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THE CUSUM MEDIAN CHART FOR KNOWN AND ESTIMATED PARAMETERS

Authors: Philippe Castagliola
– Université de Nantes & LS2N UMR CNRS 6004,
Nantes, France
philippe.castagliola@univ-nantes.fr

Fernanda Otilia Figueiredo
– Faculdade de Economia da Universidade do Porto & CEAUL,
Porto, Portugal
otilia@fep.up.pt

Petros E. Maravelakis
– Department of Business Administration, University of Piraeus,
Piraeus, Greece
maravel@unipi.gr

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Abstract:

- The usual CUSUM chart for the mean (CUSUM-$\bar{X}$) is a chart used to quickly detect small to moderate shifts in a process. In presence of outliers, this chart is known to be more robust than other mean-based alternatives like the Shewhart mean chart but it is nevertheless affected by these unusual observations because the mean ($\bar{X}$) itself is affected by the outliers. An outliers robust alternative to the CUSUM-$\bar{X}$ chart is the CUSUM median (CUSUM-$\tilde{X}$) chart, because it takes advantage of the robust properties of the sample median. This chart has already been proposed by other researchers and compared with other alternative charts in terms of robustness, but its performance has only been investigated through simulations. Therefore, the goal of this paper is not to carry out a robustness analysis but to study the effect of parameter estimation in the performance of the chart. We study the performance of the CUSUM-$\tilde{X}$ chart using a Markov chain method for the computation of the distribution and the moments of the run length. Additionally, we examine the case of estimated parameters and we study the performance of the CUSUM-$\tilde{X}$ chart in this case. The run length performance of the CUSUM-$\tilde{X}$ chart with estimated parameters is also studied using a proper Markov chain technique. Conclusions and recommendations are also given.

Key-Words:

- *average run length; CUSUM chart; estimated parameters; median; order statistics.*

AMS Subject Classification:

- 62G30, 62P30.

1. INTRODUCTION

A main objective for a product or a process is to continuously improve its quality. This goal, in statistical terms, may be expressed as variability reduction. Statistical Process Control (SPC) is a well known collection of methods aiming at this purpose and the control charts are considered as the main tools to detect shifts in a process. The most popular control charts are the Shewhart charts, the Cumulative Sum (CUSUM) charts and the Exponentially Weighted Moving Average (EWMA) charts. Shewhart type charts are used to detect large shifts in a process whereas CUSUM and EWMA charts are known to be fast in detecting small to moderate shifts.

The usual Shewhart chart for monitoring the mean of a process is the $\bar{X}$ chart. It is very efficient for detecting large shifts in a process (see for example Teoh *et al.*, 2014). An alternative chart used for the same purpose is the median ($\tilde{X}$) chart. The median chart is simpler than the $\bar{X}$ chart and it can be easily implemented by practitioners. The main advantages of the $\tilde{X}$ chart over the $\bar{X}$ chart is its robustness against outliers, contamination or small deviations from normality. This property is particularly important for processes running for a long time. Usually, in such processes, the data are not checked for irregular behaviour and, therefore, the $\tilde{X}$ chart is an ideal choice.

The CUSUM-$\bar{X}$ control chart has been introduced by Page (1954). It is able to quickly detect small to moderate shifts in a process. The CUSUM-$\bar{X}$ control chart uses information from a long sequence of samples and, therefore, it is able to signal when a persistent special cause exists (see for instance Nenes and Tagaras (2006) or Liu *et al.* (2014)). However, since it is mean-based, the CUSUM-$\bar{X}$ suffers from the inefficiency of the mean $\bar{X}$ to correctly handle outliers, contamination or small deviations from normality. A natural alternative solution to overcome this problem is the CUSUM-$\tilde{X}$ chart. This chart has already been proposed by Yang *et al.* (2010). In their paper they compare its performance with the Shewhart, EWMA and CUSUM charts for the mean under some contaminated normal distributions using only simulation procedures. This chart has also been considered by Nazir *et al.* (2013a), again in a simulation study with other CUSUM charts, in order to compare their performance for the phase II monitoring of location in terms of robustness against non-normality, special causes of variation and outliers.

In the last decades, different types of control charts for variable and count data based on CUSUM schemes, in univariate or multivariate cases, were proposed in the literature. Here we only mention some recent and less traditional works on robust and enhanced CUSUM schemes, nonparametric and adaptive CUSUM control charts and CUSUM charts for count and angular data. The interested reader could also take into account the references in the paper that follow.

Ou *et al.* (2011, 2012) carried out a comparative study to evaluate the performance and robustness of some typical $\overline{X}$, CUSUM and SPRT type control charts for monitoring either the process mean or both the mean and variance, also providing several design tables to facilitate the implementation of the optimal versions of the charts. Nazir *et al.* (2013b), following the same methodology used in Nazir *et al.* (2013a), but for monitoring the dispersion, considered several CUSUM control charts based on different scale estimators, and analyzed its performance and robustness. Qiu and Zhang (2015) investigated the performance of some CUSUM control charts for transformed data in order to accommodate deviations from the normality assumption when monitoring the process data, and they compared its efficiency with alternative nonparametric control charts. To improve the overall performance of the CUSUM charts to detect small, moderate and large shifts in the process mean, Al-Sabah (2010) and Abujiya *et al.* (2015) proposed the use of special sampling schemes to collect the data, such as the ranked set sampling scheme and some extensions of it, instead of using the traditional simple random sampling. Saniga *et al.* (2006, 2012) discussed the economic advantages of the CUSUM versus Shewhart control charts to monitor the process mean when one or two components of variance exist in a process.

