Stochastic Inequalities for the Run Length of the EWMA Chart for Long-Memory Processes

Authors: Yarema Okhrin
– Department of Statistics, University of Augsburg,
Germany (yarema.okhrin@wiwi.uni-augsburg.de)

Wolfgang Schmid
– Department of Statistics, European University Viadrina,
Frankfurt (Oder),
Germany (schmid@europa-uni.de)

Abstract:

- In this paper the properties of the modified EWMA control chart for detecting changes in the mean of an ARFIMA process are discussed. The central question is related to the false alarm probability and its behavior for different autocorrelation structures and parameters of the underlying process. It is shown under which conditions the false alarm probability of an ARFIMA(p,d,q) process is larger than that of the pure ARFIMA(0,d,0) process. Furthermore, it is shown that the false alarm probability for ARFIMA(0,d,0) and ARFIMA(1,d,1) is monotonic in d for common parameter values of the processes.

Key-Words:

- *Statistical process control, EWMA control chart, long-memory process, ARFIMA process.*

AMS Subject Classification:

- 62L10, 62M10.

1. INTRODUCTION

Detection of structural changes in a time series is an important issue and is actively tackled in the current literature. The objective is to detect the change as soon as possible after it has occurred. The change is an indication that something important has happened and the characteristics of the process have drifted from the original values. This type of questions is of particular relevance in engineering, public health, finance, environmental sciences (see, e.g., Montgomery ([18]), Lawson and Kleinman ([15]), Frisén ([7])).

The most frequently applied surveillance technique is a control chart (e.g., Montgomery ([18])), with Shewhart, EWMA and CUSUM charts being the most popular ones. Initially they were developed for monitoring independent processes. Since in many applications the process of interest appears to be time dependent, several approaches evolved to extend the above schemes to time series. One approach relies on monitoring the residuals of the fitted time series process (e.g., Alwan and Roberts ([1]), Montgomery and Mastrangelo ([19]), Wardell et al. ([32]) and ([33]) , Lu and Reynolds ([16])). This makes, however, the interpretation of the signal given by the control scheme difficult. Furthermore, the estimated residuals are not independent after a change, implying that the use of the classical charts is still erroneous (cf. Knoth and Schmid ([14])). Alternatively, we can adjust the monitoring schemes to reflect the dependence structure of the analyzed processes. This type of charts are called modified charts. The modified CUSUM charts and the related generalized likelihood ratio tests were discussed, for example, in Nikiforov ([21]), Yashchin ([34]), Schmid ([28]), Knoth and Schmid ([14]), Capizzi and Masarotto ([5]), Knoth and Frisén ([13]). The extension of the EWMA chart to time series data was suggested by Schmid ([27]). Note that the derivation of modified schemes is technically tedious, since the autocorrelation structure of the process should by explicitly taken into account while determining the design parameters of the monitoring procedure. Furthermore, most of the literature with just a few exception considers the ARMA processes. These processes are of great importance in practice, but assume inherently a short memory in the underlying data. Recently, Rabyk and Schmid ([25]) considered several control charts for long memory processes and compared these schemes within an extensive Monte Carlo study.

It is desirable for any control scheme to give a signal as soon as possible after the change has occurred, i.e. the process is out-of-control, and to give a signal as rarely as possible if no change occurred, i.e. the process is in-control. False alarms deteriorate the surveillance procedures and lead to potentially misleading inferences by practitioners. The performance of the chart in the in-control state can be quantified by the false alarm probability up to a given time point or, equivalently, the probability that the run length of the chart is longer than a given time span. Of particular importance is the impact of the process parameters on this probability. For a general family of linear processes and particularly for ARMA processes several results on stochastic ordering of the run length can be found in Schmid and Okhrin ([29]) and Morais et al. ([20]). In these papers the authors derive constraints on the autocorrelations of the observed in-control process to guarantee stochastic monotonicity of the EWMA or more general monitoring schemes. The case of nonlinear time series was treated by Pawlak and Schmid ([24]) and Gonçalves et al. ([9]).

The subject of the analysis in this paper is the one-sided EWMA chart aimed to detect an increase of the mean. The one-sided problem is of key importance in many fields, such as engineering (loading capacity, tear strength, etc.), environmental sciences (high tide, concentration of particulate matter, ozone level, etc), economics and finance (riskiness of financial assets, interest and unemployment rates). In all these examples we are interested only in increases (or only in decreases) of the quantity of interest. Its dynamics can be assessed with the tools considered here.

In this paper we discuss the stochastic ordering for the false alarm probability of modified one-sided EWMA control charts aimed to detect a shift in the mean of an ARFIMA process. First, we show that for an arbitrary ARFIMA(p,d,q) process the probability of a false signal is always larger than this probability for an i.i.d. or an ARFIMA(0,d,0) processes. To guarantee this it suffices to assume that the autocorrelation of the underlying ARMA process is always non-negative. Second, we extend the above results by showing that the false alarm probabilities are non-decreasing functions in d for ARFIMA(0,d,0) and ARFIMA(1,d,1) under specific assumptions on the process parameters. These results are of great importance, since it is well known that the parameter of fractional differencing is difficult to estimate. Thus we indicate the consequences of under- or overestimation of d for monitoring procedures.

The paper is structured as follows. Section 2 summarizes relevant results on modified EWMA control charts and on ARFIMA processes. The main results together with numerical examples and counterexamples are given in Section 3. The proofs of some results are given in the appendix.

2. The MODIFIED EWMA CHART FOR ARFIMA PROCESSES

The aim of statistical process control is to detect structural deviations in a process over time. We assume that at each time point one observation is available. The given observations $x_1, x_2, ...$ are considered to be a realization of the actual (observed) process. The underlying target process is denoted by $\{Y_t\}$. The objective of a monitoring procedure is to give a signal if the target and the observed processes differ in their characteristics. A good procedure should give a signal as soon as possible if the processes differ and give a signal as rarely as possible if the processes coincide. In the following we analyze the behavior of the modified EWMA control chart for the mean in the in-control case, i.e. if no change is present. The underlying target process is assumed to be a long-memory process, which frequently encounters in applications, for example stock marker risk in finance or environmental data (see Andersen et al. ([2]), Pan and Chen ([23])). The objective of the paper is to analyze the performance of in-control EWMA charts for different memory patterns of the long memory processes. First we introduce the control scheme and subsequently discuss the process and its features in detail.

2.1. The Modified EWMA Chart

The exponentially weighted moving average (EWMA) chart was introduced by Roberts ([26]). Contrary to the Shewhart chart all previous observations are taken into account for the decision rule. It turns out to perform better than the Shewhart control chart for detecting small and moderate shifts, namely, in the process mean (Lucas and Saccucci ([17])) or in the process variance (Crowder and Hamilton ([6])) of an independent output. An extension of the EWMA control chart to time series was given by Schmid ([27]).

The EWMA chart for monitoring the process mean is based on the statistic

$$Z_t = (1-\lambda)Z_{t-1} + \lambda X_t,\ \ t \geq 1. \tag{2.1}$$

Z_0 is the starting value. Here we choose it equal to the mean of the target process, i.e. $Z_0 = E(Y_t) = \mu_0$. The process starts in zero state, a head-start is not considered. The parameter $\lambda \in (0, 1]$ is a smoothing constant determining the influence of past observations.

