
ON ARNOLD–VILLASEÑOR CONJECTURES FOR CHARACTERIZING EXPONENTIAL DISTRIBUTION BASED ON SAMPLE OF SIZE THREE

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Abstract:

- Arnold and Villaseñor [4] obtain a series of characterizations of the exponential distribution based on random samples of size two. These results were already applied in constructing goodness-of-fit tests in [7]. Extending the techniques from [4], we prove some of Arnold and Villaseñor's conjectures for samples of size three. An example with simulated data is discussed.

Key-Words:

- *exponential distribution; characterizations; order statistics.*

AMS Subject Classification:

- 62G30, 62E10.

1. INTRODUCTION

In general, the problem of characterization of probability distributions is described as follows. Suppose a family of distributions \mathcal{F} possesses a property \mathcal{A} . If, conversely, a distribution has property \mathcal{A} only if it is a member of that family, then property \mathcal{A} characterizes the family \mathcal{F} . This result is referred to as a characterization of the distributions in \mathcal{F} . Primary motivation for characterizations problems is due to statistical applications. If a statistical procedure assumes that property \mathcal{A} holds, then the underlying distribution must be a member of the family \mathcal{F} . Naturally, first characterizations results are for the normal family of distributions. The exponential distribution is one of the non-normal distributions, which has received a lot of attention as well. Comprehensive surveys of exponential characterizations can be found in [1], [3], [5], [6], and [8].

More recently, Arnold and Villaseñor [4] obtained a series of characterizations of the exponential distribution based on random samples of size two and conjectured possible generalizations for samples of size three. They provide motivation for their results by pointing out an example of a goodness-of-fit construction. A test for exponentiality based on the characterizations in [4] was recently constructed in [7]. Another possible use of the results in [4] and their generalizations, is in verifying modeling assumptions and in simulations (see also [8]). Extending the techniques from [4], we will prove some of Arnold and Villaseñor's conjectures.

Assume throughout that X_1, X_2 , and X_3 are independent random variables with a common absolutely continuous cumulative distribution function (cdf) F , such that $F(0) = 0$ and probability density function (pdf) f . Denote $X_{2:2} := \max\{X_1, X_2\}$, $X_{3:3} := \max\{X_1, X_2, X_3\}$, and $\bar{F} = 1 - F$. Consider the relations:

$$(1.1) \quad \sum_{j=1}^3 \frac{1}{j} X_j \quad \text{has pdf} \quad \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j f(jx),$$

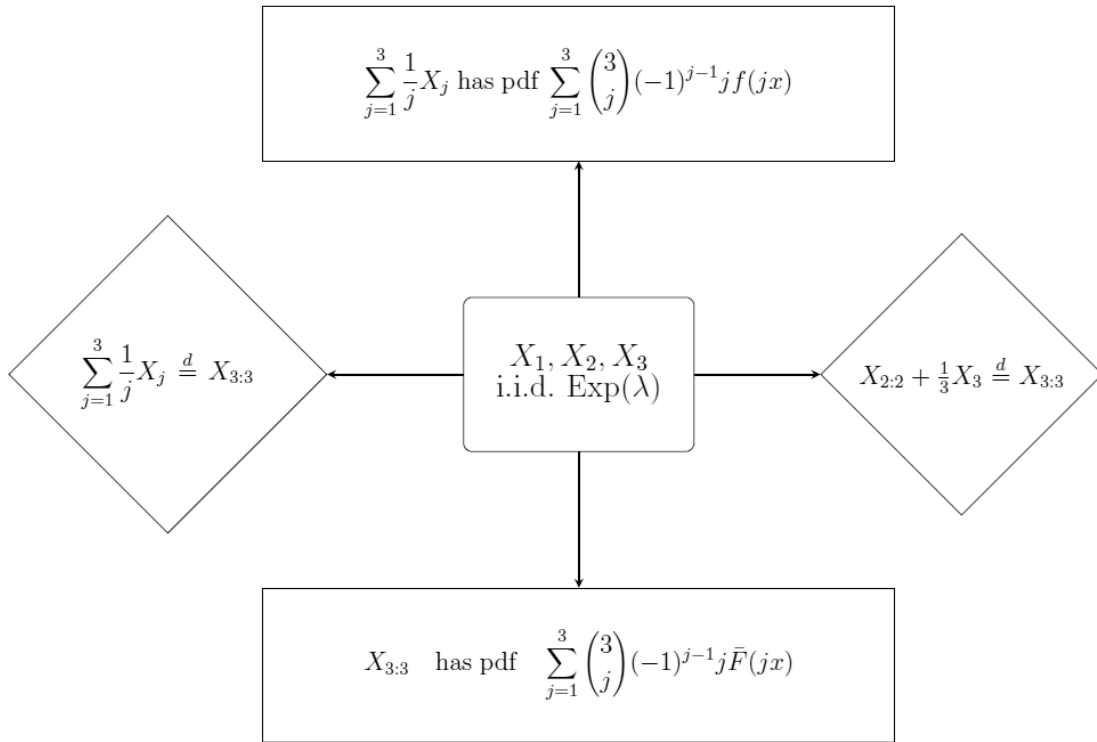
$$(1.2) \quad X_{3:3} \quad \text{has pdf} \quad \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j \bar{F}(jx),$$

