RELIABILITY ASPECTS OF PROPORTIONAL MEAN RESIDUAL LIFE MODEL USING QUANTILE FUNCTIONS *

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Abstract:

- In the present work we propose a study of the proportional mean residual life model using quantile functions instead of distribution functions employed in the traditional approach. Through various quantile-based reliability concepts ageing properties of the model are discussed. We present characterizations of the proportional mean residual life model by properties of measures of uncertainty in the residual lives of the variables. Some stochastic order properties relating to the model are also derived.

Key-Words:

- *Proportional mean residual life model; quantile function; divergence measures; characterization; stochastic orders.*

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1. INTRODUCTION

The proportional mean residual life model (PMRLM) was introduced by [11] and [14] as an alternative to the well-known proportional hazards model (PHM). These two papers explain the relevance of PMRLM and its advantages over the PHM. Let X and Y be two non-negative absolutely continuous random variables with finite expectation and survival functions $\bar{F}_X(.)$ and $\bar{F}_Y(.)$ respectively. Then the PMRLM is represented by

$$m_Y(x) = \theta m_X(x), \theta > 0 \tag{1.1}$$

where $m_X(x) = \frac{1}{\bar{F}_X(x)} \int_x^\infty \bar{F}_X(t)dt$ and $m_Y(x) = \frac{1}{\bar{F}_Y(x)} \int_x^\infty \bar{F}_Y(t)dt$ are the mean residual life functions of X and Y. Unlike the PHM, (1.1) may not be valid for all $\theta > 0$. If $m_X(x)$ is increasing then $\theta > 0$ while for a decreasing $m_X(x), 0 < \theta \leq \theta_0$, where $\theta_0^{-1} = max\,(0, -min(m_X(x)))$. The relationship between PMRLM and PHM, the ageing properties and certain bounds on residual moments and residual variance of the former in the context of reliability analysis were studied in [4]. In the same direction [9] discussed the closure properties of the ageing classes related to PMRLM and preservation of certain stochastic orders. The reliability aspects of a dynamic version of (1.1) obtained by replacing the constant θ in (1.1) by a non-negative function of x are also investigated in [10]. All the works mentioned above make use of the identity (1.1), the relationship between the survival functions derived therefrom and the properties of the mean residual life function.

An associated concept is the percentile residual life discussed in several papers like [13], [2] and their references. Instead of the distribution function, a life distribution and its desired characteristics can also be represented through the quantile function

$$Q_X(u) = \inf\{x : F_X(x) \geq u\},\ 0 \leq u \leq 1$$

and various reliability functions evaluated from $Q_X(.)$. The relevance and advantages of using $Q_X(.)$ over $\bar{F}_X(.)$ in various forms of statistical analysis are well documented in [3] and the associated methodology for reliability analysis in [8]. The present article focuses attention on studying the reliability implications of PMRLM using quantile function and the associated reliability concepts.

The factors that motivated the present work are as follows. There are many quantile functions that have simple forms capable of representing a wide variety of lifetime data. These are discussed extensively in [8]. Our work enables the induction of such quantile functions as lifetime models in the analysis of PMRLM. Many of the flexible quantile functions in literature have no tractable distribution functions to make use of them in the conventional analysis. Further, the quantile analogue of (1.1) reads

$$m_Y\left(Q_X(u)\right) = \theta m_X\left(Q_X(u)\right),\ \theta > 0 \tag{1.2}$$

in which the right side is the mean residual quantile function of X while the left side is not the mean residual quantile function of Y. Thus the analogues of the mean residual life in the quantile-based analysis are not proportional as in (1.1). This points out to the possibility of properties of PMRLM that are different from the conventional one. Because of the special properties of quantile functions we can obtain results that are difficult to obtain by the distribution function approach. Various measures of uncertainty in the residual life of a device and their association with reliability concepts are of recent interest on the premise that increase in uncertainty implies that the device becomes more unreliable. We propose two characterizations of PMRLM based on the Kullback-Leibler divergence and its cumulative form using quantile functions.

The rest of the paper contains four sections. In Section 2 we present some preliminary results required for the deliberations in the sequence. This is followed in Section 3 with discussion on the ageing concepts of Y in relation to those of X. The characterizations of the PMRLM are presented in Section 4. Finally in Section 5 some quantile-based stochastic orders associated with PMRLM are discussed.