The use of nonparametric control charts has attracted the attention of researchers and practitioners. Chatterjee and Qiu (2009) proposed a nonparametric cumulative sum control chart using a sequence of bootstrap control limits to monitor the mean, when the data distribution is non-normal or unknown. Li *et al.* (2010b) considered nonparametric CUSUM and EWMA control charts based on the Wilcoxon rank-sum test for detecting mean shifts, and they discussed the effect of phase I estimation on the performance of the chart. Mukherjee *et al.* (2013) and Graham *et al.* (2014) proposed CUSUM control charts based on the exceedance statistic for monitoring the location parameter. Chowdhury *et al.* (2015) proposed a single distribution free phase II CUSUM control chart based on the Lepage statistic for the joint monitoring of location and scale. The performance of this chart was evaluated by analyzing some moments and percentiles of the run-length distribution, and a comparative study with other CUSUM charts was provided. Wang *et al.* (2017) proposed a nonparametric CUSUM chart based on the Mann–Whitney statistic and on a change point model to detect small shifts.

Wu *et al.* (2009) and Li and Wang (2010) proposed adaptive CUSUM control charts implemented with a dynamical adjustment of the reference parameter of the chart to efficiently detect a wide range of mean shifts. Ryu *et al.* (2010) proposed the design of a CUSUM chart based on the expected weighted run length (EWRL) to detect shifts in the mean of unknown size. Li *et al.* (2010a) and Ou *et al.* (2013) considered adaptive control charts with variable sampling intervals or variable sample sizes to overcome the detecting ability of the traditional

CUSUM. Liu *et al.* (2014) proposed an adaptive nonparametric CUSUM chart based on sequential ranks that efficiently and robustly detects unknown shifts of several magnitudes in the location of different distributions. Wang and Huang (2016) proposed an adaptive multivariate CUSUM chart, with the reference value changing dynamically according to the current estimate of the process shift, that performs better than other competitive charts when the location shift is unknown but falls within an expected range.

Some CUSUM charts for count data can be found in Saghir and Lin (2014), for monitoring one or both parameters of the COM-Poisson distribution, in He *et al.* (2014), for monitoring linear drifts in Poisson rates, based on a dynamic estimation of the process mean level, and in Rakitzis *et al.* (2016), for monitoring zero-inflated binomial processes. Recently, Lombard *et al.* (2017) developed and analyzed the performance of distribution-free CUSUM control charts based on sequential ranks to detect changes in the mean direction and dispersion of angular data, which are of great importance to monitor several periodic phenomena that arise in many research areas.

To update the literature review in Jensen *et al.* (2006) about the effects of parameter estimation on control chart properties, Psarakis *et al.* (2014) provided some recent discussions on this topic. We also mention the works of Gandy and Kvaloy (2013) and Saleh *et al.* (2016), that suggest the design of CUSUM charts with a controlled conditional performance to reduce the effect of the Phase I estimation, avoiding at the same time the use of large amount of data. Such charts are designed with an in-control ARL that exceeds a desired value with a predefined probability, while guaranteeing a reduced effect on the out of control performance of the chart.

In this paper we study the CUSUM-$\tilde{X}$ chart with known and estimated parameters for monitoring the mean value of a normal process, a topic, as far as we know, not yet studied in the literature, apart from simulation. The CUSUM-$\tilde{X}$ chart is the most simple alternative to the CUSUM-$\bar{X}$ in terms of efficiency/robustness, when there is some chance of having small disturbances in the process. For instance, it is possible to have a small percentage of outliers or contamination along time, that does not affect the process location, and therefore the chart must not signal in such cases. It is important to note that the goal of this paper is not to monitor the capability of a capable but unstable process (as this has already been done in Castagliola and Vannman (2008) and Castagliola *et al.* (2009)), but to monitor the median of a process that must remain stable for ensuring the quality of the products. The paper has three aims. The first aim is to present the Markov chain methodology for the computation of the distribution and the moments of the run length for the known and the estimated parameters case. The second aim is to evaluate the performance of the CUSUM-$\tilde{X}$ chart in the known and estimated parameters case in terms of the average run length and standard deviation of the run length when the process is in- and out-of-control.

The third aim is to help practitioners in the implementation of the CUSUM-$\tilde{X}$ chart by giving the optimal pair of parameters for the chart with estimated parameters to behave like the one with known parameters.

The outline of the paper is the following. In Section 2, we present the definition and the main properties of the CUSUM-$\tilde{X}$ chart when the process parameters are known, along with the Markov chain methodology dedicated to the computation of the run length distribution of the chart and its moments. In Section 3, we study the case of estimated parameters for the CUSUM-$\tilde{X}$ chart and we also provide the modified Markov chain methodology for the computation of the run length properties. A comparison between CUSUM-$\tilde{X}$ with known v.s. estimated parameters is provided in Section 4. Finally, a detailed example is given in Section 5, followed by some conclusions and recommendations in Section 6.

2. THE CUSUM-$\tilde{X}$ CHART WITH KNOWN PARAMETERS

In this paper we will assume that $Y_{i,1}, ..., Y_{i,n}$, $i = 1, 2, ...$ is a Phase II sample of n independent normal $N(\mu_0 + \delta\sigma_0, \sigma_0)$ random variables where i is the subgroup number, μ_0 and σ_0 are the in-control mean value and standard deviation, respectively, and δ is the parameter representing the standardized mean shift, i.e. the process is assumed to be in-control (out-of-control) if $\delta = 0$ ($\delta \neq 0$).