$$(1.3) \quad \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j f(jx) = \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j \bar{F}(jx),$$

$$(1.4) \quad X_{2:2} + \frac{1}{3} X_3 \stackrel{d}{=} X_{3:3} \quad \text{and} \quad \sum_{j=1}^3 \frac{1}{j} X_j \stackrel{d}{=} X_{3:3},$$

where $\stackrel{d}{=}$ denotes equality in distribution. We will prove, under some regularity assumptions on F , that each one of these five conditions, on its own, is sufficient for X_1, X_2 , and X_3 to be exponentially distributed.

We organize this paper as follows. Using Laplace transforms, in Section 2 we prove the characterization (1.1). In Section 3, we establish characterization (1.2) utilizing the Taylor series expansion of the cdf F . In Section 4, using a recurrent relation, we prove that (1.3) is a sufficient condition for having exponential parent. Section 5 contains characterization results based on (1.4). In Section 6 we provide an example with simulated data. In the concluding section, we discuss possible extensions of the given results.



2. SUM OF THREE INDEPENDENT VARIABLES

To prove that (1.1) characterizes the exponential distribution, we will convert it into an equation for the Laplace transform $\varphi(t) := E[e^{-tX_1}]$.

Theorem 2.1. Assume $\varphi(t)$ is finite for all t in a neighbourhood of zero. If for $x > 0$

$$(2.1) \quad \sum_{j=1}^3 \frac{1}{j} X_j \quad \text{has pdf} \quad \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j f(jx),$$

then $X_1 \sim \text{exp}(\lambda)$ for some $\lambda > 0$.

Proof: It follows by (2.1), interchanging the order of summation and integration, that

$$\begin{aligned}
 \varphi(t) \varphi\left(\frac{t}{2}\right) \varphi\left(\frac{t}{3}\right) &= E \left[e^{-t \sum_{j=1}^3 \frac{1}{j} X_j} \right] \\
 &= \int_0^\infty e^{-tx} \left(\sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j f(jx) \right) dx \\
 (2.2) \quad &= \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} \int_0^\infty e^{-tx} j f(jx) dx \\
 &= \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} \varphi\left(\frac{t}{j}\right).
 \end{aligned}$$

