Some Reliability Estimates for Generalized Exponential Distribution with Presence of k-outliers

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Abstract:

- This paper studies the problem of parameters estimation for the two-parameter generalized exponential distribution in presence of outliers to estimate the reliability (R) as a measure of system performance. The maximum likelihood and Bayes estimators of R are obtained when the scale parameter λ is known and unknown. Monte Carlo simulations and Bootstap approach are performed to compare the different proposed methods.

Key-Words:

- *Bayes estimator, Bootstrap method; Generalized exponential distribution; Maximum likelihood estimator; Monte Carlo simulation; Outliers.*

AMS Subject Classification:

- 49A05, 78B26.

1. INTRODUCTION

Over the past decades, different statistical distributions and related models have been proposed for treating randomness and uncertainty, among which the exponentiated weibull distribution models is a key one [20]. Meanwhile, two-parameter generalized exponential distribution (denoted by GED) has also been proposed as a sub-model in the exponentiated weibull distribution model which model the real data in a more realistic manner. Several researchers have concentrated on applying this distribution in various fields and studied the problem of parameters estimation for GED [11]-[14], [18], [22]-[24], [27]-[30], [35].

Inferences about stress-strength model is an important and interesting fields in the reliability theory. In the mechanical reliability of a system, if we denote X as the strength of a component which is subject to the stress Y, then $R = P(Y < X)$ is known as a measure of system performance. The problem of estimating R for certain family of probability distributions, has been widely studied in the literature. In the following, we review the main studies in this context in an attempt to display the motivation for this paper.

The MLE of $P(Y < X)$, when X and Y have bivariate exponential distribution, has been considered by Awad *et al.* [1]. Church and Harris [3], Downtown [5], Woodward and Kelley [34] and Owen *et al.* [26] considered the estimation of $P(Y < X)$, when X and Y are normally distributed. Similar problem for the multivariate normal distribution has been considered by Gupta and Gupta [10]. Kelley *et al.* [16], and Sathe and Shah [32] considered the estimation of $P(Y < X)$ when X and Y are independent exponential random variables. Constantine and Karson [4] considered the estimation of $P(Y < X)$, when X and Y are independent Gamma random variables. Sathe and Dixit [31] have been estimate of $P(Y < X)$ in the negative binomial distribution. Surles and Padgett [33] considered the estimation of $P(Y < X)$, where X and Y are Burr Type random variables. Finally, Nasiri and Pazira [25] have done the estimation of $P(Y < X)$ in exponential case.

The drawback of the above mentioned models is their lack of a supporting the sample data which contain outliers due to human error in measuring or erroneous procedures. To the best of our knowledge, a few researchers investigated the statistical inference about R based on samples contain outlier observation(s). Kim and Chung [17] and Jeevanand and Nair [15] have considered the Bayesian estimation of R based on samples containing outlier from the burr-X distribution and exponential distribution, respectively. Li and Hao [19] studied the Bayesian and maximum likelihood estimation of R when X and Y are two independent generalized exponential distributions containing one outlier. Pazira and Nasiri [28] and Nasiri [21] consider the estimating parameters of R for generalized exponential istribution and Lomax distribution with presence k-outliers, respectively. Ghanizadeh [8] and Ghanizadeh et. al. [9] studied the estimation of R in the presence of k-outlier for Rayleigh and Exponentiated Gamma distribution, respectively.

In the present work, the Bayes and maximum likelihood approaches to estimate the $P(Y < X)$ are incorporated into the samples containing outliers. This paper is organized as follows: First, in Section 2, we recall the concept of GED and then formulated the problem. Then, we investigate the MLE and the Bayes estimators of R when the scale parameter is known and unknown, respectively in Section 3 and 4. The different proposed methods have been compared using Monte Carlo simulations and Bootstrap methods and their results have been reported in Section 5. An numerical example is illustrated in Section 6. Finally, a brief conclusion presented in Section 7.

2. Mathematical formulation

The two-parameter GED has the following density function

$$f(x, \alpha, \lambda) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}, \qquad x > 0 \tag{2.1}$$

where $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. We denote the two-parameter GED with the shape parameter α and scale parameter λ will be denoted by $GE(\alpha, \lambda)$.

For different values of the shape parameter, the density function can take different shape. If the scale parameter λ is equal to one, for $\alpha \leq 1$, the density function is a decreasing function and for $\alpha > 1$, it is a unimodal, skewed, right tailed similar to the $Weibull$ or $Gamma$ density function. It is observed that even for very large shape parameter (α), it is not symmetric. For this density function (2.1), the mode is at $\log \alpha$ for $\alpha > 1$ and for $\alpha \leq 1$, the mode is at α. It has the median at $-\ln(1 - 0.5^{1/\alpha})$. The mean, median and mode are non-linear functions of the shape parameter α and as this parameter goes to infinity all of them tend to infinity. For large values of α, the mean, median and mode are approximately equal to α but they converge at different rates. Figure 1 shows the shape of $f(x, \alpha)$ for different values of α when $\lambda = 1$ (For more details refer to Gupta and Kundu [11]).