The upper-sided CUSUM-$\bar{X}$ chart for detecting an increase in the process mean plots

$$Z_i^+ = \max(0, Z_{i-1}^+ + \bar{Y}_i - \mu_0 - k_z^+) \tag{2.1}$$

against i, for $i = 1, 2, ...$ where $\bar{Y}_i$ is the mean value of the quality variable for sample number i. The starting value is $Z_0^+ = z_0^+ \geq 0$ and k_z^+ is a constant. A signal is issued at the first i for which $Z_i^+ \geq h_z^+$, where h_z^+ is the upper control limit. The corresponding lower-sided CUSUM-$\bar{X}$ chart for detecting a decrease in the process mean plots

$$Z_i^- = \min(0, Z_{i-1}^- + \bar{Y}_i - \mu_0 + k_z^-) \tag{2.2}$$

against i, for $i = 1, 2, ...$ where k_z^- is a constant and the starting value is $Z_0^- = z_0^- \leq 0$. The chart signals at the first i for which $Z_i^- \leq -h_z^-$, where $-h_z^-$ is the lower control limit. There is a certain way to compute the values of k_z^+, k_z^- and h_z^+, h_z^- which is related to the distribution of Y_i's. The textbook of Hawkins and Olwell (1999) is an excellent reference on this subject.

Now, let $\tilde{Y}_i$ be the sample median of subgroup i, i.e.

$$\tilde{Y}_i = \begin{cases} Y_{i,((n+1)/2)} & \text{if } n \text{ is odd} \\ \dfrac{Y_{i,(n/2)} + Y_{i,(n/2+1)}}{2} & \text{if } n \text{ is even} \end{cases}$$

where $Y_{i,(1)}, Y_{i,(2)}, ..., Y_{i,(n)}$ is the ascendant ordered i-th subgroup. As the sample median is easier and faster to compute when the sample size n is an odd value, without loss of generality, we will confine ourselves to this case for the rest of this paper.

The upper-sided CUSUM-$\tilde{X}$ chart for detecting an increase in the process median is given by

$$U_i^+ = \max(0, U_{i-1}^+ + \tilde{Y}_i - \mu_0 - k^+) \tag{2.3}$$

where i is the sample number and $\tilde{Y}_i$ is the sample median. The starting value is $U_0^+ = u_0^+ \geq 0$ and k^+ is a constant. A signal is issued at the first i for which $U_i^+ \geq h^+$, where h^+ is the upper control limit. The corresponding lower-sided CUSUM-$\tilde{X}$ chart for detecting a decrease in the process median plots

$$U_i^- = \min(0, U_{i-1}^- + \tilde{Y}_i - \mu_0 + k^-) \tag{2.4}$$

against i, for $i = 1, 2, ...$ where k^- is a constant and the starting value is $U_0^- = u_0^- \leq 0$. The chart signals at the first i for which $U_i^- \leq -h^-$, where $-h^-$ is the lower control limit.

The mean value (ARL) and the standard deviation ($SDRL$) of the Run Length distribution are two common measures of performance of control charts that will be used in this work to design the CUSUM median chart. However we note that other methodologies recently appeared in the literature for the design of CUSUM charts. Li and Wang (2010), He *et al.* (2014) and Wang and Huang (2016), among others, suggested to design the CUSUM chart with the reference parameter dynamically adjusted according to the current estimate of the process shift, in order to improve the sensitivity of the chart to detect a wide range of shifts. Ryu *et al.* (2010) proposed the design of CUSUM charts based on the expected weighted run length, a measure of performance more appropriate than the usual ARL given that the magnitude of the shift is practically unknown. These interesting approaches are promising and will be explored in a future work.

As in the classical approach proposed by Brook and Evans (1972), the Run Length distribution of the upper-sided CUSUM-$\tilde{X}$ chart with known parameters can be obtained by considering a Markov chain with states denoted as $\{0, 1, ..., r\}$, where state r is the absorbing state (the computations for the lower-sided CUSUM-$\tilde{X}$ chart can be done accordingly). The interval from 0 to h^+ is partitioned into r subintervals $(H_j - \Delta, H_j + \Delta]$, $j \in \{0, ..., r-1\}$, each of them centered in $H_j = (2j+1)\Delta$ (the representative value of state j), with $\Delta = \frac{h^+}{2r}$. The Markov chain is in transient state $j \in \{0, ..., r-1\}$ for sample i if $U_i^+ \in (H_j - \Delta, H_j + \Delta]$, otherwise it is in the absorbing state.