Dividing both sides of (2.2) by $\varphi(t) \varphi(t/2) \varphi(t/3)$, we obtain

$$(2.3) \quad 1 = \alpha(t) \alpha\left(\frac{t}{2}\right) - 3 \alpha(t) \alpha\left(\frac{t}{3}\right) + 3 \alpha\left(\frac{t}{2}\right) \alpha\left(\frac{t}{3}\right),$$

where for $t > 0$

$$(2.4) \quad \alpha(t) := \frac{1}{\varphi(t)} = \sum_{k=0}^{\infty} a_k t^k.$$

Note that, the series in (2.4) is convergent in a neighbourhood of zero, by assumption. To prove the theorem, it is sufficient to show that

$$(2.5) \quad \alpha(t) = 1 + \lambda t, \quad \lambda > 0.$$

We will prove (2.5) by calculating the coefficients of the series in (2.4) to be: $a_0 = 1$, $a_1 = \lambda > 0$, and $a_k = 0$ for $k \geq 2$. It is clear that $a_0 = \varphi^{-1}(0) = 1$. Applying Cauchy formula for multiplication of two power series, we have for any nonzeros p and q ,

$$(2.6) \quad \alpha\left(\frac{t}{p}\right) \alpha\left(\frac{t}{q}\right) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^k \frac{1}{p^j q^{k-j}} a_j a_{k-j} \right) t^k.$$

Now, (2.3) and (2.6) yield for $k \geq 1$

$$(2.7) \quad \sum_{j=0}^k \left(\frac{1}{2^{k-j}} - \frac{3}{3^{k-j}} + \frac{3}{2^j 3^{k-j}} \right) a_j a_{k-j} = 0.$$

Setting $k = 1$ we see that equation (2.7) has as solution any a_1 . The assumption $F(0) = 0$ implies that there is $\lambda > 0$, such that $a_1 = \lambda > 0$. If $k = 2$, then (2.7) yields $a_2 = 0$. Assuming $a_j = 0$ for $2 \leq j \leq k - 1$, it follows from (2.7) that

$$\left(1 - \frac{1}{2^{k-1}}\right) a_k = 0.$$

Thus, $a_k = 0$ for any $k \geq 3$. Therefore, (2.5) holds, which completes the proof. \square

Note that, conversely, if $X_i \sim \exp(\lambda)$ for $i = 1, 2, 3$, then (2.1) holds true. To show this, it is sufficient to verify (2.2). Indeed, assuming $X_1 \sim \exp(\lambda)$, we have $\varphi(t) = (1 + \lambda t)^{-1}$. Therefore,

$$\begin{aligned} \int_0^{\infty} e^{-tx} \left(\sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j f(jx) \right) dx &= 3 \varphi(t) - 3 \varphi\left(\frac{t}{2}\right) + \varphi\left(\frac{t}{3}\right) \\ &= \frac{3}{1 + \lambda t} - \frac{6}{2 + \lambda t} + \frac{3}{3 + \lambda t} \\ &= \varphi(t) \varphi\left(\frac{t}{2}\right) \varphi\left(\frac{t}{3}\right) \\ &= E \left[e^{-t(X_1 + \frac{1}{2}X_2 + \frac{1}{3}X_3)} \right], \end{aligned}$$

which is equivalent to (2.1).

3. MAXIMUM OF THREE INDEPENDENT VARIABLES

In this section we will prove that, under some regularity assumptions on F , condition (1.2) is sufficient for X_1, X_2 , and X_3 to be exponentially distributed. The proof will be based on the Taylor series expansion of F .

Theorem 3.1. *Assume the cdf F has a power series representation for x in a neighborhood of zero. If for $x > 0$*

$$(3.1) \quad X_{3:3} \quad \text{has pdf} \quad \sum_{k=1}^3 \binom{3}{k} (-1)^{k-1} k \bar{F}(kx),$$

then $X_1 \sim \exp(1)$.

Proof: The relation (3.1) implies

$$(3.2) \quad F^2(x) f(x) + F(x) - 2F(2x) + F(3x) = 0.$$

Since $F(x) = \sum_{k=0}^{\infty} c_k x^k$ and $f(x) = \sum_{k=0}^{\infty} (k+1) c_{k+1} x^k$, Cauchy formula for the product of three power series yields

$$(3.3) \quad F^2(x) f(x) = \sum_{k=0}^{\infty} \left[\sum_{i=0}^k \sum_{j=0}^i c_j c_{i-j} (k+1-i) c_{k+1-i} \right] x^k.$$