2. THE PROPORTIONAL MEAN RESIDUAL LIFE MODEL

Let X and Y be as defined in the previous section with strictly decreasing survival functions, hazard rate functions $h_X(.)$ and $h_Y(.)$ and quantile functions $Q_X(.)$ and $Q_Y(.)$. From [4],

$$\bar{F}_Y(x) = \left[\bar{F}_X(x)\right]^{\frac{1}{\theta}} \left(\frac{m_X(x)}{\mu_X}\right)^{\frac{1}{\theta}-1}, \ \mu_X = E(X) \tag{2.1}$$

and

$$h_Y(x) - h_X(x) = \frac{1-\theta}{\theta m_X(x)}. \tag{2.2}$$

The quantile-based reliability functions of X and Y are the hazard quantile functions

$$H_X(u) = h_X\left(Q_X(u)\right) = \frac{1}{(1-u)q_X(u)}$$

and

$$H_Y(u) = \frac{1}{(1-u)q_Y(u)},$$

and the mean residual quantile functions

$$M_X(u) = m_X\left(Q_X(u)\right) = \frac{1}{(1-u)} \int_u^1 (1-p)q_X(p)dp,$$

and

$$M_Y(u) = \frac{1}{(1-u)} \int_u^1 (1-p)q_Y(p)dp$$

where $q_X(u) = \frac{dQ_X(u)}{du}$ and $q_Y(u) = \frac{dQ_Y(u)}{du}$ are the quantile density functions of X and Y. For the definitions, interpretations of these functions and interrelationships between them, we refer to [7].

Setting $x = Q_X(u)$ in (2.1) we see that

$$\bar{F}_Y\left(Q_X(u)\right) = (1-u)^{\frac{1}{\theta}} \left(\frac{M_X(u)}{\mu_X}\right)^{\frac{1}{\theta}-1}$$

or

$$Q_X(u) = Q_Y\left[1 - (1-u)^{\frac{1}{\theta}} \left(\frac{M_X(u)}{\mu_X}\right)^{\frac{1}{\theta}-1}\right] \tag{2.3}$$

Writing $A(u) = 1 - (1-u)^{\frac{1}{\theta}} \left(\frac{M_X(u)}{\mu_X}\right)^{\frac{1}{\theta}-1}$, (2.3) becomes

$$Q_X(u) = Q_Y\left(A(u)\right). \tag{2.4}$$

It is not difficult to see that when (2.3) is satisfied (1.2) also holds. Notice that $A(.)$ is a distribution function on $[0,1]$ so that $A(u)$ is increasing in u with $A(0) = 0$ and $A(1) = 1$. One can work with the identity (1.1) and obtain another relationship involving $\bar{F}_X(.)$ and $\bar{F}_Y(.)$ in the form

$$\bar{F}_X(x) = \left[\bar{F}_Y(x)\right]^{\theta} \left(\frac{m_Y(x)}{\mu_Y}\right)^{\theta-1}, \quad \mu_Y = E(Y)$$

or equivalently

$$Q_Y(u) = Q_X\left(B(u)\right) \tag{2.5}$$

where $B(u) = 1 - (1-u)^{\theta} \left(\frac{M_Y(u)}{\mu_Y}\right)^{\theta-1}$. Since Q_X is an increasing function, it is easy to see that $B(u) = A^{-1}(u)$ and $A(u) = B^{-1}(u)$.

From (2.3) by differentiation,

$$q_X(u) = q_Y\left(A(u)\right) A'(u),$$

so that

$$(1-u)q_X(u) = \frac{(1-u)A'(u)}{1-A(u)} \left(1 - A(u)\right) q_Y\left(A(u)\right)$$

giving

$$H_X(u) = \frac{1-A(u)}{(1-u)A'(u)} H_Y\left(A(u)\right) \tag{2.6}$$

and from (1.2)

$$\theta M_X(u) = m_Y\left[Q_Y A(u)\right] = M_Y\left(A(u)\right) = m_Y\left(Q_X(u)\right). \tag{2.7}$$

Equation (2.7) suggests that in general $M_X(.)$ and $M_Y(.)$ need not be proportional. Therefore unlike the distribution function approach wherein properties of $m_Y(x)$ can be directly obtained from equation (1.1), the quantile analysis does not directly provide characteristics of mean residual quantile function of Y from that of X. We give an example of the use of quantile function in the analysis of PMRLM and in modelling real data.

Remark 2.1. When the mean residual quantile function of Y is proportional to that of X,

$$\frac{1}{1-u}\int_u^1 (1-p)q_Y(p)dp = \frac{1}{1-u}\int_u^1 (1-p)q_X(p)dp$$

which is equivalent to $Q_Y(u) = \theta Q_X(u) = Q_X(\theta u)$. This is the case when Y is obtained as a change of scale in X.