The quantity Z_t can be written as weighted average

$$Z_t = \lambda \sum_{i=0}^{t-1} (1-\lambda)^i X_{t-i} + (1-\lambda)^t Z_0, \quad t = 1, 2, \ldots, \tag{2.2}$$

whose weights decrease geometrically. This shows that if λ is close to one then we have a short memory EWMA chart while for λ close to zero the preceding values get a larger weight. For $\lambda = 1$ the EWMA chart reduces to the Shewhart control chart.

Further $\{Y_t\}$ is assumed to be a stationary process with mean μ_0 and autocovariance function $\gamma(k)$. Then (see Schmid ([27])) $E(Z_t) = \mu_0$ and

$$Var(Z_t) = \lambda^2 \sum_{|i| \leq t-1} \gamma(i) \sum_{j=\max\{0,-i\}}^{\min\{t-1, t-1-i\}} (1-\lambda)^{2j+i} = \sigma^2_{e,t}.$$

In this paper we consider a one-sided EWMA chart. Our aim is to detect an increase of the mean. The process is concluded to be out of control at time point t if

$$Z_t > \mu_0 + c\sqrt{Var(Z_t)}$$

with $c > 0$. The run length of the EWMA control chart is given by

$$N_e = \inf\left\{t \in \mathbb{N}: \ Z_t > \mu_0 + c\sqrt{Var(Z_t)}\ \right\}.$$

Stochastic inequalities for the modified EWMA chart have been given in Schmid and Schöne ([30]) and Schöne et al. ([31]). Assuming that $\{Y_t\}$ is a stationary Gaussian process with non-negative autocovariances Schmid and Schöne ([30]) showed that the probability of a false signal up to a certain time point k, i.e. $P(N_e > k)$ is greater or equal to that in the i.i.d. case. Thus the dependence structure leads to an increase of the false alarm probability. Demanding further assumptions on the autocovariances Schöne et al. ([31]) proved that the false alarm probability is an increasing function in the autocorrelations provided that they satisfy a certain monotonicity condition.

2.2. The Target Process

Throughout this paper the target process is assumed to be a stationary autoregressive fractionally integrated moving average (ARFIMA) process. In many applications we are faced with processes having a long memory. The frequently applied autoregressive moving average (ARMA) modeling is not suitable in such a situation as its autocorrelation structure is geometrically decreasing.

Let L denote the lag operator, i.e. $LY_t = Y_{t-1}$ and let $\Delta = 1 - L$ be the difference operator, i.e. $\Delta Y_t = Y_t - Y_{t-1}$. In the study of non-stationary time series more generalized ARIMA(p,d,q) models are often used (cf. Box et al. ([3])). They make use of a d-multiple difference operator Δ to the original time series Y_t where d is a non-negative integer. In the approach of Granger and Joyeux ([10]) and Granger ([11]), however, d is a real number.

Let $d > -1$ then Granger and Joyeux ([10]) and Granger ([11]) define Δ^d using the binomial expansion as

$$\Delta^d = (1-L)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-1)^k L^k = \sum_{k=0}^{\infty} \frac{\Gamma(k-d)}{\Gamma(-d)\Gamma(k+1)} L^k,$$

where $\Gamma(\cdot)$ is the gamma function. Let

$$A(L) = 1 - \alpha_1 L - \ldots - \alpha_p L^p, \quad B(L) = 1 + \beta_1 L + \ldots + \beta_q L^q$$

and $\{\varepsilon_t\}$ be a white noise process, i.e.

$$E(\varepsilon_t) = 0,\ Var(\varepsilon_t) = \sigma^2,\ Cov(\varepsilon_t, \varepsilon_s) = 0 \ \ \forall\ t \neq s.$$

Now $\{Y_t\}$ is said to be an autoregressive fractionally integrated moving average process of order (p, d, q) (ARFIMA(p,d,q)) if $\{Y_t\}$ is stationary and satisfies the equation

$$A(L)\Delta^d(Y_t - \mu_0) = B(L)\varepsilon_t \tag{2.3}$$

for $d \in (-0.5, 0.5)$.

The condition on the existence and uniqueness of a stationary solution of an ARFIMA process is given in the following theorem.

Theorem 2.1. *Suppose that $\{Y_t\}$ is an ARFIMA(p,d,q) process as defined in (3). Let $d \in (-0.5, 0.5)$ and $A(\cdot)$ and $B(\cdot)$ have no common zeroes.*

a) *If $A(z) \neq 0$ for $|z| = 1$ then there is a unique purely nondeterministic stationary solution of (3) given by*

$$Y_t = \mu_0 + \sum_{j=-\infty}^{\infty} \psi_j \Delta^{-d} \varepsilon_{t-j}, \tag{2.4}$$

where $\psi(z) = \sum_{j=-\infty}^{\infty} \psi_j z^j = B(z)/A(z)$.

b) *The solution $\{Y_t\}$ is causal if and only if $A(z) \neq 0$ for $|z| \leq 1$.*

Proof: Brockwell and Davis ([4]) □

The parameter of fractional differencing d determines the strength of the process memory. Since $\rho(k) \sim ck^{2d-1}$ as $k \to \infty$ with $c \neq 0$ ARFIMA processes have a long memory for $d \in (0, 0.5)$. For $d \in (-0.5, 0)$ the process is called to have an intermediate memory. ARMA processes are referred to as short memory processes since $|\rho(k)| \leq Cr^{-k}$ for $k = 0, 1, ..$ with $C > 0$ and $0 < r < 1$.

The knowledge of the autocovariance function of an ARFIMA(p,d,q) process is crucial for the application of the monitoring techniques discussed in this paper and their theoretical properties. Nevertheless, it is difficult to obtain explicit formulas for the autocovariance and the autocorrelation functions. The autocovariance function of an ARFIMA(0,d,0) process was derived by Hosking ([12]). It holds that (see, e.g., Brockwell and Davis ([4], Theorem 13.2.1))

$$\gamma_d(0) = \sigma^2 \frac{\Gamma(1-2d)}{(\Gamma(1-d))^2}, \quad \gamma_d(k) = \gamma_d(0)\, \rho_d(k) \qquad k \in \mathbb{Z} \tag{2.5}$$

where

$$\begin{aligned} \rho_d(k) &= \frac{\Gamma(k+d)\Gamma(1-d)}{\Gamma(k-d+1)\Gamma(d)} \\ &= \prod_{i=1}^{k} \frac{i-1+d}{i-d} = \prod_{i=1}^{k}\left(1 - \frac{1-2d}{i-d}\right), \quad k = 1, 2, ... \end{aligned} \tag{2.6}$$

and $\rho_d(-k) = \rho_d(k)$.