The main aim of this paper is to focus on the inference of $R = P(Y < X)$, where $Y \sim GE(\alpha, \lambda)$, with pdf denoted in Equation (2.1) and Xhas GED with presence of k outliers, with *pdf*,

$$f(x, \beta_1, \beta_2, \lambda) = \frac{k}{n} f(x, \beta_1, \lambda) + \frac{n-k}{n} f(x, \beta_2, \lambda), \qquad x > 0, \tag{2.2}$$

where function $f(.)$ is given in Equation (2.1). For more details see Dixit [6] and Nasiri and Pazira [24]-[25].

To this end, suppose that $Y_1, Y_2, ..., Y_m$ be a random sample for Y with *pdf*

$$g(y, \alpha, \lambda) = \alpha\lambda e^{-\lambda y}(1 - e^{-\lambda y})^{\alpha-1}, \qquad y > 0 \tag{2.3}$$

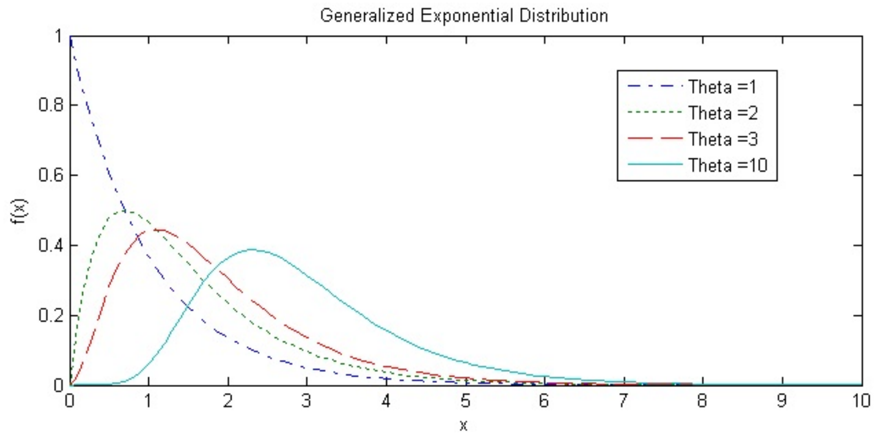

Figure 1: pdf of GE(α,1) for different values of α.

and $X_1, X_2, ..., X_n$ be random sample for X with *pdf*

$$f(x, \beta_1, \beta_2, \lambda) = \frac{k}{n} g(x, \beta_1, \lambda) + \frac{n-k}{n} g(x, \beta_2, \lambda), \qquad x > 0, \tag{2.4}$$

with presence of k outliers. The function $g(.)$ is given in Equation (2.3). Then, based on the difinition of R, we have that

$$\begin{aligned} R &= P(Y < X) = \int_0^\infty \int_0^x g(y, \alpha, \lambda) f(x, \beta_1, \beta_2, \lambda) dy dx \\ &= \frac{k}{n} . \frac{\beta_1}{\alpha + \beta_1} + \frac{n-k}{n} . \frac{\beta_2}{\alpha + \beta_2} \end{aligned} \tag{2.5}$$

Thus, in order to estimate the R, it is sufficient that we estimate the parameters α, β_1 and β_2.

3. Maximum Likelihood Estimator of R

In this section, we study the maximum likelihood estimation of the R. In order to compute the MLE of R, first we consider the joint distribution of $X_1, X_2, ..., X_n$ with presence of k outliers as follows

(3.1)

$$\begin{aligned} & f(x_1, x_2, ..., x_n) \\ &= \frac{1}{C(n,k)} \prod_{i=1}^n \left[\beta_2 \lambda e^{-\lambda x_i} (1 - e^{-\lambda x_i})^{\beta_2 - 1} \right] \sum_{\underline{A}} \prod_{r=1}^k \left(\frac{\beta_1 \lambda e^{-\lambda x_{A_r}} (1 - e^{-\lambda x_{A_r}})^{\beta_1 - 1}}{\beta_2 \lambda e^{-\lambda x_{A_r}} (1 - e^{-\lambda x_{A_r}})^{\beta_2 - 1}} \right) \\ &= \frac{1}{C(n,k)} \beta_1^k \beta_2^{n-k} \lambda^n e^{-\lambda \sum x_i} \prod_{i=1}^n \left[(1 - e^{-\lambda x_i})^{\beta_2 - 1} \right] \sum_{\underline{A}} \prod_{r=1}^k (1 - e^{-\lambda x_{A_r}})^{\beta_1 - \beta_2} \end{aligned}$$

where $C(n,k) = \binom{n}{k}$ and $\Sigma_{\underline{A}} = \Sigma_{A_1=1}^{n-k+1}\Sigma_{A_2=A_1+1}^{n-k+2}\cdots\Sigma_{A_k=A_{k-1}+1}^{n}$. (For more details see [28]).
Using Equation (3.1), the likelihood function based on two observed sample is given as follows

$$L(\alpha,\beta_1,\beta_2,\lambda) = g(y_1,y_2,...,y_m)f(x_1,x_2,...,x_n)$$

The Log-likelihood function of the observed sample is

$$\begin{aligned}\ln L(\alpha,\beta_1,\beta_2,\lambda) &= m\ln(\alpha\lambda) - \lambda\sum_{i=1}^{m} y_i + (\alpha-1)\sum_{i=1}^{m}\ln(1-e^{-\lambda y_i}) \\ &+ \ln\left[\frac{\beta_1^k\beta_2^{n-k}}{C(n,k)}\lambda^n e^{-\sum_{i=1}^{n}\lambda x_i}\prod_{i=1}^{n}\left[(1-e^{-\lambda x_i})^{\beta_2-1}\right]\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}\right] \quad (3.2)\end{aligned}$$