Example 2.1. Let the distribution of X be represented by the quantile function

$$Q_X(u) = -(\alpha+\mu)\log(1-u) - 2\alpha u, \mu > 0, -\mu < \alpha < \mu \tag{2.8}$$

Equation (2.8) specifies a family of flexible distributions that includes exponential and uniform distributions as special cases and approximates well distributions like Weibull, gamma, beta and half-normal. A detailed discussion of (2.8) is available in [5]. Notice that the general form of the distribution does not admit a closed form for its distribution function, except that the distribution function $F_X(.)$ and density function $f_X(.)$ are related through

$$f_X(x) = \frac{1-F_X(x)}{2\alpha F_X(x) + \mu - \alpha}.$$

Thus it becomes difficult to work with $m_X(.)$ and more so with $m_Y(.)$ and conclude their general properties using (1.1). From a quantile perspective we have

$$M_X(u) = \mu + \alpha u$$

$$F_Y\left(Q_X(u)\right) = 1 - (1-u)^{\frac{1}{\theta}}\left(1+\frac{\alpha u}{\mu}\right)^{\frac{1}{\theta}-1}$$

$$m_Y\left(Q_X(u)\right) = \theta(\mu + \alpha u)$$

and

$$Q_Y(u) = Q_X\left[A^{-1}(u)\right],\ \ A(u) = 1-(1-u)^{\frac{1}{\theta}}\left(1+\frac{\alpha u}{\mu}\right)^{\frac{1}{\theta}-1} \tag{2.9}$$

To illustrate the application of quantile functions in modelling proportional mean residual life using the above model we consider the data on the time to first failure of 20 electric carts given in [15].

The methodology used here is described as follows. Let $x_1 < x_2 < ... < x_n$ be the distinct observations in the sample. We estimate the sample distribution function as

$$\hat{F}_X(x_r) = u_r = \frac{r - 0.5}{n}, \ r = 1, 2, ..., n, \ \ u_r \leq x < u_{r+1}$$

by dividing the interval $(0, 1)$ with equal parts and using their midpoints to symmetrically place the u values. This gives

$$\hat{Q}_X(u_r) = x_r, \ \ u_{r-1} < u < u_r, \ \ r = 1, 2, ..., n, \ u_0 = 0$$

and

$$\begin{aligned} \hat{F}_Y \hat{Q}_X(u_r) &= \hat{F}_Y(x_r) \\ &= 1 - (1 - u_r)^{\frac{1}{\theta}} \left(\frac{M_X(u_r)}{\mu} \right)^{\frac{1}{\theta} - 1}. \end{aligned}$$

Recall $G_Y(.) = F_Y(Q_X(.))$ is a distribution function over $(0, 1)$. If $g_Y(.)$ is the probability density function of $G_Y(.)$, then

$$g_Y(u) = f_Y(Q_X(u))\, q_X(u) = \frac{f_Y Q_X(u)}{f_X Q_X(u)}.$$

We estimate the parameter of the PMRLM by minimizing

$$E = -\int_0^1 g_Y(u) \log g_Y(u) du.$$

For the given data $g_Y(.)$ is replaced by its estimated value

$$\hat{g}_Y(u) = \frac{\left(\hat{F}_Y(x_r) - \hat{F}_Y(x_{r-1}) \right)}{u_r - u_{r-1}}$$

and the minimization of E is carried out. The estimates obtained are

$$\hat{\alpha} = 0.6078, \ \hat{\mu} = 1.054 \ \ \textit{and} \ \ \hat{\theta} = 0.8438.$$

Thus

$$\hat{m}_Y \left(\hat{Q}_X(u) \right) = 0.8894 + 0.5129u.$$

for the minimum of E obtained as 1.54626×10^{-10} which shows the closeness of the fit. Further analysis of the data can be accomplished based on $G_Y(.)$ or $m_Y(Q_X(.))$ obtained above.

For all quantile functions equation, (1.2) may not be satisfied for $\theta > 0$. We present a necessary and sufficient condition for the existence of the PMRLM for a distribution.

Theorem 2.1. *A quantile funtion $Q_X(.)$ admits a PMRLM if and only if θ satisfies*

$$q_X(u) + \theta M'_X(u) \geq 0$$

where the prime denotes differentiation with respect to u.