To determine the autocovariance function of a general ARFIMA process it is convenient to deploy the splitting method. This method is based on the decomposition of the ARFIMA model into its ARMA and its fractionally integrated parts. Let $\gamma_{ARMA}(\cdot)$ be the autocovariance of the ARMA component which has a unit variance white noise and let $\gamma_d(\cdot)$ denote the autocovariance of the ARFIMA(0,d,0) process given by (2.5) and (2.6). If the conditions of Theorem 1a) are satisfied then the autocovariance of the corresponding ARFIMA process is given by the convolution of these two functions (see, e.g., Palma ([22]), Brockwell and Davis ([4], p.525, (13.2.19))

$$\gamma(k) = \sum_{i=-\infty}^{\infty} \gamma_d(i)\gamma_{ARMA}(k-i). \tag{2.7}$$

This result is obvious since $\{Y_t\}$ is an ARFIMA(p,d,q) process if and only if $\Delta^d(Y_t - \mu_0)$ is an ARMA(p,q) process.

In the following we shall frequently consider processes with $\gamma_{ARMA}(i) \geq 0$ for all $i \geq 1$, $d \in (0, 0.5)$ and $\sigma > 0$. Then it holds that

$$\gamma(k) \geq \gamma_d(k)\gamma_{ARMA}(0) > 0 \tag{2.8}$$

since $\gamma_d(k) > 0$ and $\gamma_{ARMA}(0) > 0$. Thus the autocovariance function of a general ARFIMA process is strictly positive for all k, if the autocovariances of the underlying ARMA process are non-negative.

Next we consider the special case of an ARFIMA(1,d,1) process and use the simplified notation $\alpha = \alpha_1$ and $\beta = \beta_1$. It holds that its autocovariance function is

$$\gamma(k) = \sigma^2 \sum_{i=-\infty}^{\infty} \frac{\Gamma(1-2d)}{\Gamma(d)\Gamma(1-d)} \frac{\Gamma(i+d)}{\Gamma(1+i-d)} \gamma_{ARMA}(k-i), \tag{2.9}$$

where $\gamma_{ARMA}(k)$ is the autocovariance of the ARMA(1,1) process, i.e.

$$\gamma_{ARMA}(k) = \begin{cases} \frac{1+2\alpha\beta+\beta^2}{1-\alpha^2} & \text{for } k = 0 \\ \frac{(1+\alpha\beta)(\alpha+\beta)}{1-\alpha^2}\alpha^{|k|-1} & \text{for } k \neq 0 \end{cases}.$$

3. MONOTONICITY RESULTS FOR THE MODIFIED EWMA MEAN CHART

In the following we consider the probability of a false signal assuming that the underlying process is an ARFIMA process. It is always assumed that the process is in control. We use the notation $P_{(p,d,q)}$ to denote that the probability is calculated assuming that the underlying process is an ARFIMA(p,d,q) process.

Note that for $P_{(0,0,0)}$ the in-control process is assumed to be independent and identically distributed random sequence and for $P_{(0,d,0)}$ it is a pure ARFIMA(0,d,0) process.

In this section it is always demanded that the variance of the white noise σ^2 is positive. In the case $\sigma = 0$ the process $\{Y_t\}$ is deterministic, $N_e = \infty$ and Theorem 2, Lemma 1, Theorem 3 and Theorem 4 hold without any further assumption.

3.1. Influence of the ARMA Structure on the False Alarm Probability

First, it is proved that for an ARFIMA(p,d,q) process the probability of a false signal up to a fixed time point k is always greater equal to the corresponding probability for an independent random process. This result is an immediate consequence of Schmid and Schöne ([30]).

Theorem 3.1. *Let $\{Y_t\}$ be an ARFIMA(p,d,q) process as defined in (3) with $d \in [0, 0.5)$ and let $\{\varepsilon_t\}$ be a Gaussian white noise with $\sigma > 0$. Let $A(z) \neq 0$ for $|z| = 1$ and let $\gamma_{ARMA}(v) \geq 0$ for all $v \in \mathbb{Z}$. Then*

$$P_{(p,d,q)}(N_e(c) > k) \geq P_{(0,0,0)}(N_e(c) > k), \quad k = 0, 1, 2, ...$$

Proof: Because of (2.7) we get that $\gamma(v) \geq 0$ for all v. Since $\{Y_t\}$ is a Gaussian process the result is an immediate consequence of Theorem 1 of Schmid and Schöne ([30]). □

This result gives a lower bound for the probability of a false signal. The bound itself is the probability of a false signal for an i.i.d. random sequence.

One of the crucial assumptions of Theorem 1 of Schmid and Schöne ([30]) is that all autocovariances are non-negative. Here we illustrate that the inequality may not hold for negative autocovariances. The left picture in Figure 1 shows the autocorrelations up to lag 5 of an ARFIMA(0,d,0) process. We see that for positive d's the autocorrelations are large and positive, but they become negative and small for negative d's. The right hand side picture shows the probability of no signal up to the time point k. The probabilities are computed by numerical integration of the $k+1$-dimensional normal density with the covariance matrix determined using Lemma 1 of Schmid and Schöne ([30]). The integration utilizes the Genz-Bretz algorithm (see Genz ([8])). The solid black line stands for the case $d = 0$ and thus for the i.i.d. process. Positive d's (grey lines) induce positive autocovariances, fulfill the assumptions of Theorem 2 and lead to probabilities larger than for the i.i.d. case. However, the negative autocorrelations

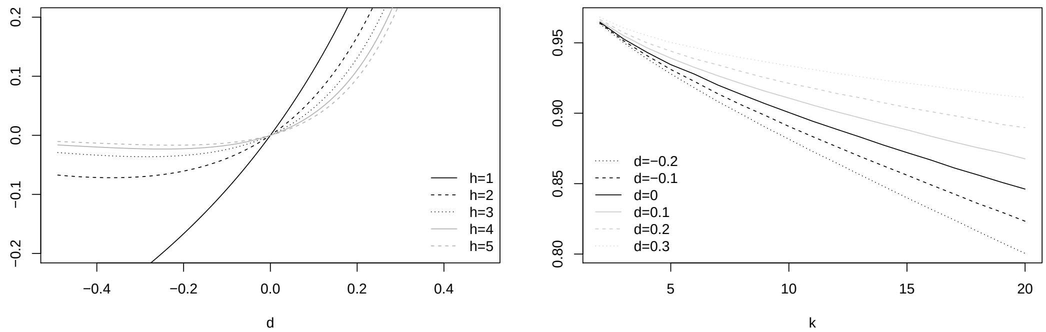

Figure 1: The first five autocorrelations $\rho_d(h)$ (left side) and the probability of no signal up to the time point k (right side) for the modified EWMA chart with $\lambda = 0.1$ and $c = 2.04$ applied to an ARFIMA(0,d,0) process.

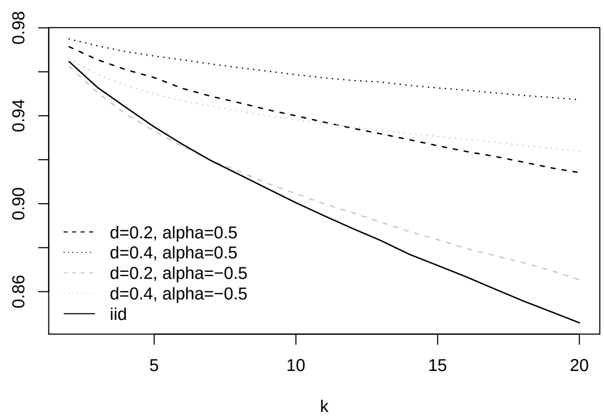

Figure 2: The probability of no signal up to the time point k for the modified EWMA chart with $\lambda = 0.1$ and $c = 2.04$ applied to an ARFIMA(1,d,0) process.

for $d < 0$ (black lines) lead to smaller probabilities of no signal. Thus we have a counterexample for the case if the assumptions of Theorem 2 are not satisfied.