It is well known that, in order to compute the The MLE's of α say $\hat{\alpha}$, we must obtain the solution of following equation

$$\frac{\partial \ln L}{\partial\alpha} = \frac{m\lambda}{\alpha\lambda} + \sum_{i=1}^{m}\ln(1-e^{-\lambda y_i}) = 0,$$

or

$$\frac{m}{\alpha} = -\sum_{i=1}^{m}\ln(1-e^{-\lambda y_i}).$$

Hence,

$$\hat{\alpha} = \frac{-m}{\sum_{i=1}^{m}\ln(1-e^{-\hat{\lambda} y_i})} \quad (3.3)$$

In similar way, the MLE's of β_1, β_2 and λ, say $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\lambda}$ respectively, obtained as the solutions of

$$\begin{aligned}\frac{\partial \ln L}{\partial\beta_1} &= \frac{k}{\beta_1} + \frac{\frac{\partial}{\partial\beta_1}\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}}{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}} = 0 \\ &= \frac{k}{\beta_1} + \frac{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}\ln(1-e^{-\lambda x_{A_r}})}{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}} = 0 \quad (3.4)\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial\beta_2} &= \frac{n-k}{\beta_2} + \sum_{i=1}^{n}\ln(1-e^{-\lambda x_i}) + \frac{\frac{\partial}{\partial\beta_2}\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}}{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}} = 0 \\ &\frac{n-k}{\beta_2} + \sum_{i=1}^{n}\ln(1-e^{-\lambda x_i}) - \frac{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}\ln(1-e^{-\lambda x_{A_r}})}{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}} = 0 \quad (3.5)\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \lambda} &= \frac{m}{\lambda} - \sum_{i=1}^{m} y_i + \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{m}\frac{y_i e^{-\lambda y_i}}{1-e^{-\lambda y_i}} \\ &+ (\beta_2-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}} + \frac{\frac{\partial}{\partial\lambda}\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}}{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}} = 0 \\ &= \frac{m}{\lambda} - \sum_{i=1}^{m} y_i + \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (\alpha-1)\sum_{i=1}^{m}\frac{y_i e^{-\lambda y_i}}{1-e^{-\lambda y_i}} \\ &+ (\beta_2-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}} + \frac{\sum_{\underline{A}}\prod_{r=1}^{k}(\beta_1-\beta_2)x_{A_r}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2-1}}{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2}} = 0 \end{aligned} \tag{3.6}$$

From Equations (3.4)-(3.6), we obtain the $\hat{\beta}_1, \hat{\beta}_2$ and $\hat{\lambda}$ as the solution of non-linear equations.

Since ML estimators are invariant, so the MLE of R becomes

$$\hat{R} = \frac{k}{n}\frac{\hat{\beta}_1}{\hat{\alpha}+\hat{\beta}_1} + \frac{n-k}{n}\frac{\hat{\beta}_2}{\hat{\alpha}+\hat{\beta}_2}. \tag{3.7}$$

Note 3.1. For $\beta_1 = \beta_2 = \beta$ in case of no outliers presence, $\hat{\alpha}$ and $\hat{\beta}$ can be obtain as

$$\hat{\alpha} = \frac{-m}{\sum_{i=1}^{m}\ln(1-e^{-\hat{\lambda} y_i})}, \qquad \hat{\beta} = \frac{-n}{\sum_{i=1}^{n}\ln(1-e^{-\hat{\lambda} x_i})}$$

and $\hat{\lambda}$ can be obtained as the function of the non-linear equation

$$\begin{aligned} g(\lambda) &= \frac{m+n}{\lambda} - \frac{n}{\sum_{i=1}^{n}\ln(1-e^{-\lambda x_i})}\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{(1-e^{-\lambda x_i})} \\ &- \frac{m}{\sum_{i=1}^{m}\ln(1-e^{-\lambda y_i})}\sum_{i=1}^{m}\frac{y_i e^{-\lambda y_i}}{(1-e^{-\lambda y_i})} - \sum_{i=1}^{n}\frac{x_i}{(1-e^{-\lambda x_i})} - \sum_{i=1}^{m}\frac{y_i}{(1-e^{-\lambda y_i})} = 0 \end{aligned}$$

are given by Kundu and Gupta [18].