Let $\mathbf{Q}$ be the (r, r) submatrix of probabilities $Q_{j,k}$ corresponding to the r transient states defined for the upward CUSUM-$\tilde{X}$ chart, i.e.

$$\mathbf{Q} = \begin{pmatrix} Q_{0,0} & Q_{0,1} & \cdots & Q_{0,r-1} \\ Q_{1,0} & Q_{1,1} & \cdots & Q_{1,r-1} \\ \vdots & \vdots & \vdots & \vdots \\ Q_{r-1,0} & Q_{r-1,1} & \cdots & Q_{r-1,r-1} \end{pmatrix}.$$

By definition, we have $Q_{j,k} = P(U_i^+ \in (H_k - \Delta, H_k + \Delta] | U_{i-1}^+ = H_j)$, where $j \in \{0, ..., r-1\}$ and $k \in \{1, ..., r-1\}$. This is actually equivalent to $Q_{j,k} = P(H_k - \Delta < \tilde{Y} + H_j - \mu_0 - k^+ \leq H_k + \Delta)$. This equation can be written as

$$\begin{aligned} Q_{j,k} &= P\left(\tilde{Y} \leq H_k - H_j + \Delta + k^+ + \mu_0\right) - P\left(\tilde{Y} \leq H_k - H_j - \Delta + k^+ + \mu_0\right) \\ &= F_{\tilde{Y}}\left(H_k - H_j + \Delta + k^+ + \mu_0 \middle| n\right) - F_{\tilde{Y}}\left(H_k - H_j - \Delta + k^+ + \mu_0 \middle| n\right), \end{aligned}$$

where $F_{\tilde{Y}}(...|n)$ is the cumulative distribution function (c.d.f.) of the sample median $\tilde{Y}_i$, $i \in \{1, 2, ...\}$. For the computation of $Q_{j,0}$, $j \in \{0, ..., r-1\}$ we have that

$$\begin{aligned} Q_{j,0} &= P\left(\tilde{Y} \leq -H_j + \Delta + k^+ + \mu_0\right) \\ &= F_{\tilde{Y}}\left(-H_j + \Delta + k^+ + \mu_0 \middle| n\right). \end{aligned}$$

The c.d.f. of the sample median $\tilde{Y}$ is given by

$$F_{\tilde{Y}}(y|n) = F_\beta\left(\Phi\left(\frac{y - \mu_0}{\sigma_0} - \delta\right) \middle| \frac{n+1}{2}, \frac{n+1}{2}\right),$$

where $\Phi(x)$ and $F_\beta(x|a, b)$ are the c.d.f. of the standard normal distribution and the beta distribution with parameters (a, b) (here, we have $a = b = \frac{n+1}{2}$), respectively.

Let $\mathbf{q} = (q_0, q_1, ..., q_{r-1})^T$ be the $(r, 1)$ vector of initial probabilities associated with the r transient states $\{0, ..., r-1\}$, where

$$q_j = \begin{cases} 0 \text{ if } U_0^+ \notin (H_j - \Delta, H_j + \Delta] \\ 1 \text{ if } U_0^+ \in (H_j - \Delta, H_j + \Delta] \end{cases}.$$

Using this method, the Run Length (RL) properties of the CUSUM-$\tilde{X}$ chart with known parameters can be accurately evaluated if the number r of subintervals in matrix $\mathbf{Q}$ is sufficiently large. In this paper, we have fixed $r = 200$. Using the results in Neuts (1981) or Latouche and Ramaswami(1999) concerning the fact that the number of steps until a Markov chain reaches the absorbing state is a *Discrete PHase-type* (or DPH) random variable, the probability mass function (p.m.f.) $f_{RL}(\ell)$ and the c.d.f. $F_{RL}(\ell)$ of the RL of the CUSUM-$\tilde{X}$ chart with known parameters are respectively equal to

$$\begin{aligned} f_{RL}(\ell) &= \mathbf{q}^T \mathbf{Q}^{\ell-1} \mathbf{c}, \\ F_{RL}(\ell) &= 1 - \mathbf{q}^T \mathbf{Q}^{\ell} \mathbf{1}, \end{aligned}$$

where $\mathbf{c} = \mathbf{1} - \mathbf{Q1}$ with $\mathbf{1} = (1, 1, ..., 1)^T$. Using the moment properties of a DPH random variable also allows to obtain the mean (ARL), the second non-central moment $E2RL = E(RL^2)$ and the standard-deviation ($SDRL$) of the RL

$$\begin{aligned} ARL &= \nu_1 \\ E2RL &= \nu_1 + \nu_2 \\ SDRL &= \sqrt{E2RL - ARL^2}, \end{aligned}$$

where ν_1 and ν_2 are the first and second factorial moments of the RL, i.e.

$$\begin{aligned} \nu_1 &= \mathbf{q}^T (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}, \\ \nu_2 &= 2\mathbf{q}^T (\mathbf{I} - \mathbf{Q})^{-2} \mathbf{Q1}. \end{aligned}$$

3. THE CUSUM-$\tilde{X}$ CHART WITH ESTIMATED PARAMETERS

In real applications the in-control process mean value μ_0 and the standard deviation σ_0 are usually unknown. In such cases they have to be estimated from a Phase I data set, having $i = 1, ..., m$ subgroups $\{X_{i,1}, ..., X_{i,n}\}$ of size n. Following Montgomery's (2009, p. 193 and p. 238) recommendations, these subgroups must be formed from observations taken in a time-ordered sequence in order to allow the estimation of between-sample variability, i.e., the process variability over time. The observations within a subgroup must be taken at the same time from a single and stable process, or at least as closely as possible to guarantee independence between them, to allow the estimation of the within-sample variability, i.e., the process variability at a given time. Here we assume that there is independence within and between subgroups, and also that $X_{i,j} \sim N(\mu_0, \sigma_0)$. The estimators that are usually used for μ_0 and σ_0 are

$$\hat{\mu}_0 = \frac{1}{m} \sum_{i=1}^{m} \bar{X}_i, \tag{3.1}$$

$$\hat{\sigma}_0 = \frac{1}{c_4(n)} \left(\frac{1}{m} \sum_{i=1}^{m} S_i \right), \tag{3.2}$$