It is important to note that in the case $\gamma_{ARMA}(v) \leq 0$ for all $v \in \mathbb{Z}$ the inequality in Theorem 2 does not with the reversed inequality. This is illustrated in Figure 2. It is shown that for an ARFIMA(1,d,0) process with $d = 0.2$ and $\alpha = -0.5$ the probability of no signal up to time point k is sometimes larger and sometimes smaller than that of the i.i.d. case depending on the choice of k.

Next, we try to improve the lower bound. It is analyzed for which ARFIMA(p,d,q) processes it can be replaced by the probability of a false signal of an ARFIMA(0,d,0) process.

Let $\{\rho_{ARMA}(h)\}$ denote the autocorrelation function of an ARMA(p,q) process and let $\{\rho_d(h)\}$ denote the autocorrelation function of an ARFIMA(0,d,0) process.

Let $k \in I\!N$, $1 \leq v \leq k-1$ and

$$\begin{aligned} I_v &= \sum_{i=1}^{v-1} i\ \rho_{ARMA}(i)\ (\frac{\rho_d(v-1+i)}{v+i-d} - \frac{\rho_d(v-1-i)}{v-i-d}), \\ II_v &= v\ \rho_{ARMA}(v) \left(\frac{\rho_d(2v-1)}{2v-d} + \frac{1}{1-d}\right), \\ III_v &= \sum_{i=v+1}^{\infty} i\ \rho_{ARMA}(i) \left(\frac{\rho_d(i-v)}{i-v+1-d} + \frac{\rho_d(i+v-1)}{v+i-d}\right). \end{aligned}$$

Lemma 3.1. *Let $k \in I\!N$ and let $\{Y_t\}$ be an ARFIMA(p,d,q) process as defined in (3) with $d \in [0, 0.5)$ and let $\{\varepsilon_t\}$ be a Gaussian white noise with $\sigma > 0$. Let $A(z) \neq 0$ for $|z| = 1$ and $\gamma_{ARMA}(v) \geq 0$ for all v. If additionally*

$$I_v + II_v + III_v \geq 0\ , \qquad v = 1, .., k-1 \tag{3.1}$$

then $P_{(p,d,q)}(N_e(c) > k) \geq P_{(0,d,0)}(N_e(c) > k)$.

Proof: see Appendix. □

Keeping in mind the conditions of Lemma 1, it can be seen that II_v and III_v are non-negative while I_v is non-positive because

$$\frac{\rho_d(v-1+i)}{v+i-d} \leq \frac{\rho_d(v-1+i)}{v-i-d} \leq \frac{\rho_d(v-1-i)}{v-i-d}$$

using Lemma 3c of the Appendix. Thus it is not clear for which processes this condition is fulfilled at all. Next the condition (3.1) is analyzed for various processes.

Lemma 3.2. *Suppose that the conditions of Lemma 1 are fulfilled. Then it holds that:*

a) *For $k = 2$ condition (3.1) is always fulfilled.*

b) *For $k = 3$ condition (3.1) is satisfied if $2\rho_{ARMA}(2) \geq \rho_{ARMA}(1)$.*

c) *Let $k \geq 2$. If $\{Y_t\}$ is an ARFIMA(1,d,0) process with autoregressive coefficient $\alpha \in [0, 1)$ and $\alpha \geq (k-2)/(k-1)$ then condition (3.1) is satisfied.*

Proof: see Appendix. □

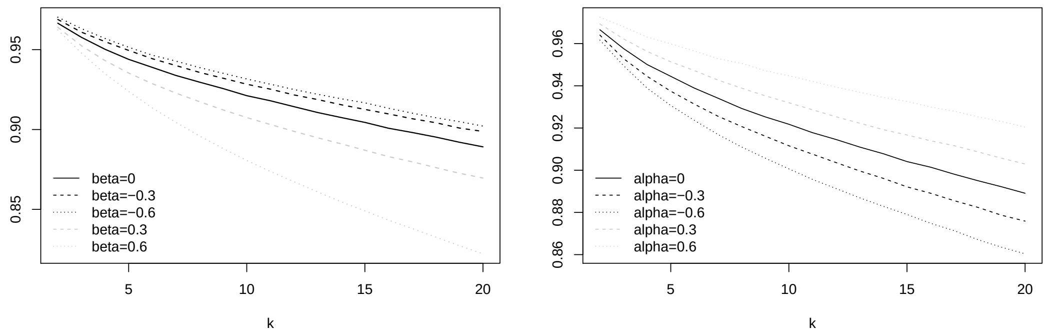

Figure 3: The probability of no signal up to the time point k for the EWMA control chart with $\lambda = 0.1$ and $c = 2.04$. The target process is an ARFIMA(0,d,1) process with $d = 0.2$ (left side) and an ARFIMA(1,d,0) process with $d = 0.2$ (right side), respectively.

Note that from Lemma 1 is an extension of Theorem 2. It shows that the presence of the ARMA part with specific parameters (see Lemma 2) leads to an increase of the probability of no signal. The condition in b) does not hold for an ARFIMA(0,d,1) process with positive coefficient. As it is shown in the left picture in Figure 3 for $k = 3$ the probabilities of no signal are larger for the ARFIMA(0,d,1) process than for ARFIMA(0,d,0) for negative β's and smaller for positive β's. The same holds, however, also for higher values of k too, i.e. no signals for longer time intervals.

The case of ARFIMA(1,d,0) is particularly important from practical perspective. The right picture of Figure 3 reveals a similar pattern as we observed for ARFIMA(0,d,1). The probabilities of no signal are larger for the ARFIMA(1,d,0) process than for ARFIMA(0,d,0) for positive α's and smaller for negative α's. However, part c) of Lemma 2 contains an additional constraint which makes the set where (3.1) holds very small. It stems from a statement about the magnitude of a hypergeometric function in α which is hard to obtain. Nevertheless, numerically we can argue that the monotonicity also holds for $0 \leq \alpha \leq (k-2)/(k-1)$. For the left picture in Figure 4 the selected α's are small and satisfy $\alpha \leq (k-2)/(k-1)$ for $k \geq 3$. Despite the condition in part c) of Lemma 2 is not fulfilled, we observe that $P_{(1,d,0)}(N_e(c) > k) \geq P_{(0,d,0)}(N_e(c) > k)$ still holds. As a counterexample consider an ARFIMA(1,d,1) process with $d = 2$ and $\beta = -0.8$ and the right picture in Figure 4. The probability of no signal up to time point k is sometimes larger for $\alpha = 0.6$ than for $\alpha = 0.0$, sometimes smaller. This depends on the value of k. The probabilities for the discussed figures are determined by numerical integration as above.