Using (3.2) and (3.3), we obtain for any $k \geq 0$

$$(3.4) \quad \sum_{i=0}^k \sum_{j=0}^i c_j c_{i-j} (k+1-i) c_{k+1-i} + c_k (1 - 2^{k+1} + 3^k) = 0.$$

Since $F(0) = 0$, we have $c_0 = 0$. Also (3.4) with $k = 1$ yields $c_0^2 c_1 = 0$, which in turn implies that c_1 is undetermined. Let us set $c_1 = \delta$, where $-\infty < \delta < \infty$. Equation (3.4) with $k = 2$ yields $c_1^3 + 2c_2 = 0$. Hence, $c_2 = \delta^3/2$. We will prove by induction that

$$(3.5) \quad c_k = (-1)^{k-1} \frac{\delta^{2k-1}}{k!}, \quad k = 1, 2, 3, \dots$$

Indeed, assuming (3.5) holds true for $1, 2, \dots, k$, we have

$$(3.6) \quad \sum_{i=0}^{k+1} \sum_{j=0}^i c_j c_{i-j} (k+2-i) c_{k+2-i} = \sum_{i=2}^{k+1} \sum_{j=1}^{i-1} \frac{(-1)^{k+1} \delta^{2k+1}}{j! (i-j)! (k+1-i)!}.$$

Observe that

$$(3.7) \quad \begin{aligned} \sum_{i=2}^{k+1} \sum_{j=1}^{i-1} \frac{1}{j! (i-j)! (k+1-i)!} &= \sum_{i=2}^{k+1} \frac{1}{i! (k+1-i)!} \sum_{j=1}^{i-1} \frac{i!}{j! (i-j)!} \\ &= \frac{1}{(k+1)!} \sum_{i=2}^{k+1} \frac{(k+1)!}{i! (k+1-i)!} (2^i - 2) \\ &= \frac{1}{(k+1)!} \left[\sum_{i=2}^{k+1} \binom{k+1}{i} 2^i - 2 \sum_{i=2}^{k+1} \binom{k+1}{i} \right] \\ &= \frac{1}{(k+1)!} (3^{k+1} - 2^{k+2} + 1). \end{aligned}$$

It follows from (3.4), (3.6) and (3.7) that

$$(-1)^{k+1} \frac{\delta^{2k+1}}{(k+1)!} \left(3^{k+1} - 2^{k+2} + 1\right) + c_{k+1}(1 - 2^{k+2} + 3^{k+1}) = 0.$$

Therefore,

$$c_{k+1} = (-1)^k \frac{\delta^{2k+1}}{(k+1)!},$$

which completes the induction and hence proves (3.5).

Now, we have

$$F(x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\delta^{2k-1}}{k!} x^k = \frac{1}{\delta} \left(1 - e^{-\delta^2 x}\right).$$

Since $\lim_{x \rightarrow \infty} F(x) = 1$, we obtain $\delta = 1$. The proof is complete. □

It is not difficult to see that, conversely, if $X_1 \sim \exp(1)$, then (3.1) holds. Indeed, under the assumption of unit exponential parent variable, for the pdf of $X_{3:3}$ we obtain

$$3F^2(x) f(x) = 3(1 - e^{-x})^2 e^{-x} = 3\bar{F}(x) - 6\bar{F}(2x) + 3\bar{F}(3x),$$

which is equivalent to (3.1).

4. SUMS OF DENSITY AND DISTRIBUTION FUNCTIONS

In this section we will prove that (1.3) is a sufficient condition for X_1 to be exponentially distributed. It is straightforward that (1.3) is a necessary condition as well.

Theorem 4.1. *Assume that f is right-continuous at zero. If for $x > 0$*

$$(4.1) \quad \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j f(jx) = \sum_{j=1}^3 \binom{3}{j} (-1)^{j-1} j \bar{F}(jx),$$

then $X_1 \sim \exp(1)$.