Proof: The relationship (2.4) is satisfied for some $Q_Y(.)$ if and only if there exists a quantile function $Q_X(.)$ for which $q_X(.) > 0$ since $Q_X(.)$ must be an increasing function. Now

$$\begin{aligned} q_X(u) = \quad & \frac{q_Y\left(A(u)\right)(1-u)^{\frac{1}{\theta}-1}}{\theta}\left(\frac{M_X(u)}{\mu_X}\right)^{\frac{1}{\theta}-1} \\ & -q_Y\left(A(u)\right)(1-u)^{\frac{1}{\theta}}\left(\frac{1}{\theta}-1\right)\left(\frac{M_X(u)}{q_X(u)}\right)^{\frac{1}{\theta}-2}\frac{M'_X(u)}{\mu_X} \geq 0 \\ \Leftrightarrow \quad & \frac{M_X(u)}{\theta} - \left(\frac{1}{\theta}-1\right)M'_X(u) > 0 \\ \Leftrightarrow \quad & M_X(u) - (1-u)M'_X(u) + (1-u)\theta M'_X(u) > 0 \\ \Leftrightarrow \quad & \frac{1}{H_X(u)} + (1-u)\theta M'_X(u) > 0 \\ \Leftrightarrow \quad & q_X(u) + \theta M'_X(u) > 0. \end{aligned}$$

□

Remark 2.2. In view of Theorem 1, we see that

(i) if $M'_X(.) \geq 0$ then PMRLM holds for all $\theta > 0$ and

(ii) if $M'_X(.) < 0$, X admits PMRLM only when the range of θ is limited to $[0, \theta_0]$, $\theta_0 = \max\left(0, -\min \frac{q_X(u)}{M'_X(u)}\right)$.

3. AGEING PROPERTIES

There are situations when the distribution of Y specified by $Q_Y(.)$ may not have tractable form to study the ageing properties of Y analytically. For example see $Q_Y(.)$ given in (2.9). This does not pose any problems to data analysis since $F_Y\left(Q_X(.)\right)$ can be employed for inferential purposes. Generally, the baseline distribution is one for which the ageing criteria is known or can be evaluated and therefore results that enable the inference of ageing characteristics of Y in terms of those of X become useful. In this section we prove some theorems in this direction. For this we need the following definitions. The definitions and results are given only for positive ageing concepts as it is easy to deduce their negative ageing counterparts by reversing the monotonicity or the inequality in each case. The random variable X is said to be (i) increasing hazard rate (IHR) if $H_X(.)$ is increasing (ii) new better than used in hazard rate (NBUHR) if $H_X(0) \leq H_X(u)$ for all u and new better than used in hazard rate average (NBUHRA) if $-\frac{\log(1-u)}{Q_X(u)} \geq H_X(u)$ (iii) increasing hazard rate average (IHRA) if $\frac{Q_X(u)}{-\log(1-u)}$ is decreasing in u (iv) decreasing mean residual life (DMRL) if $M_X(.)$ is decreasing (v) decreasing mean residual life in harmonic average (DMRLHA) if $\frac{1}{Q_X(u)} \int_0^u \frac{q_X(p)}{M_X(p)} dp$ is decreasing in u, (vi) new better than used (NBU) if $Q_X(u+v-uv) \leq Q_X(u) + Q_X(v)$ for $0 \leq u < v < 1$ and (vii) new better than used in expectation (NBUE) if $M_X(u) \leq \mu_X$. It may be noticed the definitions of the above concepts in the distribution function approach and the quantile function approaches are equivalent. However the results pertaining to PMRLM in the two differ at least in some cases. For a detailed discussion of the characteristics of various quantile ageing classes, see [8].

Theorem 3.1. *If X is IHR, $\theta > 1$ and $M_X(.)$ is logconvex then Y is IHR.*

Proof: Recall that

$$H_Y\left(A(u)\right) = \frac{(1-u)A'(u)}{1-A(u)} H_X(u).$$

Also

$$T(u) = \frac{(1-u)A'(u)}{1-A(u)} = \frac{1}{\theta} - \left(\frac{1}{\theta} - 1\right)(1-u)\frac{d \log M_X(u)}{du},$$

gives

$$T'(u) = \left(1 - \frac{1}{\theta}\right)(1-u)\frac{d^2 \log M_X(u)}{du^2} + \left(\frac{1}{\theta} - 1\right)\frac{d \log M_X(u)}{du}.$$

When X is IHR, it is also DMRL. Under the conditions of the theorem $T(.)$ is increasing and so is $H_X(.)$. Thus $H_Y\left(A(u)\right)$ is an increasing function of $A(u)$ and hence of u, showing that $H_Y(u)$ is increasing. □

Remark 3.1. It can be shown that if Y is IHR, $\theta < 1$ and $M_Y(.)$ is logconcave, then X is IHR. To prove this we use (2.5) and work with $B(u)$ in the same manner as with $A(u)$. Various results concerning other ageing properties proved below can also have parallel results relating Y with X. Because of similarity they are not pursued further.

We give two examples, one demonstrating the usefulness of Theorem 2 and the other to show that the conditions imposed on the reliability functions of X are essential.