Note 3.2. The estimation of R when λ is known was studied by Pazira and Nasiri [28]. In this case, the MLE estimation of R is given as Equation (3.7) in which $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ given as follows

$$\hat{\alpha} = \frac{-m}{\sum_{i=1}^{m}\ln(1-e^{-y_i})} \tag{3.8}$$

$$\frac{\partial \ln L}{\partial \beta_1} == \frac{k}{\beta_1} + \frac{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-x_{A_r}})^{\beta_1-\beta_2}\ln(1-e^{-x_{A_r}})}{\sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-x_{A_r}})^{\beta_1-\beta_2}} = 0 \tag{3.9}$$

$$\frac{\partial \ln L}{\partial \beta_2} = \frac{n-k}{\beta_2} + \sum_{i=1}^{n} \ln(1-e^{-x_i}) - \frac{\sum_{\underline{A}} \prod_{r=1}^{k} (1-e^{-x_{A_r}})^{\beta_1-\beta_2} \ln(1-e^{-x_{A_r}})}{\sum_{\underline{A}} \prod_{r=1}^{k} (1-e^{-x_{A_r}})^{\beta_1-\beta_2}} = 0 \quad (3.10)$$

3.1. Bootstrap Method

In this subsection, we propose the percentile bootstrap method based on the idea of Efrom [7] in two cases of parameter λ is known and unknown. The algorithms for estimating the R in these cases are illustrated below.

When λ is Unknown

`Step 1:` From the sample $\{y_1, ..., y_m\}$ and $\{x_1, ..., x_n\}$, compute $\hat{\alpha}$, $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\lambda}$ from equations (3.3), (3.4) and (3.5) and (3.6) respectively.

`Step 2:` Using $\hat{\alpha}$ and $\hat{\lambda}$, we generate a bootstrap sample $\{y_1^*, ..., y_m^*\}$ and similarly using $\hat{\beta}_1$, $\hat{\beta}_2$ and $\hat{\lambda}$, generate a bootstrap sample $\{x_1^*, ..., x_n^*\}$. Based on $\{y_1^*, ..., y_m^*\}$ and $\{x_1^*, ..., x_n^*\}$ compute R.

`Step 3:` Repeat `step 2`, NBOOT times.

When λ is Known

`Step 1:` From the sample $\{y_1, ..., y_m\}$ and $\{x_1, ..., x_n\}$, compute $\hat{\alpha}$, $\hat{\beta}_1$ and $\hat{\beta}_2$ from Equations (3.8), (3.9) and (3.10) respectively.

`Step 2:` Using $\hat{\alpha}$, we generate a bootstrap sample $\{y_1^*, ..., y_m^*\}$ and similarly using $\hat{\beta}_1$ and $\hat{\beta}_2$, generate a bootstrap sample $\{x_1^*, ..., x_n^*\}$. Based on $\{y_1^*, ..., y_m^*\}$ and $\{x_1^*, ..., x_n^*\}$ compute R.

`Step 3:` Repeat `step 2`, NBOOT times.

4. Bayes Estimator of R

In this section, we obtain the Bayes estimation of R under assumption that the parameters β_1, β_2, α and λ are random variables. We mainly obtain the Bayes estimate of R under the squared error loss. It is assumed that the parameters β_1, β_2, α and λ have independent gamma priors with the parameters $\beta_1 \sim Gamma(a_1, b_1)$, $\beta_2 \sim Gamma(a_2, b_2)$, $\alpha \sim Gamma(a_3, b_3)$ and $\lambda \sim Gamma(a_4, b_4)$. Based on the above assumptions, the joint density of the data, β_1, β_2, α and λ can be obtained as

$$\begin{aligned} &L(data, \beta_1, \beta_2, \alpha, \lambda) = L(data; \beta_1, \beta_2, \alpha, \lambda)\pi(\beta_1), \pi(\beta_2), \pi(\alpha)\pi(\lambda) \\ &= C_1\, \beta_1^{k+a_1-1} \beta_2^{n-k+a_2-1} \alpha^{m+a_3-1} \lambda^{n+m+a_4-1}\, h\left(\beta_1, \beta_2, \alpha\right) h\left(\beta_1, \beta_2, \alpha, \lambda\right) \end{aligned}$$

where

$$\begin{aligned} &C_1 = \prod_{i=1}^{4} \left(\frac{b_i^{a_i}}{\Gamma(a_i)} \right) \frac{1}{C(n,k)}, \\ &h\left(\beta_1, \beta_2, \lambda\right) = \textstyle\sum_{\underline{A}} \prod_{r=1}^{k} (1 - e^{-\lambda x_{A_r}})^{\beta_1 - \beta_2}, \\ &h\left(\beta_1, \beta_2, \alpha, \lambda\right) = e^{-b_1\,\beta_1\, - \beta_2 \left(b_2 - \sum\limits_{i=1}^{n} \ln\left(1 - e^{-\lambda\, x_i}\right)\right) - \alpha \left(b_3 - \sum\limits_{j=1}^{m} \ln\left(1 - e^{-\lambda\, y_j}\right)\right)} e^{-\lambda(n\bar{x} + m\bar{y})}. \end{aligned}$$