where $\bar{X}_i$ and S_i are the sample mean and the sample standard deviation of subgroup i, respectively. Constant $c_4(n) = E(\frac{S_i}{\sigma_0})$ can be computed for different sample sizes n under normality. Although these estimators are usually used in the mean ($\bar{X}$) chart, they are not a straightforward choice with the median chart. Keeping in mind that the median chart is based on order statistics, a more typical selection of estimators based on order statistics is the following

$$\hat{\mu}'_0 = \frac{1}{m}\sum_{i=1}^{m}\tilde{X}_i, \tag{3.3}$$

$$\hat{\sigma}'_0 = \frac{1}{d_2(n)}\left(\frac{1}{m}\sum_{i=1}^{m}R_i\right), \tag{3.4}$$

where $\tilde{X}_i$ and $R_i = X_{i,(n)} - X_{i,(1)}$ are the sample median and the range of subgroup i, respectively, and $d_2(n) = E(\frac{R_i}{\sigma_0})$ is a constant tabulated assuming a normal distribution. Instead of the range we could have considered an estimator for the standard deviation based on quantiles to achieve higher level of robustness against outliers. The analysis of the properties of such CUSUM median chart is cumbersome and we will apply this approach in a future work.

The standardised versions of the lower-sided and the upper-sided CUSUM-$\tilde{X}$ chart with estimated parameters are given by

$$G_i^- = \min\left(0, G_{i-1}^- + \frac{\tilde{Y}_i - \hat{\mu}'_0}{\hat{\sigma}'_0} + k_g^-\right), \tag{3.5}$$

$$G_i^+ = \max\left(0, G_{i-1}^+ + \frac{\tilde{Y}_i - \hat{\mu}'_0}{\hat{\sigma}'_0} - k_g^+\right), \tag{3.6}$$

respectively, where $G_0^- = g_0^- \leq 0$, $G_0^+ = g_0^+ \geq 0$ with k_g^- and k_g^+ being two constants to be fixed. For the lower-sided (upper-sided) CUSUM-$\tilde{X}$ chart with estimated parameters a signal is issued at the first i for which $G_i^- \leq h_g^-$ ($G_i^+ \geq h_g^+$), where h_g^- (h_g^+) is the lower (upper) control limit.

Equations (3.5) and (3.6) can be equivalently written as

$$G_i^- = \min\left(0, G_{i-1}^- + \frac{\frac{\tilde{Y}_i - \mu_0}{\sigma_0} + \frac{\mu_0 - \hat{\mu}'_0}{\sigma_0}}{\frac{\hat{\sigma}'_0}{\sigma_0}} + k_g^-\right), \tag{3.7}$$

$$G_i^+ = \max\left(0, G_{i-1}^+ + \frac{\frac{\tilde{Y}_i - \mu_0}{\sigma_0} + \frac{\mu_0 - \hat{\mu}'_0}{\sigma_0}}{\frac{\hat{\sigma}'_0}{\sigma_0}} - k_g^+\right). \tag{3.8}$$

If we define the random variables V and W as $V = \frac{\hat{\mu}'_0 - \mu_0}{\sigma_0}$ and $W = \frac{\hat{\sigma}'_0}{\sigma_0}$, both G_i^+ and G_i^- can be written as

$$G_i^+ = \max\left(0, G_{i-1}^+ + \frac{\frac{\tilde{Y}_i - \mu_0}{\sigma_0} - V}{W} - k_g^+\right), \tag{3.9}$$

$$G_i^- = \min\left(0, G_{i-1}^- + \frac{\frac{\tilde{Y}_i - \mu_0}{\sigma_0} - V}{W} + k_g^-\right). \tag{3.10}$$

Apparently, the decision about when a process is declared as out of control does not change.

Both $\hat{\mu}_0'$ and $\hat{\sigma}_0'$ are random variables, therefore V and W are also random variables. Assuming that $\hat{\mu}_0'$ and $\hat{\sigma}_0'$ have fixed values, which actually means that V and W have fixed values, the conditional p.m.f. (denoted as $\hat{f}_{RL}(\ell)$) of RL, the conditional c.d.f. (denoted as $\hat{F}_{RL}(\ell)$) of RL and the conditional factorial moments (denoted as $\hat{\nu}_1$ and $\hat{\nu}_2$) can be computed through the equations given in section 2. Therefore, if the joint p.d.f. $f_{(V,W)}(v, w|m, n)$ of V and W is known, then the unconditional p.d.f. $f_{RL}(\ell)$ and the unconditional c.d.f. $F_{RL}(\ell)$ of the Run Length of the upper-sided CUSUM-$\tilde{X}$ chart with estimated parameters are equal to

$$f_{RL}(\ell) = \int_{-\infty}^{+\infty}\int_0^{+\infty} f_{(V,W)}(v, w|m, n)\hat{f}_{RL}(\ell)dwdv, \tag{3.11}$$

$$F_{RL}(\ell) = \int_{-\infty}^{+\infty}\int_0^{+\infty} f_{(V,W)}(v, w|m, n)\hat{F}_{RL}(\ell)dwdv. \tag{3.12}$$

Now we are ready to compute the unconditional ARL that is equal to

$$ARL = \int_{-\infty}^{+\infty}\int_0^{+\infty} f_{(V,W)}(v, w|m, n)\hat{\nu}_1 dwdv. \tag{3.13}$$