Example 3.1. Consider the linear hazard quantile function distribution with quantile function

$$Q_X(u) = \frac{1}{a+b}\log\left(\frac{a+bu}{a(1-u)}\right),\ a>0,\ b>0. \tag{3.1}$$

whose properties and applications are studied in [6]. The hazard and the mean residual quantile functions of (3.1) are

$$H_X(u) = a + bu$$

and

$$M_X(u) = \frac{1}{b(1-u)}\log\left(\frac{a+b}{a+bu}\right).$$

Obviously X is IHR and

$$\frac{d\log M_X(u)}{du} = \frac{1}{1-u} - \frac{b}{(a+bu)\left[\log(a+b)-\log(a+bu)\right]}$$

is increasing and hence by Theorem 2, Y is IHR. This method looks easier than evaluating the reliability aspects directly from

$$Q_Y(u) = \frac{1}{a+b}\log\left(\frac{a+bA^{-1}(u)}{a(1-A^{-1}(u))}\right)$$

with

$$A(u) = 1-(1-u)^{\frac{1}{\theta}}\left[\frac{\log(a+b)-\log(a+bu)}{(1-u)\left(\log(a+b)-\log a\right)}\right]^{\frac{1}{\theta}-1}$$

derived from (2.4).

Example 3.2. The quantile function

$$Q_X(u) = \frac{3\alpha\beta u^2}{2} + \alpha u(2-\beta),\quad \alpha>0,\ \beta=\frac{1}{2}$$

has

$$H_X(u) = \left[\alpha(1-u)\left(3\beta u+2-\beta\right)\right]^{-1}$$

and

$$M_X(u) = \alpha(1-u)(1+\beta u).$$

Differentiating $H_X(u)$, the sign of $H_X'(u)$ depends on the sign of $3\beta u - 2\beta + 1$ which is positive for $\beta = \frac{1}{2}$. Hence X is IHR. Further

$$M_X'(u) = \alpha(\beta-1) - \alpha\beta u$$

so that $M_X(.)$ is decreasing and concave. Thus the conditions of the theorem are not satisfied. Form (2.6) and

$$A(u) = 1-(1-u)^{\frac{1}{\theta}}\left[(1-u)(1+\beta u)\right]^{\frac{1}{\theta}-1},$$

and

$$H_Y\left(A(u)\right) = \frac{\left(\frac{2}{\theta}-1\right)(1+\beta u) - \left(\frac{1}{\theta}-1\right)\beta(1-u)}{\alpha(1+\beta u)(1-u)(3\beta u+2-\beta)},\ \beta=\frac{1}{2}$$

splitting up the terms we see that $H_Y(u)$ is not increasing for all u.

Theorem 3.2. *(i) If X is IHRA, $\theta > 1$ and $\left(\frac{\log\left(\frac{M_X(u)}{\mu_X}\right)}{\log(1-u)}\right)$ is decreasing then Y is IHRA.*

(ii) If X is IHR and $\theta > 1$, then Y is NBUHR.

(iii) IF X is NBUHRA, $\theta > 1$ and $\left(\frac{\log\frac{M_X(u)}{\mu_X}}{\log(1-u)} + \frac{M'_X(0)}{\mu_X}\right) \leq 0$, then Y is NBUHRA.

Proof: (i) First we note that

$$\begin{aligned}\frac{Q_Y(u)}{-\log(1-u)} &= \frac{Q_X\left(A^{-1}(u)\right)}{-\log(1-u)} = \frac{Q_X(u)}{-\log\left(1-A(u)\right)}\\ &= \frac{Q_X(u)}{-\log(1-u)}\frac{\log(1-u)}{\log\left(1-A(u)\right)}.\end{aligned} \tag{3.2}$$

The sign of $\frac{\log(1-u)}{\log(1-A(u))}$ depends on

$$\begin{aligned}D(u) &= \frac{A"(u)\log(1-u)}{1-A(u)} - \log\frac{1-A(u)}{1-u}\\ &= \frac{\left(\frac{1}{\theta}-1\right)}{1-u}\left[-\log\frac{M_X(u)}{\mu_X} - (1-u)\log(1-u)\frac{M'_X(u)}{M_X(u)}\right]\\ &= -\left(\frac{1}{\theta}-1\right)(\log(1-u))^2\frac{d}{du}\left(\frac{\log\left(\frac{M_X(u)}{\mu_X}\right)}{\log(1-u)}\right).\end{aligned}$$

When X is IHRA, the first term on the right of (3.2) increases and the second term increases when $\theta > 1$ and $\frac{\log\left(\frac{M_x(u)}{\mu_X}\right)}{\log(1-u)}$ increases. Hence Y is IHRA.

