{D}$ can be calculated as in the following table:

Table 4: Stratum efficiency factors for $\mathcal{D}$.

Type of contrasts	Number of contrasts	Strata				
		I	II	III	IV	V
A	$v_A - 1$	0	0	ω_j	$1 - \omega_j$	0
B	$m(s-1)$	$1/m$	0	0	0	$1 - 1/m$
	$s^2 - m(s-1) - 1$	0	0	0	0	1
$A \times B$	$m(v_A - 1)(s-1)$	0	0	ξ_{ij}	$1/m - \xi_{ij}$	$1 - 1/m$
	$(v_A - 1)\{s^2 - m(s-1) - 1\}$	0	0	0	0	1

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, v_A - 1$, where A and B denote the basic contrasts among the main effects of whole plot and subplot treatments, respectively,

$A \times B$ denotes the basic contrasts among the interaction effects, $\xi_{ij} = \theta_j^{(i)}/(mk_A^2)$ and $\omega_j = \sum_{i=1}^m \xi_{ij}$. The eigenvectors of (2), (3)–(4) and (5)–(6) define the basic contrasts A, B and $A \times B$, respectively. We use Table 4 in order to improve the estimators for the basic contrasts of the treatment effects combining the estimators obtained from the strata I, III, IV and V. This procedure was proposed by Nelder (1965a, 1965b) and Houtman and Speed (1983). Especially, we see that some basic contrasts of B and $A \times B$ are estimable with full efficiency.

Example 3.1. For the nested row-column design $\mathcal{D}$ with split units given in Example 2.1, $m = 2$, $v_A = 6$, $k_A = 3$, $s = 3$, $\theta_1^{(1)} = 4$, $\theta_2^{(1)} = 0$, $\theta_3^{(1)} = 1$, $\theta_4^{(1)} = 0$, $\theta_5^{(1)} = 4$, $\theta_1^{(2)} = 1$, $\theta_2^{(2)} = 3$, $\theta_3^{(2)} = 1$, $\theta_4^{(2)} = 3$ and $\theta_5^{(2)} = 1$. Thus, by use of Table 4, the stratum efficiency factors can be calculated as in the following table:

Table 5: Stratum efficiency factors for $\mathcal{D}$ given in Example 2.1.

Type of contrasts	Number of contrasts	Strata				
		I	II	III	IV	V
A	1	0	0	1/9	8/9	0
	2	0	0	1/6	5/6	0
	2	0	0	5/18	13/18	0
B	4	1/2	0	0	0	1/2
	4	0	0	0	0	1
$A \times B$	4	0	0	0	1/2	1/2
	4	0	0	1/6	1/3	1/2
	4	0	0	2/9	5/18	1/2
	8	0	0	1/18	4/9	1/2
	20	0	0	0	0	1

4. REMARKS

In the design of experiments at least a few aspects play crucial roles. The first one concerns proper use of available structure of experimental units. The general rule, for example, in field agricultural experiments constitutes that smaller units better satisfy requirements concerning homogeneity of stratum units. In addition, usually smaller errors are associated after randomizations with these units.

The second aspect concerns statistical properties of designs. Using complete, orthogonal designs leads to the best unbiased estimators of the estimable functions of linear model parameters. In this work, we use a randomization-derived linear model (random block effect describing structure of units) with treatment (combination) effects being fixed. The structure of units and randomization performed lead to a design which possesses orthogonal block structure.

In a complete case, the estimators of all estimable treatment effect functions are BLUEs. This means that the design is optimal from a point of view of statistical properties. Such a design can be used for our experiment if it is possible. However, many times there exist some limitations in available structure of experimental units (material). Then in our experiment some incomplete design can be applied only.

The new problem concerns how to choose an incomplete design that fits to the structure of experimental units, is optimal for the most interesting treatment effect functions, and is not so expensive (utilizes small as possible number of units of proper size). In the worse case we can use any incomplete design. Then it is difficult to describe the statistical properties of the proposed design.

The experimenter usually makes a ranking of linear functions of treatment effects (contrasts) with respect to a scientific interest and an aim of the experiment. It would be helpful to have a design with known efficiencies of all estimable treatment effect functions. This property has a generally balanced design (see, for example, Mejza (1992) and Bailey (1994)). General balance aids interpretation; the design which is generally balanced with respect to meaningful contrasts may be superior to a technically optimal design. For generally balanced designs, we can identify the meaning of the treatment effect contrasts and their efficiency factors (cf. Table 2, Table 3 and Table 4). Hence we restrict our searching in the class of generally balanced designs.

Those considered here (nested row-column designs with split units) can be characterized by a few component block designs. We are looking for methods allowing for generation of new row-column designs with split units by using some known incomplete block designs instead of component designs. The Kronecker product of the component incomplete block designs is often used for constructing new designs with split units. The final design possesses optimal properties, but it utilizes many experimental units (high cost of the experiment). To overcome this problem (size of the experiment) we proposed to use of the semi-Kronecker product as defined in Section 2 instead of the ordinary Kronecker product. The final design is much smaller and also possesses desirable statistical properties (see Example 2.1). Moreover using the semi-Kronecker product to generate new designs leads to much smaller number of units and smaller size. In the Example 2.1, one block of the complete design will have 6 rows and 6 columns while the whole plot consists of 9 units. For example, in agricultural field experiments (where such designs are very often used) it would be difficult to find so many homogeneous plots. In these cases the use of an incomplete design is recommended. In this paper, we construct a nested row-column design with split units by the semi-Kronecker product of the incidence matrices of a cyclic design for the whole plot treatments and a square lattice design for the subplot treatments. We give the stratum efficiency factors for such a nested row-column design with split units having the general balance property.

Although we proposed the new method for constructing the design in the class of nested row-column designs with split units, we still need new methods for constructing designs in the considered class which will lead to general balanced designs with desirable statistical properties and will have reasonable size. Naturally, in the future work for construction optimal row-column designs with nested structures someone can look for new methods and for another class of incomplete block designs as considered in the paper.

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