Therefore, the joint posterior density of given the data is

$$L(\beta_1, \beta_2, \alpha, \lambda | data) = \frac{\beta_1^{k+a_1-1} \beta_2^{n-k+a_2-1} \alpha^{m+a_3-1} \lambda^{n+m+a_4-1}\, h\left(\beta_1, \beta_2, \alpha\right) h\left(\beta_1, \beta_2, \alpha, \lambda\right)}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \beta_1^{k+a_1-1} \beta_2^{n-k+a_2-1} \alpha^{m+a_3-1} \lambda^{n+m+a_4-1}\, h\left(\beta_1, \beta_2, \alpha\right) h\left(\beta_1, \beta_2, \alpha, \lambda\right)\, d\beta_1\, d\beta_2\, d\alpha\, d\lambda} \tag{4.1}$$

Finally, the Bayes estimator of R, denoted by $\hat{R}_B$, given as follows

$$\begin{aligned} \hat{R}_B = &\frac{k}{n} \frac{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty u(\alpha, \beta_1) \beta_1^{k+a_1-1} \beta_2^{n-k+a_2-1} \alpha^{m+a_3-1} \lambda^{n+m+a_4-1}\, h\left(\beta_1, \beta_2, \alpha\right) h\left(\beta_1, \beta_2, \alpha, \lambda\right) d\beta_1\, d\beta_2\, d\alpha\, d\lambda}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \beta_1^{k+a_1-1} \beta_2^{n-k+a_2-1} \alpha^{m+a_3-1} \lambda^{n+m+a_4-1}\, h\left(\beta_1, \beta_2, \alpha\right) h\left(\beta_1, \beta_2, \alpha, \lambda\right)\, d\beta_1\, d\beta_2\, d\alpha\, d\lambda} \\ &+ \frac{n-k}{n} \frac{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty u(\alpha, \beta_2) \beta_1^{k+a_1-1} \beta_2^{n-k+a_2-1} \alpha^{m+a_3-1} \lambda^{n+m+a_4-1}\, h\left(\beta_1, \beta_2, \alpha\right) h\left(\beta_1, \beta_2, \alpha, \lambda\right) d\beta_1\, d\beta_2\, d\alpha\, d\lambda}{\int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \beta_1^{k+a_1-1} \beta_2^{n-k+a_2-1} \alpha^{m+a_3-1} \lambda^{n+m+a_4-1}\, h\left(\beta_1, \beta_2, \alpha\right) h\left(\beta_1, \beta_2, \alpha, \lambda\right)\, d\beta_1\, d\beta_2\, d\alpha\, d\lambda}, \end{aligned} \tag{4.2}$$

where $u(\alpha, \beta_i) = \frac{\beta_i}{\alpha + \beta_i}, i = 1, 2.$

Furthermore, in the case of λ known, the Bayes estimator of R is given by

(4.3)

$$\hat{R}_B = \frac{k}{n}\frac{\int_0^\infty\int_0^\infty\int_0^\infty u(\alpha,\beta_1)\beta_1^{k+a_1-1}\beta_2^{n-k+a_2-1}\alpha^{m+a_3-1}\, g\left(\beta_1,\beta_2\right) g\left(\beta_1,\beta_2,\alpha\right) d\beta_1\, d\beta_2\, d\alpha}{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k+a_1-1}\beta_2^{n-k+a_2-1}\alpha^{m+a_3-1}\, g\left(\beta_1,\beta_2\right) g\left(\beta_1,\beta_2,\alpha\right)\, d\beta_1\, d\beta_2\, d\alpha}$$
$$+\frac{n-k}{n}\frac{\int_0^\infty\int_0^\infty\int_0^\infty u(\alpha,\beta_2)\beta_1^{k+a_1-1}\beta_2^{n-k+a_2-1}\alpha^{m+a_3-1}\, g\left(\beta_1,\beta_2\right) g\left(\beta_1,\beta_2,\alpha\right) d\beta_1\, d\beta_2\, d\alpha}{\int_0^\infty\int_0^\infty\int_0^\infty \beta_1^{k+a_1-1}\beta_2^{n-k+a_2-1}\alpha^{m+a_3-1}\, g\left(\beta_1,\beta_2\right) g\left(\beta_1,\beta_2,\alpha\right)\, d\beta_1\, d\beta_2\, d\alpha},$$

where

$$g\left(\beta_1,\beta_2\right) = \sum_{\underline{A}}\prod_{r=1}^{k}(1-e^{-\lambda x_{A_r}})^{\beta_1-\beta_2},$$
$$g\left(\beta_1,\beta_2,\alpha\right) = e^{-b_1\,\beta_1\,-\beta_2\left(b_2-\sum\limits_{i=1}^{n}\ln\left(1-e^{-\lambda\, x_i}\right)\right)-\alpha\left(b_3-\sum\limits_{j=1}^{m}\ln\left(1-e^{-\lambda\, y_j}\right)\right)}.$$

Since Equations 4.2 and 4.3 can not be obtained analytically, we adopt the Gibbs sampling technique to compute the Bayes estimate of R. Moreover, to compute different Bayes estimates, we prefer to use the non-informative prior, because we do not have any prior information on R. On the other hand, the non-informative prior provides prior distributions which are not proper, we adopt the suggestion of Congdon [2] and Kundu and Gupta [18].