Proof: The relation (4.1) leads to

$$(4.2) \quad [f(3x) - \bar{F}(3x)] - [f(2x) - \bar{F}(2x)] = [f(2x) - \bar{F}(2x)] - [f(x) - \bar{F}(x)].$$

Denoting $Q(y) = f(y) - \bar{F}(y)$, we rewrite (4.2) as

$$Q(y) - Q\left(\frac{2}{3}y\right) = Q\left(\frac{2}{3}y\right) - Q\left(\frac{1}{3}y\right).$$

Iterating this equation k times and taking limit as $k \rightarrow \infty$, we obtain

$$Q(y) - Q\left(\frac{2}{3}y\right) = Q\left(\frac{2}{3}y\right) - Q\left(\frac{1}{3}y\right) = \lim_{k \rightarrow \infty} Q\left(\left(\frac{2}{3}\right)^k y\right) - Q\left(\left(\frac{1}{3}\right)^k y\right) = 0.$$

This implies $Q(y) = Q(2y/3)$ and thus,

$$(4.3) \quad Q(y) = Q\left(\frac{2}{3}y\right) = \lim_{k \rightarrow \infty} Q\left(\left(\frac{2}{3}\right)^k y\right) = f(0+) - \bar{F}(0+) = f(0+) - 1.$$

On the other hand,

$$(4.4) \quad \lim_{y \rightarrow \infty} f(y) = \lim_{y \rightarrow \infty} f(y) - \lim_{y \rightarrow \infty} \bar{F}(y) = \lim_{y \rightarrow \infty} Q(y) = f(0+) - 1.$$

But since f is integrable, we have $\lim_{y \rightarrow \infty} f(y) = 0$, and therefore, by (4.3) and (4.4), $Q(x) = 0$. Thus, $f(x) = \bar{F}(x)$ for every $x \geq 0$. This, in turn, implies $X_1 \sim \exp(1)$. \square

5. SUM AND MAXIMUM OF THREE VARIABLES

It is known (e.g., Arnold *et al.* (2008), p. 77) that if $X \sim \exp\{\lambda\}$, then

$$(5.1) \quad \sum_{j=1}^3 \frac{1}{j} X_j \stackrel{d}{=} X_{3:3} \quad \text{and} \quad X_{2:2} + \frac{1}{3} X_3 \stackrel{d}{=} X_{3:3}.$$

We will prove that both relations in (5.1) are also characterization properties of the exponential distribution. Next lemma provides the key argument in the proof of Theorem 1 in [4] and of the theorem below.

Lemma 5.1. *If $F(0) = 0$, the pdf f has a Taylor series expansion for $x > 0$, and*

$$(5.2) \quad f^{(m)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{m-1} f'(0), \quad m = 1, 2, \dots,$$

then $X_1 \sim \exp\{\lambda\}$ for some $\lambda > 0$.

Proof: For the Taylor series of $f(x)$, using (5.2), we have for $x > 0$

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} x^m = f(0) + f(0) \sum_{m=1}^{\infty} \left[\frac{f'(0)}{f(0)} \right]^m \frac{x^m}{m!} = f(0) \exp\left\{ \frac{f'(0)}{f(0)} x \right\}.$$

Since $f(x)$ is a pdf, we have $f'(0)/f(0) < 0$. Denoting $\lambda = -f'(0)/f(0) > 0$ and setting $\int_0^{\infty} f(x) dx = 1$, we obtain $\lambda = f(0)$. Therefore, $f(x) = \lambda e^{-\lambda x}$. \square

Next theorem can be obtained as a particular case of the results in [9]. We include it here since it complements the other results for samples of size three given in Theorems 2.1–4.1 and thus provides an easily reference.

Theorem 5.1. *Assume the cdf F admits a power series representation in a neighborhood of zero and $F(0) = 0$.*

(i) *If*

$$(5.3) \quad X_{2:2} + \frac{1}{3} X_3 \stackrel{d}{=} X_{3:3},$$

then $X_1 \sim \exp\{\lambda\}$ for some $\lambda > 0$.

(ii) If

$$(5.4) \quad \sum_{j=1}^3 \frac{1}{j} X_j \stackrel{d}{=} X_{3:3},$$

then $X_1 \sim \exp\{\lambda\}$ for some $\lambda > 0$.