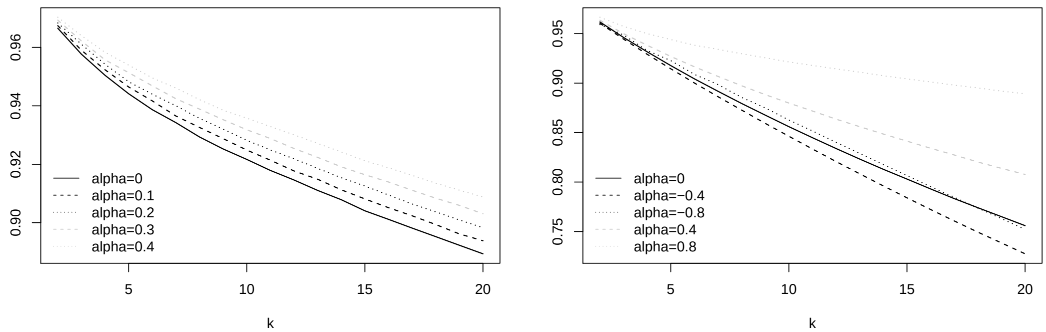

Figure 4: The probability of no signal up to the time point k for the EWMA control chart with $\lambda = 0.1$ and $c = 2.04$. The target process is an ARFIMA(1,d,0) process with $d = 0.2$ (left side) and an ARFIMA(1,d,1) process with $d = 0.2$ and $\beta = -0.8$ (right side), respectively.

3.2. Behavior of the False Alarm Probability as a Function of the Fractional Parameter

In the previous subsection the probability of a false signal for an ARFIMA process was compared with that of an independent random process and an ARFIMA(0,d,0) process, respectively. In this subsection we want to analyze how the probability of a false signal behaves as a function of the fractional parameter d.

In a first stage we consider the simplest case, an ARFIMA(0,d,0) process.

Theorem 3.2. *Let $\{Y_t\}$ be an ARFIMA(0,d,0) process as defined in (3) with $d \in [0, 0.5)$ and let $\{\varepsilon_t\}$ be a Gaussian white noise with $\sigma > 0$. Then $P_{(0,d,0)}(N_e(c) > k)$ is a non-decreasing function in d.*

Proof: First we observe that the autocovariances of an ARFIMA(0,d,0) process can be easily recursively calculated. It holds that $\gamma_d(k) = \frac{k+d-1}{k-d}\gamma_d(k-1)$ for $k \geq 1$ (cf. Lemma 3b of the appendix).

Next let $0 < d_1 < d_2 < 1/2$. Then it holds for $k \geq 1$ that

$$\frac{\gamma_{d_2}(k)}{\gamma_{d_2}(k-1)} = \frac{k + d_2 - 1}{k - d_2} \geq \frac{k + d_1 - 1}{k - d_1} = \frac{\gamma_{d_1}(k)}{\gamma_{d_1}(k-1)}.$$

The result follows with Theorem 1 of Schöne et al. ([31]).

If $d_1 = 0$ the result is a special case of Schmid and Schöne ([30]). □

Figure 5 illustrates the result of Theorem 3. It shows that the probabilities of no signal up to the time point k are increasing in d.

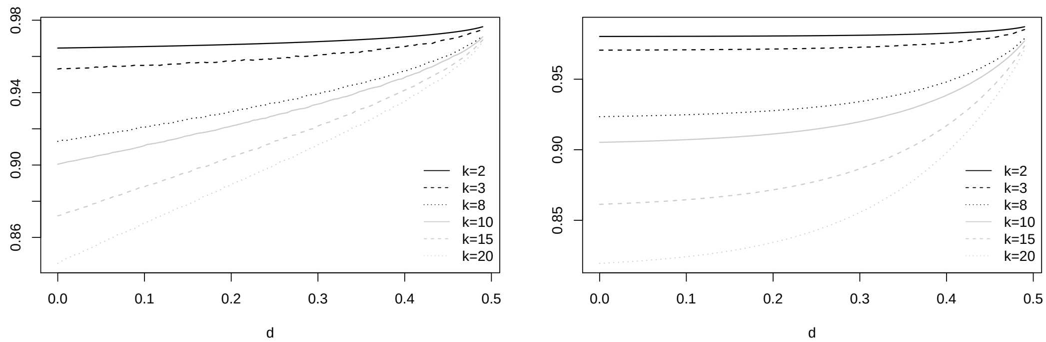

Figure 5: Probabilities of no signal up to the time point k as a function of d. The target process is an ARFIMA(0,d,0) process. The EWMA parameters are $c = 2.04$, $\lambda = 0.1$ (left), and $c = 2.33$, $\lambda = 1.0$ (right).

Next we want to study the behavior of the false alarm probability for an ARFIMA(1,d,1) process.

Theorem 3.3. *Let $\{Y_t\}$ be an ARFIMA(1,d,1) process as defined in (3) with $d \in [0, 0.5)$ and let $\{\varepsilon_t\}$ be a Gaussian white noise with $\sigma > 0$. Suppose that $0 \leq \alpha < 1$ and $\beta \geq 0$. Then it follows that for all $k \in I\!N$ the quantity $P_{(1,d,1)}(N_e(c) > k)$ is a non-decreasing function in d.*

Proof: see Appendix. □

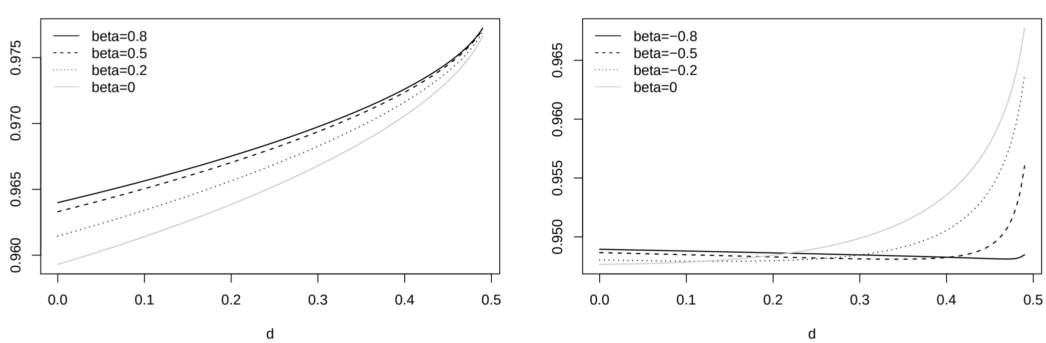

Figure 6: Probabilities of no signal up to the time point k as a function of d for ARFIMA(1,d,1) processes. We set $\alpha = 0.4$ for the left figure, $\alpha = -0.8$ for the right one and choose $c = 2.04$, $\lambda = 0.1$.

This result is quite remarkable. It says that the probability of a false signal is increasing with the fractional parameter d for positive parameters α and β. In the left plot of Figure 6 we visualize this effect for $\alpha = 0.4$ and several values of the MA parameter. On the right hand side figure we show a counterexample of nonmonotonicity if the assumptions of the theorem are not fulfilled. Here α is set equal to -0.8.