5. Simulation results

In this section, we present some results based on Monte Carlo simulations to compare the performance of the different methods. We consider two cases separately to draw inference on R, namely when (i) the common scale parameter λ is known, (ii) the common scale parameter λ is unknown. In both cases we consider the following small sample size

$$(n,m) = (15,15),(20,20),(25,25),(15,20),(20,15),(15,25),(25,15),(20,25),(25,20).$$

Moreover, in both cases we take $\alpha = 1.50$, $\beta_1 = 2.50$ and $\beta_2 = 2.75$. Without loss of generality we take $\lambda = 1$ in the case λ is known. Here we present a complete analysis of a simulated data, and the results are given in tables 1 to 4 for $k = 1$ and tables 5 to 8 for $k = 2$.

It is observed that the maximum likelihood estimator of R, when λ is known and unknown works quite well. We report the average estimates and the MSEs based on 5000 replications. The results are reported in Tables 1 and 2 for $k = 1$ and, 5 and 6 for $k = 2$. In this case, as we expected, when $m = n$ and m , n increase then the average biases and the MSEs decrease. For fixed m as n increase the MSEs decrease and also for fixed n as m increases the MSEs decrease.

Based on obtained results, it is clear that the estimator of R using Bootstrap method, when λ is known and unknown works quite well. We report the average estimates and the MSEs based on 100 replications. The results are reported in Tables 3 and 4 for $k = 1$ and, 7 and 8 for $k = 2$. In this case, as we expected, when $m = n$ and m , n increase then the average biases and the MSEs decrease. For fixed m as n increase the MSEs decrease and also for fixed n as m increases the MSEs decrease.

(n,m)	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{R}$	$Bias(\hat{R})$	$MSE(\hat{R})$
(15,15)	1.8444	2.5000	3.3135	0.6278	-0.0178	0.0217
(20,20)	1.6075	2.5000	2.7086	0.6237	-0.0222	0.0063
(25,25)	1.8233	2.5000	2.7277	0.6074	-0.0388	0.0082
(15,20)	1.6851	2.5000	3.1127	0.6445	-0.0011	0.0023
(20,15)	1.4864	2.5000	3.4041	0.6959	0.0500	0.0034
(15,25)	1.7807	2.5000	2.6832	0.6071	-0.0385	0.0088
(25,15)	1.4213	2.5000	2.7206	0.6490	0.0028	0.0033
(20,25)	1.6360	2.5000	2.8331	0.6333	-0.0126	0.0030
(25,20)	1.5888	2.5000	2.6093	0.6249	-0.0213	0.0073

Table 1: MLE when $k = 1,\ \alpha = 1.5,\ \beta_1 = 2.5,\ \beta_2 = 2.75$ and $\lambda = 1$

(n,m)	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{R}$	$Bias(\hat{R})$	$MSE(\hat{R})$
(15,15)	0.9877	1.5666	2.5000	2.7500	0.6443	-0.0012	0.0051
(20,20)	0.9899	1.6447	2.5000	2.7500	0.6338	-0.0122	0.0050
(25,25)	1.0223	1.6172	2.5000	2.7500	0.6344	-0.0118	0.0036
(15,20)	1.0108	1.6242	2.5000	2.7500	0.6365	-0.0091	0.0050
(20,15)	1.0209	1.6831	2.5000	2.7500	0.6291	-0.0168	0.0060
(15,25)	1.0165	1.6402	2.5000	2.7500	0.6359	-0.0097	0.0052
(25,15)	1.0037	1.6527	2.5000	2.7500	0.6324	-0.0138	0.0054
(20,25)	0.9974	1.5571	2.5000	2.7500	0.6425	-0.0034	0.0032
(25,20)	1.0251	1.6440	2.5000	2.7500	0.6325	-0.0137	0.0044

Table 2: MLE when $k = 1,\ \alpha = 1.5,\ \beta_1 = 2.5,$ and $\beta_2 = 2.75$

6. Numerical example

In this section an numerical example is illustrated and the results of different methods are compared. to do this, the data has been generated using $k = 2$, $m = n = 15$, $\alpha = 1.50, \beta_1 = 2.50, \beta_2 = 2.75$ and $\lambda = 1$. The data has been truncated after four decimal places and it has been presented below. The Y values are

(n,m)	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{R}$	$Bias(\hat{R})$	$MSE(\hat{R})$
(15,15)	1.6774	14.6622	59.6942	0.9076	0.2620	0.0735
(20,20)	1.7030	9.7609	180.1635	0.8881	0.2421	0.0780
(25,25)	0.3793	2.5000	12.5714	0.9010	0.2548	0.0796
(15,20)	4.5208	2.5000	13.3324	0.6994	0.0539	0.0105
(20,15)	1.5245	6.5037	89.0338	0.9411	0.2952	0.0899
(15,25)	3.4078	6.1308	272.3902	0.8519	0.2063	0.0503
(25,15)	2.6388	2.5401	112.7504	0.8501	0.2039	0.0489
(20,25)	1.6082	2.5000	7.5632	0.7984	0.1525	0.0324
(25,20)	0.5908	2.5000	2.1251	0.8065	0.1603	0.0692