Proof: (i). The pdf of the left-hand side of (5.3) is

$$(5.5) \quad \begin{aligned} f_{X_{2:2}+X_{3/3}}(x) &= \int_0^x f_{X_{3/3}}(y) f_{X_{2:2}}(x-y) dy \\ &= \int_0^x 3f(3y) \frac{d}{dx} [F^2(x-y)] dy \\ &= 6 \int_0^x f(3y) F(x-y) f(x-y) dy. \end{aligned}$$

For the pdf of the right-hand side of (5.3), we have

$$(5.6) \quad f_{X_{3:3}}(x) = 3F^2(x) f(x) = 6f(x) \int_0^x F(y) f(y) dy.$$

Let $G(x) := F(x) f(x)$. It follows from (5.5) and (5.6) that (5.3) is equivalent to

$$(5.7) \quad \int_0^x f(3y) G(x-y) dy = f(x) \int_0^x G(y) dy.$$

Differentiating the left-hand side of (5.7) n times with respect to x , we obtain

$$\frac{d^n}{dx^n} \int_0^x f(3y) G(x-y) dy = \sum_{i=1}^n f^{(n-i)}(3x) G^{(i-1)}(0) + \int_0^x f(3y) G^{(n)}(x-y) dy.$$

Applying the Leibniz rule for the n -th derivative of a product of two functions to the right-hand side of (5.7), we obtain

$$\frac{d^n}{dx^n} \left[f(x) \int_0^x G(y) dy \right] = \sum_{i=1}^n \binom{n}{i} f^{(n-i)}(x) G^{(i-1)}(x) + f^{(n)}(x) \int_0^x G(y) dy.$$

In the last two equations letting $x = 0$, we have

$$(5.8) \quad \sum_{i=1}^n 3^{n-i} f^{(n-i)}(0) G^{(i-1)}(0) = \sum_{i=1}^n \binom{n}{i} f^{(n-i)}(0) G^{(i-1)}(0).$$

Since $G(0) = 0$ and $G'(0) = f^2(0)$, the above equation is equivalent to

$$(5.9) \quad \left[3^{n-2} - \binom{n}{2} \right] f^{(n-2)}(0) f^2(0) = \sum_{i=3}^n \left[\binom{n}{i} - 3^{n-i} \right] f^{(n-i)}(0) G^{(i-1)}(0),$$

where $n \geq 4$. We will prove that (5.9) implies (5.2). Equation (5.2) is trivially true for $m = 1$. To proceed by induction, assume (5.2) holds true for all $1 \leq m \leq n-3$, where $n \geq 4$. We need to prove it for $m = n-2$. Using the induction assumption, it is not difficult to obtain for $j = 1, 2, \dots, n-2$

$$G^{(j)}(0) = \sum_{i=0}^j \binom{j}{i} F^{(i)}(0) f^{(j-i)}(0) = f^2(0) \left[\frac{f'(0)}{f(0)} \right]^{j-1} (2^j - 1).$$

Therefore, using the induction assumption again, we have for $i = 3, 4, \dots, n - 1$

$$(5.10) \quad f^{(n-i)}(0) G^{(i-1)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{n-3} f'(0) f^2(0) (2^{i-1} - 1).$$

Substituting this in the right-hand side of (5.9) yields

$$\left[3^{n-2} - \binom{n}{2} \right] f^{(n-2)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{n-3} f'(0) \sum_{i=3}^n \left[\binom{n}{i} - 3^{n-i} \right] (2^{i-1} - 1).$$

To complete the proof of (5.2), it is sufficient to show that

$$3^{n-2} - \binom{n}{2} = \sum_{i=3}^n \left[\binom{n}{i} - 3^{n-i} \right] (2^{i-1} - 1),$$

which can be easily verified. This proves (5.2). The claim in (i) follows from (5.2) and the lemma. \square