4. SUMMARY

In this paper we consider the stochastic properties of the run length of an EWMA monitoring scheme if applied to an ARFIMA process. Particularly we are interested in the monotonic behavior of the probability of no signal up to an arbitrary time point as a function of the fractional differencing parameter d. We compare the probability of no signal for an ARFIMA(p,d,q) process with non-negative autocovariances with the probability of no signal for a sequence of i.i.d. variables and for an ARFIMA(0,d,0) process, respectively. It is analyzed under what conditions the probability of no signal of an ARFIMA(p,d,q) process is greater or equal to that of an i.i.d. sequence and of an ARFIMA(0,d,0) process. Furthermore, we prove that for ARFIMA(0,d,0) and ARFIMA(1,d,1) with positive parameters the probability of no signal is increasing in d.

5. APPENDIX

In the following lemma some useful properties on the behavior of the autocorrelation function $\rho_d(k)$ of an ARFIMA(0,d,0) process are summarized which will be used in the proofs.

Lemma 5.1. *Suppose that $\{Y_t\}$ is an ARFIMA(0,d,0) process with $d \in (-0.5, 0.5)$.*

a) *Let $k \in \mathbb{N}$. Then $\rho_d(k) > 0$ for $d > 0$, $\rho_d(k) = 0$ for $d = 0$, and $\rho_d(k) < 0$ for $d < 0$.*

b)

$$\rho_d(k) = \frac{k-1+d}{k-d}\rho_d(k-1) = \left(1 - \frac{1-2d}{k-d}\right)\rho_d(k-1), \qquad k = 1, 2, \ldots. \tag{5.1}$$

c) *Let $0 \leq d < 0.5$. Then it holds that $\rho_d(k)$ is a non-increasing function in k.*

d) *Let $k \in \mathbb{N} \cup \{0\}$. Then*

$$\rho_d'(k) = \rho_d(k)\ A_d(k)$$

with

$$A_d(k) = \sum_{i=1}^{k}\left(\frac{1}{i-1+d} + \frac{1}{i-d}\right).$$

e) *Let $0 \le d < 0.5$. Then $\rho_d(k)$ is a non-decreasing function in d.*

Proof: Parts a) and b) are obvious.
c) The statement follows from the fact that $\frac{1-2d}{k-d} \ge 0$, since $d \in [0, 0.5)$ and $k \ge 1$.
d) Since

$$log(\rho_d(k) = \sum_{i=1}^{k}(log(i-1+d) - log(i-d))$$

the result follows by building the derivative with respect to d.
e) Follows from d). □

Proof of Lemma 1:

Proof: Note that for $d = 0$ Lemma 1 reduces to Theorem 2 which was already proved. Thus it can be assumed that $d > 0$ in the following. In order to prove Lemma 1 we apply Theorem 1 of Schöne et al. ([31]) and the comment after the theorem. In order to do that we need that $\gamma(v) > 0$ for $v = 1, .., k-1$ but this was already proved in (2.8).

Following Theorem 1 of Schöne et al. ([31]) it holds that $P_{(p,d,q)}(N_e(c) > k) \ge P_{(0,d,0)}(N_e(c) > k)$ if all autocovariances are positive and if for all $1 \le v \le k-1$ it holds that

$$\gamma_d(v-1) \sum_{i=-\infty}^{\infty} \gamma_d(v-i)\gamma_{ARMA}(i) \ge \gamma_d(v) \sum_{i=-\infty}^{\infty} \gamma_d(v-1-i)\gamma_{ARMA}(i).$$

This condition is equivalent to

$$\sum_{i=-\infty}^{\infty} \rho_{ARMA}(i)(\rho_d(v-i)\rho_d(v-1) - \rho_d(v)\rho_d(v-1-i)) \ge 0, \quad v = 1, .., k-1$$

and

$$\sum_{i=1}^{\infty} \rho_{ARMA}(i)((\rho_d(v-i)+\rho_d(v+i))\rho_d(v-1)-(\rho_d(v-1-i)+\rho_d(v-1+i))\rho_d(v)) \ge 0,$$

for $v = 1, .., k-1$ and

$$\begin{aligned} &\sum_{i=1}^{v-1} \rho_{ARMA}(i)\left[(\rho_d(v-i) + \rho_d(v+i))\rho_d(v-1) - (\rho_d(v-1-i) + \rho_d(v-1+i))\rho_d(v)\right] \\ &+ \rho_{ARMA}(v)(1 + \rho_d(2v))\rho_d(v-1) - [\rho_d(1) + \rho_d(2v-1)]\rho_d(v)) \\ &+ \sum_{i=v+1}^{\infty} \rho_{ARMA}(i)\left[(\rho_d(i-v) + \rho_d(i+v))\rho_d(v-1) - (\rho_d(i-v+1) + \rho_d(i+v-1))\rho_d(v)\right] \ge 0 \end{aligned}$$

for $v = 1, ..., k-1$ respectively. Using the recursion property of the autocorrelations of an ARFIMA(0,d,0) process (cf. (Lemma 3b of the Appendix) the last condition can be rewritten as follows

$$\text{(5.2)} \qquad \frac{\rho_d(v-1)(1-2d)}{v-d} \; [I_v + II_v + III_v] \geq 0, \qquad v = 1, .., k-1$$

with

$$\begin{aligned} I_v &= \sum_{i=1}^{v-1} i \; \rho_{ARMA}(i) \; (\frac{\rho_d(v-1+i)}{v+i-d} - \frac{\rho_d(v-1-i)}{v-i-d}), \\ II_v &= v \; \rho_{ARMA}(v) \left(\frac{\rho_d(2v-1)}{2v-d} + \frac{1}{1-d} \right), \\ III_v &= \sum_{i=v+1}^{\infty} i \; \rho_{ARMA}(i) \; \left(\frac{\rho_d(i-v)}{i-v+1-d} + \frac{\rho_d(i+v-1)}{v+i-d} \right). \end{aligned}$$

□

Proof of Lemma 2:

Proof: a) The proof is obvious.
b) Since

$$I_2 = -\rho_{ARMA}(1) \frac{6}{(3-d)(2-d)}, \quad II_2 = 24 \rho_{ARMA}(2) \frac{d^2 - 2d + 2}{(4-d)(3-d)(2-d)(1-d)}$$

it holds that $I_2 + II_2 \geq 0$ if $\rho_{ARMA}(1) \leq 2\rho_{ARMA}(2)$.
c) For an ARFIMA(1,d,0) process with coefficient α it holds that

$$\begin{aligned} I_v &= \sum_{i=v}^{2v-2} \frac{\rho_d(i)}{i+1-d}(i-v+1)\alpha^{i-v+1} - \sum_{i=0}^{v-2} \frac{\rho_d(i)}{i+1-d}(v-1-i)\alpha^{v-1-i}, \\ III_v &= \sum_{i=0}^{v-2} \frac{\rho_d(i)}{i+1-d}(i+v)\alpha^{i+v} + \sum_{i=v-1}^{\infty} \frac{\rho_d(i)}{i+1-d}(i+v)\alpha^{i+v} \\ &+ \sum_{i=2v}^{\infty} \frac{\rho_d(i)}{i+1-d}(i-v+1)\alpha^{i-v+1}. \end{aligned}$$