Table 3: Bootstrap method when $k = 1$, $\alpha = 1.5$, $\beta_1 = 2.5$, $\beta_2 = 2.75$ and $\lambda = 1$

(n,m)	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{R}$	$Bias(\hat{R})$	$MSE(\hat{R})$
(15,15)	0.0942	47.7459	2.5000	2.7500	0.3846	-0.2609	0.1497
(20,20)	0.1831	4.0096	2.5000	2.7500	0.4931	-0.1528	0.0571
(25,25)	1.8805	9.7125	2.5000	2.7500	0.3571	-0.2890	0.1195
(15,20)	0.7036	8.4306	2.5000	2.7500	0.4055	-0.2401	0.0980
(20,15)	1.9512	6.6062	2.5000	2.7500	0.3598	-0.2861	0.1075
(15,25)	0.8380	2.2379	2.5000	2.7500	0.6275	-0.0181	0.0474
(25,15)	0.2228	10.3235	2.5000	2.7500	0.3642	-0.2820	0.1196
(20,25)	1.4792	2.4191	2.5000	2.7500	0.5761	-0.0698	0.0277
(25,20)	0.4040	1.7287	2.5000	2.7500	0.6455	-0.0006	0.0198

Table 4: Bootstrap method when $k = 1$, $\alpha = 1.5$, $\beta_1 = 2.5$, and $\beta_2 = 2.75$

(n,m)	$\hat{\alpha}$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{R}$	$Bias(\hat{R})$	$MSE(\hat{R})$
(15,15)	1.8567	2.8847	2.2264	0.5198	-0.1244	0.0217
(20,20)	1.5968	2.5165	2.5660	0.5848	-0.0601	0.0071
(25,25)	1.6174	1.9278	2.4801	0.6004	-0.0449	0.0056
(15,20)	1.5593	3.9744	2.8183	0.6380	-0.0061	0.0111
(20,15)	1.6675	3.1053	2.6176	0.6174	-0.0274	0.0166
(15,25)	1.6831	2.9464	2.3125	0.5741	-0.0700	0.0106
(25,15)	1.5960	1.7858	2.6101	0.6034	-0.0419	0.0071
(20,25)	1.6159	3.1946	2.7131	0.6198	-0.0250	0.0060
(25,20)	1.4687	3.1153	2.8283	0.6556	0.0103	0.0034

Table 5: MLE when $k = 2$, $\alpha = 1.5$, $\beta_1 = 2.5$, $\beta_2 = 2.75$ and $\lambda = 1$

0.1656 1.4907 0.1297 0.1890 1.0442 0.2366 2.0775 2.0741
1.6354 0.3315 1.4178 1.0370 4.0119 1.3847 1.9806

and the corresponding X values are

(n,m)	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta_1}$	$\hat{\beta_2}$	$\hat{R}$	$Bias(\hat{R})$	$MSE(\hat{R})$
(15,15)	1.0637	1.6394	2.7108	2.3042	0.5491	-0.0950	0.0106
(20,20)	0.9842	1.6196	2.6559	2.4108	0.5779	-0.0670	0.0110
(25,25)	0.9916	1.8235	2.6076	2.4758	0.5824	-0.0629	0.0159
(15,20)	0.9449	1.3620	2.5189	2.3047	0.5885	-0.0557	0.0037
(20,15)	1.0560	1.9955	2.8272	2.4110	0.5297	-0.1152	0.0185
(15,25)	0.9265	1.4211	3.7023	2.3022	0.5857	-0.0584	0.0051
(25,15)	0.9867	1.5628	2.2423	2.4753	0.5911	-0.0542	0.0056
(20,25)	1.0057	1.4799	2.4455	2.4105	0.5931	-0.0517	0.0056
(25,20)	0.9153	1.5412	2.7855	2.4752	0.5947	-0.0506	0.0045

Table 6: MLE when $k = 2,\ \alpha = 1.5,\ \beta_1 = 2.5,$ and $\beta_2 = 2.75$

(n,m)	$\hat{\alpha}$	$\hat{\beta_1}$	$\hat{\beta_2}$	$\hat{R}$	$Bias(\hat{R})$	$MSE(\hat{R})$
(15,15)	2.21351	0.65190	1.30161	0.348023	-0.254428	0.0671290
(20,20)	2.42581	0.46529	1.21980	0.315847	-0.297756	0.0922301
(25,25)	1.95369	3.91502	2.95105	0.601363	-0.018931	0.0036032
(15,20)	1.14788	5.47549	3.46179	0.748267	0.145816	0.0250333
(20,15)	1.50094	6.10851	3.89440	0.709387	0.095784	0.0205170
(15,25)	1.42394	0.66237	1.25841	0.439300	-0.163151	0.0443621
(25,15)	4.55750	1.89003	2.24141	0.362190	-0.258104	0.0943967
(20,25)	1.09092	1.39501	1.75356	0.603435	-0.010168	0.0032842
(25,20)	1.53441	1.77145	2.36214	0.589008	-0.031286	0.0134582