The unconditional $SDRL$ is derived using the well known relationship

$$SDRL = \sqrt{E2RL - ARL^2}, \tag{3.14}$$

where

$$E2RL = \int_{-\infty}^{+\infty}\int_0^{+\infty} f_{(V,W)}(v, w|m, n)(\hat{\nu}_1 + \hat{\nu}_2)dwdv.$$

Assuming normality, it is known that $\bar{X}_i$ and S_i^2 are two independent statistics. Consequently, $\hat{\mu}_0$ and $\hat{\sigma}_0$ in equations (3.1) and (3.2) are also independent statistics. However, $\tilde{X}_i$ and R_i are dependent statistics and so are $\hat{\mu}_0'$ and $\hat{\sigma}_0'$ in equations (3.3) and (3.4). Hogg (1960) proved that "*an odd location statistic like the sample median and an even scale location-free statistic like the sample range are uncorrelated when sampling from a symmetric distribution*". On account of the fact that we assume we are sampling from a normal distribution, the sample median $\tilde{X}_i$ and the sample range R_i are uncorrelated statistics. Since $\hat{\mu}_0'$ and $\hat{\sigma}_0'$

are averaged quantities of $\tilde{X}_i$ and R_i respectively, the central limit theorem can be used to conclude that their joint distribution asymptotically converges to a bivariate normal distribution as m increases. Moreover, the fact that these statistics are uncorrelated, leads us to the conclusion that the statistics $\hat{\mu}'_0$ and $\hat{\sigma}'_0$ are asymptotically independent (and so are V and W). Therefore, the joint p.d.f. $f_{(V,W)}(v, w|m, n)$ in equations (3.11)–(3.14) is well approximated by the product of the marginal p.d.f. $f_V(v|m, n)$ of V and $f_W(w|m, n)$ of W, i.e.

$$f_{(V,W)}(v, w|m, n) \simeq f_V(v|m, n) \times f_W(w|m, n). \tag{3.15}$$

To evaluate how large n has to be for equation (3.15) to hold, or instead, to get approximately independence between the median and the range statistics in case of normal data, we did some simulations, using the following algorithm.

1. We generated 100000 samples of size $n = 3, 5, 7, 9, 11, 13, 15$. Each observation X_{ij} $(i = 1, ..., 100000,\ j = 1, ..., n)$ follows a N(0,1) distribution;

2. Then, we computed the median ($\tilde{X}$) and the range (R) for these 100000 samples, i.e., we got the values $\tilde{X}_i$ and R_i, $i = 1, ..., 100000$.

3. Afterwards, we estimated the c.d.f. of the statistics $\tilde{X}$ and R, and the joint c.d.f. of $(\tilde{X}, R)$, i.e., the functions
- $F_{\tilde{X}}(xm) = P(\tilde{X} \leq xm)$, for several values of xm,
- $F_R(xr) = P(R \leq xr)$, for several values of xr,
- $F_{\tilde{X},R}(xm, xr) = P(\tilde{X} \leq xm \cap R \leq xr)$, for the combinations of (xm, xr).

4. Finally, we computed the difference $|F_{\tilde{X},R}(xm, xr) - F_{\tilde{X}}(xm) \times F_R(xr)|$ for all the combinations of (xm, xr), and we kept the maximum of these differences.

Table 1: Maximum difference between the joint c.d.f. and the product of the two marginal c.d.f. for some small sample sizes n.

Sample size n	$\max \lvert F_{\tilde{X},R}(xm, xr) - F_{\tilde{X}}(xm) \times F_R(xr)\rvert$
3	0.01134
5	0.00838
7	0.00640
9	0.00529
11	0.00481
13	0.00402
15	0.00402

The obtained results are presented in Table 1. As we can see, the difference between the joint c.d.f. and the product of the two marginal c.d.f.'s is very small and it gets smaller as n increases. This is not a proof of independence between $\tilde{X}$ and R, but for sure these statistics seem to be almost independent for small values of n. We also notice that the estimates for the nominal values of the process are the average of the medians and of the ranges, and consequently, the convergence to independence is faster.

For the computation of $f_V(v|m,n)$ and $f_W(w|m,n)$ there is no known closed-form, however, suitable approximations with satisfactory results are provided in Castagliola and Figueiredo (2013):

- The marginal p.d.f. $f_V(v|m,n)$, can be computed through the equation

$$f_V(v|m,n) \simeq \frac{b}{\sqrt{(v-\delta)^2+d^2}}\phi\left(b\sinh^{-1}\left(\frac{v-\delta}{d}\right)\right),$$

where $\phi(x)$ is the p.d.f. of the standard normal distribution, and

$$\begin{aligned} b &= \sqrt{\frac{2}{\ln(\sqrt{2(\gamma_2(V)+2)}-1)}}, \\ d &= \sqrt{\frac{2\mu_2(V)}{\sqrt{2(\gamma_2(V)+2)}-2}}, \end{aligned}$$

with

$$\begin{aligned} \mu_2(V) &\simeq \frac{1}{m}\left(\frac{\pi}{2(n+2)}+\frac{\pi^2}{4(n+2)^2}+\frac{\pi^2\left(\frac{13}{24}\pi-1\right)}{2(n+2)^3}\right), \\ \gamma_2(V) &\simeq \frac{2(\pi-3)}{m(n+2)}; \end{aligned}$$