Consequently

$$\begin{aligned} I_v + II_v + III_v \;=\; & \alpha \left[\sum_{i=0}^{v-2} \frac{\rho_d(i)}{i+1-d} \left((i+v)\alpha^{i+v-1} - (v-1-i)\alpha^{v-2-i} \right) \right. \\ & + \sum_{i=v}^{2v-1} \frac{\rho_d(i)}{i+1-d}(i-v+1)\alpha^{i-v} + \sum_{i=2v}^{\infty} \frac{\rho_d(i)}{i+1-d}(i-v+1)\alpha^{i-v} \\ & \left. + \sum_{i=v-1}^{\infty} \frac{\rho_d(i)}{i+1-d}(i+v)\alpha^{i+v-1} \right]. \end{aligned}$$

This quantity is non-negative if $(i+v)\alpha^{i+v-1} - (v-1-i)\alpha^{v-2-i} \geq 0$ for $i = 0,..,v-2$. This is fulfilled if $\alpha \geq \left(1 - \frac{2i+1}{i+v}\right)^{1/(2i+1)}$ for all $i = 0,..,v-2$ since $\left(1 - \frac{2i+1}{i+v}\right)^{1/(2i+1)} \leq 1 - \frac{1}{v}$. Using mathematical induction we shall prove that $(1-\frac{1}{v})^{2i+1} \geq 1 - \frac{2i+1}{i+v}$ for all $i = 0,..,v-2$. For $i = 0$ it is obvious. Next we consider the induction step. Note that

$$\begin{aligned} (1-\frac{1}{v})^{2i+3} \;\geq\; & (1 - \frac{2i+1}{i+v})(1-\frac{1}{v})^2 \\ =\; & 1 - \left(\frac{2i+1}{i+v} + \frac{2v-1}{v^2} - \frac{(2v-1)(2i+1)}{v^2(i+v)} \right) \\ \geq\; & 1 - \frac{2i+3}{i+1+v} \end{aligned}$$

since after some calculations it can be seen that the last inequality is equivalent to $i^2 + 2i + 1 \geq 0$.

Since $v \leq k-1$ we finally get that $\alpha \geq (k-2)/(k-1)$ for $k \geq 2$.

□

Proof of Theorem 4:

Proof: Note that

$$\rho_{ARMA}(k) = \alpha^{k-1}\rho_1, \quad k \geq 1, \quad \rho_1 = \frac{(1+\alpha\beta)(\alpha+\beta)}{1+2\alpha\beta+\beta^2}.$$

Since $0 \leq \alpha < 1$ and $\beta \geq 0$ it follows that $\rho_1 \geq \alpha$.

In Theorem 2 it was proved that for an ARFIMA process the in-control probability of a false signal up to a given time point is greater or equal than for an independent random sequence. Thus we may assume in the following that $d > 0$. As shown in the proof of Lemma 1 this implies that $\gamma(v) > 0$.

a) Let $\{\gamma(h)\}$ denote the autocovariance function of an ARFIMA(1,d,1) process. In order to prove the result we make use of the comment after Theorem 1 of Schöne et al. ([31]) which says that it is sufficient to show that $\gamma(k)/\gamma(k-1)$ is a non-decreasing function in d. Suppose that $\alpha > 0$. Let $\rho^* = \rho_1/\alpha$. Then

$$\begin{aligned}\frac{\gamma(k)}{\gamma_{ARMA}(0)\gamma_d(0)} &= \sum_{i=-\infty}^{\infty} \rho_d(i)\rho_{ARMA}(k-i) = a_d(k) + b_d(k)\\ &= (1-\rho^*)\rho_d(k) + \alpha\, a_d^*(k-1) + \frac{1}{\alpha} b_d(k-1).\end{aligned}$$

with

$$a_d(k) = \sum_{i=-\infty}^{k} \rho_d(i)\rho_{ARMA}(k-i) = (1-\rho^*)\rho_d(k) + a_d^*(k) = \rho_d(k) + \alpha a_d^*(k-1),$$

$$b_d(k) = \sum_{i=k+1}^{\infty} \rho_d(i)\rho_{ARMA}(k-i) = \rho^* \sum_{i=k+1}^{\infty} \rho_d(i)\alpha^{i-k} = \frac{1}{\alpha} b_d(k-1) - \rho^*\rho_d(k).$$

$$a_d^*(k) = \rho^* \sum_{i=-\infty}^{k} \rho_d(i)\alpha^{k-i}.$$

b) The numerator of the derivative of $\gamma(k)/\gamma(k-1)$ with respect to d is equal to

$$\begin{aligned}&(\frac{1}{\alpha} - \alpha)\ \left[a_d^*(k-1)b_d'(k-1) - a_d^{*\prime}(k-1)b_d(k-1)\right]\\ &+(1-\rho^*)\ a_d^{*\prime}(k-1)\ [\alpha\rho_d(k-1) - \rho_d(k)] + (1-\rho^*)\ a_d^*(k-1)\ [-\alpha\rho_d'(k-1) + \rho_d'(k)]\\ &+(1-\rho^*)\ b_d'(k-1)\ [\frac{1}{\alpha}\rho_d(k-1) - \rho_d(k)] + (1-\rho^*) b_d(k-1)[-\frac{1}{\alpha}\rho_d'(k-1) + \rho_d'(k)]\\ &+(1-\rho^*)^2[\rho_d(k-1)\rho_d'(k) - \rho_d'(k-1)\rho_d(k)].\end{aligned}$$

It is sufficient to prove that this quantity is not negative.

Let $\gamma = (1-\rho^*)/(1/\alpha - \alpha)$. Note that $\gamma \leq 0$. An equivalent representation is

$$\begin{aligned}&\left[a_d^*(k-1)b_d'(k-1) - a_d^{*\prime}(k-1)b_d(k-1)\right]\\ &+\ \gamma\ \alpha\ [\rho_d(k-1)a_d^{*\prime}(k-1) - \rho_d'(k-1)a_d^*(k-1)]\\ &+\ \gamma\ [\rho_d'(k)a_d^*(k-1) - \rho_d(k)a_d^{*\prime}(k-1)] + \frac{\gamma}{\alpha}[\rho_d(k-1)b_d'(k-1) - \rho_d'(k-1)b_d(k-1)]\\ &+\ \gamma[\rho_d'(k)b_d(k-1) - \rho_d(k)b_d'(k-1)] + \gamma(1-\rho^*)[\rho_d(k-1)\rho_d'(k) - \rho_d'(k-1)\rho_d(k)]\\ &=\ I + II + III + IV + V + VI.\end{aligned}$$

c) Next we apply Lemma 3d. Defining $A_d(-h) = A_d(h)$ for $h \geq 1$ we get that $\rho_d'(h) = \rho_d(h)A_d(h)$ for all $h \in \mathbb{Z}$.