Proof: (ii). Equation (5.4) is equivalent to

$$(5.11) \quad 6 \int_0^z f(y) \int_0^{z-y} f(2x) f(3(z-y-x)) dx dy = 6 f(z) \int_0^z F(y) f(y) dy.$$

Denoting

$$(5.12) \quad H(z-y) := \int_0^{z-y} f(2x) f(3(z-y-x)) dx,$$

we write (5.11) as

$$(5.13) \quad \int_0^z f(y) H(z-y) dy = f(z) \int_0^z G(y) dy.$$

Similarly to the proof of (i), differentiating n times both sides of (5.13) with respect to z and setting $z = 0$, we have

$$\sum_{i=1}^{n-1} f^{(n-1-i)}(0) H^{(i)}(0) = \sum_{i=1}^{n-1} \binom{n}{i+1} f^{(n-1-i)}(0) G^{(i)}(0).$$

Since $H'(0) = G'(0) = f^2(0)$, the last equation can be written for $k = n - 1$ as

$$(5.14) \quad \left[1 - \binom{k+1}{2} \right] f^{(k-1)}(0) f^2(0) = \sum_{i=2}^k \left[\binom{k+1}{i+1} G^{(i)}(0) - H^{(i)}(0) \right] f^{(k-i)}(0).$$

Now we are in a position to prove (5.2) by induction. (5.2) holds true for $m = 1, 2, \dots, k - 2$. Differentiating (5.12) with respect to z and setting $z = y$, we have

$$(5.15) \quad H^{(n)}(0) = \sum_{i=1}^n 2^{n-i} f^{(n-i)}(0) 3^{i-1} f^{(i-1)}(0).$$

Under the induction assumption, (5.15) implies for $j = 1, 2, \dots, n - 2$

$$H^{(j)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{j-1} f^2(0) (3^j - 2^j).$$

Using the induction assumption again, we have for $i = 3, 4, \dots, n - 1$

$$f^{(n-i)}(0) H^{(i-1)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{n-3} f'(0) f^2(0) (3^{i-1} - 2^{i-1}).$$

Recalling (5.10) from the proof of (i), we rewrite (5.14) as (note that $i = n$ corresponds to a 0 term)

$$\left[1 - \binom{n}{2} \right] f^{(n-2)}(0) = \left[\frac{f'(0)}{f(0)} \right]^{n-3} f'(0) \sum_{i=3}^n \left[\binom{n}{i} (2^{i-1} - 1) - (3^{i-1} - 2^{i-1}) \right].$$

Thus, to prove (5.2) for $k = n - 2$ it is sufficient to show that

$$1 - \binom{n}{2} = \sum_{i=3}^n \left[\binom{n}{i} (2^{i-1} - 1) - (3^{i-1} - 2^{i-1}) \right],$$

which verifies. This proves (5.2), which referring to the lemma, completes the proof of (ii). \square

6. EXAMPLE

We will illustrate a possible application of Theorem 5.1 with an example (see also [4]). Assume we have a simple random sample X_1, X_2, \dots, X_n for $n \geq 6$. Let us randomly divide the data set into six subsets, relabeled as

$$\begin{aligned} U_1, U_2, \dots, U_{n/6}, & \quad V_1, V_2, \dots, V_{n/6}, & \quad W_1, W_2, \dots, W_{n/6}, \\ X_1, X_2, \dots, X_{n/6}, & \quad Y_1, Y_2, \dots, Y_{n/6}, & \quad Z_1, Z_2, \dots, Z_{n/6}. \end{aligned}$$

Define for $i = 1, 2, \dots, n/4$

$$R_i := U_i + \frac{1}{2} V_i + \frac{1}{3} W_i, \quad S_i := \max\{U_i, V_i\} + \frac{1}{3} W_i \quad \text{and} \quad T_i := \max\{X_i, Y_i, Z_i\}.$$

Then, according to Theorem 5.1, the R 's, the S 's, and the T 's will have a common distribution if and only if the original X 's follow an exponential distribution.

Let us simulate a sample of size $n = 180$ from a parent variable with $\exp(1)$ distribution. The values of $R_i, S_i,$ and T_i for $i = 1, 2, \dots, 30$ are presented in Table 1.