(ii) From (2.2)

$$H_Y\left(A(u)\right) - H_X(u) = \left(\frac{1}{\theta}-1\right)\left[M_X(u)\right]^{-1}.$$

This gives

$$H_Y\left(A(u)\right) - H_Y\left(A(0)\right) = H_X(u) - H_X(0) + \left(\frac{1}{\theta}-1\right)\left(\frac{1}{M_X(u)} - \frac{1}{\mu_X}\right).$$

Since IHR implies NBUHR and NBUE $H_X(u) \geq H_X(0)$ and $\frac{1}{M_X(u)} \geq \frac{1}{\mu_X}$. Hence $H_Y\left(A(u)\right) \geq H_Y\left(A(0)\right) = H_Y(0)$ for all $A(u)$ implies that $H_Y(u) \geq H_Y(0)$, $0 \leq u \leq 1$ and Y is NBUHR.

(iii) From (3.2)

$$\frac{Q_Y(u)}{-\log(1-u)} = \frac{Q_Y(u)}{-\log(1-u)}\frac{\log(1-u)}{\log\left(1-A(u)\right)}$$

and

$$\frac{-\log(1-u)}{Q_Y(u)} - H_Y(u) = \frac{-\log(1-u)}{Q_X(u)} \frac{\log(1-A(u))}{\log(1-u)} - H_Y(0). \tag{3.3}$$

Also (2.6) leads to

$$\begin{aligned} H_X(0) &= \frac{H_Y(0)}{A'(0)} \quad since \quad A(0)=0, \\ &= \frac{1}{\theta} - \left(\frac{1}{\theta}-1\right)\frac{M'_X(0)}{\mu_X} \end{aligned} \tag{3.4}$$

and

$$\log(1-A(u)) = \frac{1}{\theta}\log(1-u) + \left(\frac{1}{\theta}-1\right)\log\frac{M_X(u)}{\mu_X}. \tag{3.5}$$

Using (3.4) and (3.5) in (3.3),

$$\frac{-\log(1-u)}{Q_Y(u)} - H_Y(u) = \frac{-\log(1-u)}{Q_X(u)}\left[\frac{\frac{1}{\theta}+\left(\frac{1}{\theta}-1\right)\log\frac{M_X(u)}{\mu_X}}{\log(1-u)}\right] - A'(0)H_X(0)$$

$$\begin{aligned} &\geq H_X(0)\left[\frac{\frac{1}{\theta}+\left(\frac{1}{\theta}-1\right)\log\frac{M_X(u)}{\mu_X}}{\log(1-u)} - \left(\frac{1}{\theta}-\left(\frac{1}{\theta}-1\right)\right)\frac{M'_X(0)}{\mu_X}\right] \\ &= H_X(0)\left(\frac{1}{\theta}-1\right)\left[\frac{\log\left(\frac{M_X(u)}{\mu_X}\right)}{\log(1-u)} + \frac{M'_X(0)}{\mu_X}\right]. \end{aligned}$$

Under the conditions assumed in Theorem 3, Y is NBUHRA. □

Theorem 3.3. *If X is DMRL and $\theta < 1$, then Y is DMRL.*

Proof: First we notice that

$$A'(u) = \frac{(1-u)^{\frac{1}{\theta}-1}\left(M_X(u)\right)^{\frac{1}{\theta}-2}}{(\mu_X)^{\frac{1}{\theta}-1}}\left[\frac{1}{\theta}M_X(u) - (1-u)\left(\frac{1}{\theta}-1\right)M'_X(u)\right]$$

implies that $A(u)$ is increasing and X is DMRL only when $\theta < 1$. Now

$$M'_X(u) = \theta M'_Y(A(u))\,A'(u)$$

provides $M'_Y(A(u)) \leq 0$. When $M'_Y(A(u))$ is decreasing so does $M_Y(u)$ and hence Y is DMRL. □

Remark 3.2. In the distribution function approach Y is DMRL if and only if X is DMRL ([9]) irrespective of the value of θ. In our case the restriction on θ cannot be dropped. For example, when X is beta with $\bar{F}_X(x) = (1-x)^2, 0 \leq x \leq 1$ and $\theta = 4$

$$M_X(u) = \frac{1}{3}(1-u)^{\frac{1}{2}}$$

which is decreasing, while

$$M_Y(u) = \frac{4}{3}(1-u)^{-4}$$

is increasing.

Theorem 3.4. *If X is DMRLHA if and only if Y is DMRLHA.*

Proof: X is DMRLHA $\Leftrightarrow \frac{1}{Q_X(u)} \int_0^u \frac{q_X(p)dp}{M_X(p)}$ is decreasing in u.