- The marginal p.d.f. $f_W(w|m,n)$, can be computed through the equation

$$f_W(w|m,n) \simeq \frac{2\nu d_2^2(n)w}{c^2}f_{\chi^2}\left(\frac{\nu d_2^2(n)w^2}{c^2}|\nu\right),$$

where $f_{\chi^2}(x|\nu)$ is the p.d.f. of the χ^2 distribution with ν degrees of freedom with

$$\begin{aligned} \nu &\simeq \left(-2+2\sqrt{1+2\left(\frac{B}{A^2}+\frac{\left(-2+2\sqrt{1+\frac{2B}{A^2}}\right)^3}{16}\right)}\right)^{-1}, \\ c &\simeq A\left(1+\frac{1}{4\nu}+\frac{1}{32\nu^2}-\frac{1}{128\nu^3}\right), \end{aligned}$$

and $A = d_2(n)$, $B = \frac{d_3^2(n)}{m}$ where $d_2(n)$ and $d_3(n)$ are constants tabulated for the normal case.

4. A COMPARISON

In this work we compute the mean and the standard deviation of the Run Length distribution under the assumption that the scheme starts at the specified initial value, and we denote them as $ARL_{0,m}$ and $SDRL_{0,m}$ when the process is in-control, and as $ARL_{1,m}$ and $SDRL_{1,m}$, when the process is out-of-control, where m is the number of samples used in Phase I. We compare several control charts implemented with known and estimated parameters, all with the same in-control ARL value, here assumed equal to $ARL_{0,m} = 370.4$. The chart that exhibits the best performance to detect a specific shift size δ among its counterparts is the one that has the smaller $ARL_{1,m}$ value for this specific shift. Due to the symmetry of the Gaussian distribution, the performance of the charts are similar to detect either an upward or a downward shift of the same magnitude in the process mean value. Therefore we only concentrate our analysis on the performance of the charts for $\delta \geq 0$.

Using equations (3.13) and (3.14) we computed the $ARL_{1,m}$ and $SDRL_{1,m}$ values for several combinations of the sample size n, the number of samples m, and the shift size δ. These values $(ARL_{1,m}, SDRL_{1,m})$ are present in Table 2, together with the optimal set of parameters (H, K) for the specific n and δ values. The value $m = +\infty$ is associated with the known parameters case. In the estimated parameters case the number of subgroups considered in the estimation is $m = 5, 10, 20, 50, 100$. The pairs (H, K) given in each line are optimal in the sense that, among all the possible values of H and K, the noted pair gives the smallest $ARL_{1,m}$ value for the case $m = +\infty$. Setting $h^- = h^+ = H$ and $k^- = k^+ = K$ in the CUSUM-$\tilde{X}$ charts defined in (2.3) and (2.4) allow us to obtain a control chart with the ARL behavior described in Table 2. From Table 2 the following conclusions are easily observed:

- The $ARL_{1,m}$ and $SDRL_{1,m}$ values in the known and estimated parameters cases are significantly different when the shift size δ or the number of subgroups m is small. For example, in case of $n = 3$ and $\delta = 0.1$, the $ARL_{1,m}$ and $SDRL_{1,m}$ values are larger than 10^5 if $m = 5$, but for $m = \infty$ we have $ARL_{1,\infty} = 98.7$ and $SDRL_{1,\infty} = 69.9$.
- If we take $m = 20$ subgroups of size $n = 5$ for the estimation of the unknown process parameters, as usually happens, only for $\delta \geq 1$ we get $ARL_{1,m} \simeq ARL_{1,\infty}$.
- For δ small, even if m is relatively large, the $ARL_{1,m}$ values are larger than the ones obtained in the case of known parameters. See, for example, the case of $m = 100$ subgroups of size $n = 9$ (an overall sample of size $n \times m = 900$ observations): for $\delta = 0.1$, we get an $ARL_{1,m}$ value about 50% larger than the corresponding $ARL_{1,\infty}$.

Table 2: $ARL_{1,m}$, $SDRL_{1,m}$, H, K values for different combinations of n, m and δ.

δ	(H,K)	$n=3$					
		$m=5$	$m=10$	$m=20$	$m=50$	$m=100$	$m=\infty$
0.1	(8.003, 0.0501)	$(>10^5,>10^5)$	$(>10^5,>10^5)$	$(>10^5,>10^5)$	(636.9, 60171.3)	(165.7, 659.3)	(98.7, 69.9)
0.2	(6.003, 0.0999)	$(>10^5,>10^5)$	$(>10^5,>10^5)$	$(2723.1,>10^5)$	(76.7, 407.1)	(55.9, 66.2)	(46.5, 30.4)
0.3	(4.813, 0.1497)	$(>10^5,>10^5)$	$(>10^5,>10^5)$	(98.3, 14407.4)	(34.4, 49.9)	(30.3, 25.5)	(27.5, 17.1)
0.5	(3.444, 0.2489)	$(>10^5,>10^5)$	(63.6, 62917.8)	(17.7, 47.5)	(14.5, 11.3)	(13.9, 9.2)	(13.3, 7.7)
0.7	(2.666, 0.3478)	$(1039.6,>10^5)$	(11.8, 165.9)	(9.1, 8.6)	(8.4, 5.5)	(8.2, 4.9)	(8.0, 4.4)
1.0	(1.965, 0.4951)	(8.3, 2204.2)	(5.3, 5.7)	(4.9, 3.3)	(4.7, 2.7)	(4.7, 2.6)	(4.6, 2.4)
1.5	(1.319, 0.7432)	(2.8, 3.3)	(2.6, 1.7)	(2.5, 1.4)	(2.5, 1.3)	(2.5, 1.3)	(2.5, 1.2)
2.0	(0.934, 0.9963)	(1.7, 1.1)	(1.6, 0.9)	(1.6, 0.8)	(1.6, 0.8)	(1.6, 0.8)	(1.6, 0.8)