It holds that

$$I/\rho^{*2} = \sum_{i=-\infty}^{k-1}\sum_{j=k}^{\infty}\rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j)-A_d(i)) = I_1 + I_2$$

with

$$I_1 = \sum_{j=k}^{\infty}\sum_{i=-\infty}^{-j-1}\rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j)-A_d(-i)),$$
$$I_2 = \sum_{j=k}^{\infty}\sum_{i=-j}^{k-1}\rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j)-A_d(|i|)).$$

Now

$$\begin{aligned} I_1 &= -\sum_{j=k}^{\infty}\sum_{i=j+1}^{\infty}\rho_d(i)\rho_d(j)\alpha^{j+i}(A_d(i)-A_d(j)) \\ &= -\sum_{j=k+1}^{\infty}\sum_{i=k}^{j-1}\rho_d(i)\rho_d(j)\alpha^{j+i}(A_d(j)-A_d(i)), \end{aligned}$$

$$\begin{aligned} I_2 &= \sum_{j=k}^{\infty}\sum_{i=-k+1}^{k-1}\rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j)-A_d(|i|)) \\ &+ \sum_{j=k}^{\infty}\sum_{i=-j}^{-k}\rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j)-A_d(|i|)) \\ &= \sum_{j=k}^{\infty}\sum_{i=-k+1}^{k-1}\rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j)-A_d(|i|)) - I_1. \end{aligned}$$

Thus

$$I/\rho^{*2} = \sum_{j=k}^{\infty}\sum_{i=-k+1}^{k-1}\rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j)-A_d(|i|)). \tag{5.3}$$

d) Since

$$
\begin{aligned}
I &= \rho^{*2} \left(\sum_{j=k}^{\infty} \sum_{i=-k+1}^{k-1} \rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j) - A_d(|i|)) \right) \\
&= \rho^{*2} \left(\sum_{j=k+1}^{\infty} \sum_{i=-k+2}^{k-2} \rho_d(i)\rho_d(j)\alpha^{j-i}(A_d(j) - A_d(|i|)) \right. \\
&+ \rho_d(k) \sum_{i=-k+1}^{k-1} \rho_d(i)\alpha^{k-i}(A_d(k) - A_d(|i|)) \\
&+ \rho_d(k-1) \sum_{j=k+1}^{\infty} \rho_d(j)\alpha^{j-k+1}(A_d(j) - A_d(k-1)) \\
&+ \left. \rho_d(k-1) \sum_{j=k+1}^{\infty} \rho_d(j)\alpha^{j+k-1}(A_d(j) - A_d(k-1)) \right) \\
&= I_3 + I_4 + I_5 + I_6, \qquad I_i \geq 0, i = 3, .., 6 \\
II &= -\gamma\alpha\rho^*\rho_d(k-1) \sum_{i=-k+2}^{k-2} \rho_d(i)\alpha^{k-1-i}(A_d(k-1) - A_d(|i|)) \\
&+ \gamma\alpha\rho^*\rho_d(k-1) \sum_{j=k}^{\infty} \rho_d(j)\alpha^{k-1+j}(A_d(j) - A_d(k-1)) = II_1 + II_2, \\
&\quad \text{with} \qquad II_1 \geq 0, II_2 \leq 0, \\
III &= \gamma\rho^*\rho_d(k) \sum_{i=-k+1}^{k-1} \rho_d(i)\alpha^{k-1-i}(A_d(k) - A_d(|i|)) \\
&+ \gamma\rho^*\rho_d(k) \sum_{j=k+1}^{\infty} \rho_d(j)\alpha^{k-1+j}(A_d(k) - A_d(j)) \\
&= III_1 + III_2, \qquad III_1 \leq 0, III_2 \geq 0, \\
IV &= \frac{\gamma\rho^*}{\alpha}\rho_d(k-1) \sum_{j=k}^{\infty} \rho_d(j)\alpha^{j-k+1}(A_d(j) - A_d(k-1)) \leq 0
\end{aligned}
$$

we get that

$$\begin{aligned}
VII = I_4 + III_1 &= (\rho^{*2} + \gamma\rho^*/\alpha)\rho_d(k) \sum_{i=-k+1}^{k-1} \rho_d(i)\alpha^{k-i}(A_d(k) - A_d(|i|)) \geq 0, \\
I_6 + II_2 + VI &= (\rho^{*2} + \gamma\alpha\rho^*)\rho_d(k-1) \sum_{j=k+1}^{\infty} \rho_d(j)\alpha^{j+k-1}(A_d(j) - A_d(k-1)) \\
&+ (\gamma\rho^*\alpha^{2k} + \gamma(1-\rho^*)\rho_d(k-1)\rho_d(k)(A_d(k) - A_d(k-1)) \\
&= VIII_1 + VIII_2, \quad \text{with} \quad VIII_1 \geq 0 \\
I_5 + IV &= (\rho^{*2} + \gamma\rho^*/\alpha)\rho_d(k-1) \sum_{j=k+1}^{\infty} \rho_d(j)\alpha^{j-k+1}(A_d(j) - A_d(k-1)) \\
&+ \gamma\rho^*\rho_d(k-1)\rho_d(k)(A_d(k) - A_d(k-1)) = IX_1 + IX_2, \\
&\quad \text{with} \quad IX_1 \geq 0, IX_2 \leq 0, \\
VIII_2 + IX_2 &= \gamma(1 + \rho^*\alpha^{2k})\rho_d(k-1)\rho_d(k)(A_d(k) - A_d(k-1)) = X.
\end{aligned}$$

Now

$$\begin{aligned}
VII &\geq (\rho^{*2} + \frac{\gamma\rho^*}{\alpha})(\alpha + \alpha^{2k-1})\rho_d(k-1)\rho_d(k)(A_d(k) - A_d(k-1)), \\
&= \frac{\rho_1^2 + \gamma\rho_1}{\alpha}(1 + \alpha^{2k-2})\ \rho_d(k-1)\rho_d(k)(A_d(k) - A_d(k-1))
\end{aligned}$$

and

$$VII + X \geq (\gamma(1 + \rho^*\alpha^{2k}) + \frac{\rho_1^2 + \gamma\rho_1}{\alpha}(1 + \alpha^{2k-2}))\ \rho_d(k-1)\rho_d(k)(A_d(k) - A_d(k-1)).$$

Since

$$\begin{aligned}
\gamma(1 + \rho^*\alpha^{2k}) + \frac{\rho_1^2 + \gamma\rho_1}{\alpha}(1 + \alpha^{2k-2}) &= \gamma(1 + \rho_1\alpha^{2k-1}) + \frac{\rho_1^2 + \gamma\rho_1}{\alpha}(1 + \alpha^{2k-2}) \\
&\geq \frac{\rho_1^2 + \gamma\rho_1}{\alpha}(1 + \alpha^{2k-1}) + \gamma(1 + \alpha^{2k-1}) \\
&= \frac{1 + \alpha^{2k-1}}{\alpha}\ (\rho_1^2 + \gamma\rho_1 + \gamma\alpha) \\
&= \frac{1 + \alpha^{2k-1}}{\alpha(1-\alpha^2)}\ (\rho_1^2(1-\alpha^2) + \rho_1(\alpha - \rho_1) + \alpha(\alpha - \rho_1)) \\
&= \frac{\alpha(1-\rho_1^2)(1 + \alpha^{2k-1})}{1 - \alpha^2} \geq 0
\end{aligned}$$

it holds that $VII + X \geq 0$.

Moreover, we get that

$$\begin{aligned}
V &= \gamma\rho^* \sum_{i=k}^{\infty} \rho_d(k)\rho_d(i)\alpha^{i-k+1}(A_d(k) - A_d(i)) \geq 0, \\
VI &= \gamma(1-\rho^*)\rho_d(k-1)\rho_d(k)(A_d(k) - A_d(k-1)) \geq 0.
\end{aligned}$$

Thus the result is proved. □

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