$\Leftrightarrow \frac{1}{Q_Y(A(u))} \int_0^u \frac{q_Y(A(p))A'(p)dp}{M_Y(A(p))}$ is decreasing in u

$\Leftrightarrow \frac{1}{Q_Y(u)} \int_0^u \frac{q_Y(p)dp}{M_Y(p)}$ is decreasing in u

$\Leftrightarrow Y$ is DMRLHA. □

Theorem 3.5. *If X is NBU and $\theta < 1$ and $-\log \frac{M_X(u)}{\mu_X}$ is super additive, then Y is NBU.*

Proof:

$$-\log \frac{(1-u)M_X(u)}{\mu_X} = -\log(1-u) - \log \frac{M_X(u)}{\mu_X}.$$

Since X is NBU, $-\log \frac{(1-u)M_X(u)}{\mu_X}$ is super additive.

$$S_Y\left(Q_X(u)\right) = (1-u)\left[\frac{(1-u)M_X(u)}{\mu_X}\right]^{\frac{1}{\theta}-1}.$$

The right side is the product of two survival functions each of which is NBU and therefore $S_Y(.)$ is NBU, which proves the result. □

4. CHARACTERIZATIONS

In this section we attempt two characterization theorems of the PMRLM by properties of measures of uncertainty in the residual lives of X and Y. The first measure is the Kullback-Leibler divergence between the residual life distributions of X and Y given by

$$i(t) = \frac{1}{\bar{F}_X(x)} \int_x^\infty \left(\log \frac{f_X(t)}{f_Y(t)} \right) f_X(t)dt + \log \frac{\bar{F}_Y(x)}{\bar{F}_X(x)}. \tag{4.1}$$

The quantile version of (4.1),

$$i\left(Q_X(u)\right) = I(u) = \log \left(\frac{\bar{F}_Y\left(Q_X(u)\right)}{1-u} \right) + \frac{1}{1-u} \int_u^1 \log \frac{d\bar{F}_Y\left(Q_X(p)\right)}{dp} dp \tag{4.2}$$

was studied by [12] and several properties including characterization of PHM were obtained by them. In the following theorem we investigate the distributions satisfying PMRLM for which $I(u) = C$, a constant.

Theorem 4.1. *Let X and Y be continuous non-negative random variables as defined in Section 1 satisfying PMRLM. Then $I(u) = C$, a constant if and only if the distribution of X is either exponential with quantile function*

$$Q_E(u) = \lambda^{-1} \left(-\log(1-u)\right), \ \lambda > 0$$

or Pareto with

$$Q_P(u) = \alpha \left[(1-u)^{-\frac{1}{c}} - 1 \right], \ c > 1, \alpha > 0$$

or beta having quantile function

$$Q_B(u) = \beta \left[1 - (1-u)^{\frac{1}{a}} \right], \ a > 0, \beta > 0.$$

Proof: In the case of PMRLM, the divergence measure (4.2) reduces to

$$I(u) = \log \frac{1-A(u)}{1-u} - \frac{1}{1-u} \int_u^1 \log A'(p)dp.$$

When X is exponential,

$$A(u) = 1 - (1-u)^{\frac{1}{\theta}}$$

and hence

$$I(u) = \theta - \log \theta - 1.$$

In the case of $Q_P(u)$, $A(u) = 1 - (1-u)^{\frac{c+\theta-1}{c\theta}}$ and

$$I(u) = \frac{c+\theta-1}{c\theta} - \log \frac{c+\theta-1}{c\theta} - 1$$

and similarly for the beta distribution $A(u) = 1 - (1-u)^{\frac{a-\theta+1}{a\theta}}$ gives

$$I(u) = \frac{a-\theta+1}{a\theta} - \log \frac{a-\theta+1}{a\theta} - 1.$$

Thus $I(u)$ is a constant for all the three distributions. Conversely, when $I(u) = C$,

$$\log \frac{1-A(u)}{1-u} - \frac{1}{1-u} \int_u^1 \log A'(p)dp = C$$

takes the form

$$\int_u^1 \log A'(p)dp = (1-u) \left[\log \frac{1-A(u)}{1-u} - C \right].$$

Differentiating with respect to u and simplifying

$$P(u) = C + 1 + \log P(u) \tag{4.3}$$

where $P(u) = \frac{(1-u)A'(u)}{1-A(u)}$. Differentiating (4.3)

$$P'(u) \left[1 - \frac{1}{P(u)} \right] = 0$$

which leaves two solutions $P(u) = K$, a constant or $P(u) = 1$. Of these $P(u) = 1$ leads to $\frac{M'_X(u)}{M_X(u)} = \frac{1}{1-u}$ or $M_X(u) = \frac{K}{1-u}$ which cannot be mean residual quantile function of a proper distribution. The second solution $P(u) = K$, simplifies to $M_X(u) = K(1-u)^{-b}$ which is the mean residual quantile function of the exponential or Pareto or beta distribution according as $b = 0$ or $b > 0$ or $b < 0$. This completes the proof. □