Table 7: Bootstrap method when $k = 2,\ \alpha = 1.5,\ \beta_1 = 2.5,\ \beta_2 = 2.75$ and $\lambda = 1$

(n,m)	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta_1}$	$\hat{\beta_2}$	$\hat{R}$	$Bias(\hat{R})$	$MSE(\hat{R})$
(15,15)	1.5127	2.0224	2.2197	2.3071	0.5339	-0.1102	0.0150
(20,20)	1.0199	1.5270	1.5802	2.4121	0.6068	-0.0380	0.0047
(25,25)	1.1685	2.2125	2.3427	2.4754	0.5302	-0.1151	0.0150
(15,20)	2.2332	1.8661	2.5324	2.3066	0.5534	-0.0908	0.0110
(20,15)	1.3911	0.6630	1.7005	2.4124	0.7806	0.1357	0.0209
(15,25)	1.3795	3.0205	3.4349	2.3031	0.4660	-0.1781	0.0424
(25,15)	0.6082	1.1873	2.5821	2.4751	0.6773	0.0321	0.0020
(20,25)	0.7843	1.9834	2.8723	2.4099	0.5562	-0.0887	0.0103
(25,20)	1.1818	2.1626	2.0703	2.4756	0.5314	-0.1139	0.0142

Table 8: Bootstrap method when MLE when $k = 2,\ \alpha = 1.5,\ \beta_1 = 2.5,$ and $\beta_2 = 2.75$

3.5641	3.5056	4.9680	2.4494	2.6494	2.7850	3.3939	5.0067
4.8371	2.3331	3.4162	3.7709	3.4634	1.8660	1.7731	

Now, we obtain the MLE estimates of α, β_1, β_2 and R as, $\hat{\alpha} = 2.234, \hat{\beta_1} = 2.5,\ \hat{\beta_2} = 10.43, R = 0.6441$ and therefore $\hat{R} = 0.7542$. Also, using Equation 4.3

the Bayes estimation becomes $\hat{R}_B = 0.7623$.

In case (ii), when λ is unknown, the Y values are

1.5746	0.1059	0.5531	0.1378	0.2374	2.1082	1.5347	0.6255
3.3972	0.1119	0.8613	0.7467	1.8130	1.9542	0.3958	

and the corresponding X values are

3.6642	3.5416	4.1511	4.3893	4.5871	3.0850	4.2729	4.1823
2.7502	2.5972	3.6886	6.5070	3.2589	1.6457	0.7974	

Then $\hat{\alpha} = 0.7731, \hat{\beta}_1 = 2.5,\ \hat{\beta}_2 = 2.75, \hat{\lambda} = 0.5405, R = 0.6441$ and $\hat{R} = 0.7783$. Also, the Bayes estimation becomes $\hat{R}_B = 0.7763$ using Equation 4.2.

For the Bootstrap method when λ is known, the Y values are

0.2550	1.3994	0.9810	1.8751	1.6076	2.7293	2.6022	0.6569
1.5485	0.4147	0.1028	1.7211	0.9942	0.9493	2.7400	

and the corresponding X values are

4.0273	4.0531	5.2043	4.8492	3.9213	2.8151	2.9842	5.4328
2.1106	3.6646	2.7675	7.1520	4.4030	1.4194	1.3471	

Then $\hat{\alpha} = 1.7297, \hat{\beta}_1 = 2.5,\ \hat{\beta}_2 = 6.206, R = 0.6441$ and $\hat{R} = 0.7566$.
In the Bootstrap method when λ is unknown, the Y values are

1.9301	3.3788	0.6447	1.4552	0.8611	2.1686	1.8280	0.3618
2.3616	4.9962	1.0273	2.5419	1.2103	0.3400	0.4183	

and the corresponding X values are

3.4369	4.5594	4.9697	4.7634	3.2003	3.7920	2.4787	2.5690
2.6606	4.2689	3.6796	2.8361	3.6791	0.6259	0.3760	

Then $\hat{\alpha} = 1.7886, \hat{\beta}_1 = 2.5,\ \hat{\beta}_2 = 2.75, \hat{\lambda} = 0.7535, R = 0.6441$ and $\hat{R} = 0.6029$.

7. Conclusion

In this paper, we have studied the estimation of $P(Y < X)$ for the GED. We assume that the sample from each population contains k-outlier. Two cases scale parameter is known or unknown are considered in this context. The MLE and Bayes estimator of R are obtained in each case.

When the common scale parameter is unknown, it is observed that the maximum likelihood estimator works quite well. Based on the simulation results, when the sample size is very small, we recommend to use the parametric Bootstrap percentile method. The similar results was obtained in the case of the common scale parameter is known.

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