Using the non-parametric two-sample Wicoxon rank test, we compare the sample distribution functions of the R 's and T 's on one hand and the S 's and T 's on another. The test results provide evidence supporting an exponential underlying distribution. Namely, the hypothesis that the distributions of the R 's and the T 's are the same cannot be rejected with p -value 0.7635 ($W = 471$). The hypothesis that the distributions of the S 's and the T 's are the same cannot be rejected with p -value 0.9357 ($W = 444$).

Table 1: Values R_i , S_i , and T_i for $i = 1, 2, \dots, 30$.

R	3.56	0.70	0.62	3.33	0.30	0.78	2.29	0.97	1.59	0.50
	0.83	2.27	0.69	2.95	0.32	4.12	0.74	0.91	2.66	0.48
	2.87	2.19	2.32	1.08	3.69	1.98	1.13	1.32	3.37	2.73
S	2.98	1.23	0.77	2.75	0.44	0.75	1.97	1.08	1.43	0.50
	0.76	1.65	0.58	2.39	0.27	3.41	0.73	0.89	2.63	0.40
	2.22	1.87	4.25	1.07	2.72	1.74	1.07	1.11	2.71	3.87
T	2.07	0.60	0.97	0.47	2.84	0.84	1.02	1.84	0.57	2.88
	1.39	1.92	8.46	1.77	2.60	1.42	1.50	0.47	0.26	2.17
	1.92	1.67	2.87	1.06	2.24	6.63	0.52	1.09	1.33	1.07

7. CONCLUDING REMARKS

In this paper we proved characterizations of the exponential distribution conjectured by Arnold and Villaseñor in [4]. Furthermore, under the assumptions of Theorem 2.1 and using the same technique of proof, it can be seen that if $X_1 + \frac{1}{2}X_2 + \frac{1}{3}X_3$ has as its density any one of the following seven forms, then X_i 's are exponential:

$$\begin{aligned}
 &3f(x) - 6f(2x) + 3\bar{F}(3x), & 3f(x) - 6\bar{F}(2x) + 3f(3x), \\
 &3\bar{F}(x) - 6f(2x) + 3f(3x), & 3f(x) - 6\bar{F}(2x) + 3\bar{F}(3x), \\
 &3F(x) - 6f(2x) + 3\bar{F}(3x), & 3\bar{F}(x) - 6\bar{F}(2x) + 3f(3x), \\
 &3\bar{F}(x) - 6\bar{F}(2x) + 3\bar{F}(3x).
 \end{aligned}$$

Likewise, under the assumptions of Theorem 3.1 and using the same technique of proof, it can be obtained that if $X_{3;3}$ has as its density any one of the preceding seven forms, then X_i 's are exponential.

The results presented here can be extended in several directions. Naturally, one would like to explore the general case of samples of size n for any $n \geq 4$. As we mentioned earlier, generalizations of Theorem 5.1 for arbitrary sample size are proved in [9]. Here we would like to propose as open problems the following two characterizations, which would extend Theorem 2.1 and Theorem 3.1, respectively.

Proposition 7.1. *Let X_1, X_2, \dots, X_n be i.i.d. random variables, where $n \geq 4$. Assume $\varphi(t)$ is finite for all t in a neighbourhood of zero. If for $x > 0$*

$$\sum_{j=1}^n \frac{1}{j} X_j \quad \text{has pdf} \quad \sum_{j=1}^n \binom{n}{j} (-1)^{j-1} j f(jx),$$

then $X_1 \sim \exp(\lambda)$ for some $\lambda > 0$.

Proposition 7.2. Let X_1, X_2, \dots, X_n be i.i.d. random variables, where $n \geq 4$. Assume the cdf F has a power series representation in a neighborhood of zero. If for $x > 0$

$$X_{n:n} \quad \text{has pdf} \quad \sum_{j=1}^n \binom{n}{j} (-1)^{j-1} j \bar{F}(jx),$$

then $X_1 \sim \exp(1)$.

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