In the second theorem the choice of the uncertainty measure is the cumulative Kullback-Leibler divergence proposed by [1] for the residual lives of X and Y as

$$j(x) = \left(\log \frac{\bar{F}_Y(x)}{\bar{F}_X(x)} - 1 \right) m_X(x) + \frac{1}{\bar{F}_X(x)} \int_x^\infty \log \frac{\bar{F}_X(t)}{\bar{F}_Y(t)} \bar{F}_X(t)dt + m_Y(x). \tag{4.4}$$

In terms of quantile functions, we can write (4.4) in the form

$$J(u) = j\left(Q_X(u)\right) = M_Y(u) - M_X(u) + \frac{1}{1-u} \int_u^1 \left(1 - \frac{H_Y\left(A(p)\right)}{H_X(p)} \right) M_X(p)dp.$$

Here $Q_X(.)$ is taken as representing the true distribution and $Q_Y(.)$ an arbitrary reference model. The measure $J(.)$ provides the relative amount of uncertainty in the residual life of Y in comparison with that of X. We prove a theorem that identifies the class of distributions for which this relative entropy is a constant.

Theorem 4.2. *The cumulative divergence measure $J(u) = C$ for all u in $[0,1]$ if and only if the quantile function of Y admits the representation*

$$Q_Y(u) = Q_1(u) + Q_2(u)$$

where $Q_1(.)$ is the quantile function of the exponential distribution with mean C and $Q_2(.)$ is the quantile function of $\frac{X}{\theta}$.

Proof: When $J(u) = C$, we have

$$M_Y(u) - M_X(u) + \frac{1}{1-u}\int_u^1 \left(1 - \frac{h_Y\left(Q_X(p)\right)}{H_X(p)}\right) M_X(p)dp = C \tag{4.5}$$

From (2.2)

$$h_Y\left(Q_X(u)\right) = H_X(u) + \frac{1-\theta}{\theta M_X(u)}$$

and so $1 - \frac{h_Y(Q_X(u))}{H_X(u)} = \frac{\theta-1}{\theta}\frac{1}{M_X(u)H_X(u)}$. Hence

$$\begin{aligned}
\int_u^1 \left(1 - \frac{h_Y\left(Q_X(p)\right)}{H_X(p)}\right) M_X(p)dp &= \frac{\theta-1}{\theta}\int_u^1 \frac{1}{H_X(p)}dp, \\
&= \frac{\theta-1}{\theta}\int_u^1 -\frac{d}{dp}(1-p)M_X(p)dp, \\
&= \frac{\theta-1}{\theta}(1-u)M_X(u). \qquad (4.6)
\end{aligned}$$

Inserting (4.6) in (4.5) and simplifying

$$M_Y(u) = C + \frac{M_X(u)}{\theta} \tag{4.7}$$

or

$$\int_u^1 (1-p)q_Y(p)dp = C(1-u) + \frac{1}{\theta}\int_u^1 (1-p)q_X(p)dp$$

giving

$$q_Y(u) = \frac{C}{1-u} + \frac{1}{\theta}q_X(u).$$

Integrating from 0 to u

$$\begin{aligned}
Q_Y(u) &= -C\log(1-u) + \frac{1}{\theta}Q_X(u) \\
&= Q_1(u) + Q_2(u), \qquad (4.8)
\end{aligned}$$

as stated. Conversely assuming (4.8) we have (4.7) and substituting this in the expression on the left side of (4.5) we have the result stated and this completes the proof. □

Corollary 4.1. *1. When X is exponential (λ), Y is also exponential with parameter $\frac{1+C\lambda\theta}{\lambda\theta}$.*

2. When X has linear mean residual quantile distribution ([5]) with

$$Q_X(u) = -(\alpha+\mu)\log(1-u) - 2\alpha u, \quad \mu > 0, -\mu < \alpha < \mu$$

and

$$M_X(u) = \alpha u + \mu$$

Y also has the same form of distribution with linear mean residual quantile function

$$M_X(u) = \frac{\alpha}{\theta}u + \left(C + \frac{\mu}{\theta}\right).$$

It is noted that (4.8) gives a class of distributions many of which do not possess a closed form distribution functions, so that it is difficult to arrive such forms using the distribution function approach.

We conclude this work by noting that here we have proposed an alternative approach in analysing PMRLM through quantile functions. This brings in some new results and models that are sometimes difficult to arrive at by using the traditional approach.

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