δ	(H,K)	$n=5$					
		$m=5$	$m=10$	$m=20$	$m=50$	$m=100$	$m=\infty$
0.1	(5.903, 0.0500)	$(>10^5,>10^5)$	$(>10^5,>10^5)$	$(>10^5,>10^5)$	(275.2, 8232.2)	(115.9, 281.0)	(79.1, 54.7)
0.2	(4.269, 0.0999)	$(>10^5,>10^5)$	$(>10^5,>10^5)$	(266.2, 49881.9)	(48.3, 107.4)	(39.9, 38.2)	(35.1, 22.2)
0.3	(3.349, 0.1496)	$(>10^5,>10^5)$	$(829.8,>10^5)$	(37.1, 474.7)	(23.4, 23.6)	(21.6, 16.0)	(20.2, 12.1)
0.5	(2.329, 0.2487)	$(1035.0,>10^5)$	(16.5, 316.0)	(11.3, 12.8)	(10.1, 6.9)	(9.8, 6.0)	(9.5, 5.3)
0.7	(1.767, 0.3473)	(14.5, 3462.5)	(6.9, 10.9)	(6.2, 4.5)	(5.9, 3.5)	(5.8, 3.2)	(5.7, 3.0)
1.0	(1.270, 0.4949)	(4.1, 11.7)	(3.6, 2.6)	(3.4, 2.0)	(3.3, 1.8)	(3.3, 1.7)	(3.3, 1.7)
1.5	(0.812, 0.7467)	(1.9, 1.3)	(1.8, 1.0)	(1.8, 0.9)	(1.8, 0.9)	(1.8, 0.9)	(1.7, 0.9)
2.0	(0.511, 0.9990)	(1.3, 0.6)	(1.2, 0.5)	(1.2, 0.5)	(1.2, 0.5)	(1.2, 0.5)	(1.2, 0.4)

δ	(H,K)	$n=7$					
		$m=5$	$m=10$	$m=20$	$m=50$	$m=100$	$m=\infty$
0.1	(4.749, 0.0500)	$(>10^5,>10^5)$	$(>10^5,>10^5)$	$(27291.2,>10^5)$	(170.3, 2633.7)	(90.9, 166.3)	(67.0, 45.5)
0.2	(3.348, 0.0999)	$(>10^5,>10^5)$	$(17933.4,>10^5)$	(100.8, 5616.9)	(36.2, 54.4)	(31.6, 27.1)	(28.6, 17.8)
0.3	(2.586, 0.1496)	$(>10^5,>10^5)$	(123.7, 36763.2)	(24.0, 102.8)	(18.0, 15.6)	(17.0, 11.8)	(16.2, 9.5)
0.5	(1.762, 0.2487)	(60.7, 50167.3)	(10.5, 42.8)	(8.5, 7.4)	(7.8, 5.0)	(7.7, 4.5)	(7.5, 4.1)
0.7	(1.316, 0.3472)	(7.0, 114.4)	(5.1, 4.9)	(4.7, 3.1)	(4.6, 2.6)	(4.5, 2.5)	(4.5, 2.3)
1.0	(0.926, 0.4963)	(3.0, 3.3)	(2.7, 1.8)	(2.6, 1.5)	(2.6, 1.4)	(2.6, 1.3)	(2.6, 1.3)
1.5	(0.557, 0.7484)	(1.5, 0.8)	(1.4, 0.7)	(1.4, 0.7)	(1.4, 0.6)	(1.4, 0.6)	(1.4, 0.6)
2.0	(0.286, 0.9998)	(1.1, 0.3)	(1.1, 0.3)	(1.1, 0.3)	(1.1, 0.3)	(1.1, 0.3)	(1.1, 0.3)

δ	(H,K)	$n=9$					
		$m=5$	$m=10$	$m=20$	$m=50$	$m=100$	$m=\infty$
0.1	(4.007, 0.0500)	$(>10^5,>10^5)$	$(>10^5,>10^5)$	$(7728.6,>10^5)$	(123.1, 1171.1)	(75.7, 115.1)	(58.7, 39.3)
0.2	(2.770, 0.0999)	$(>10^5,>10^5)$	$(2714.7,>10^5)$	(58.7, 1396.9)	(29.4, 35.5)	(26.4, 21.0)	(24.3, 14.9)
0.3	(2.114, 0.1496)	$(24658.7,>10^5)$	(48.4, 4394.4)	(18.2, 42.2)	(14.8, 11.7)	(14.2, 9.4)	(13.6, 7.9)
0.5	(1.416, 0.2487)	(19.3, 2775.4)	(7.9, 15.3)	(6.9, 5.3)	(6.5, 4.0)	(6.4, 3.6)	(6.3, 3.4)
0.7	(1.044, 0.3474)	(5.0, 20.6)	(4.1, 3.3)	(3.9, 2.4)	(3.8, 2.1)	(3.7, 2.0)	(3.7, 1.9)
1.0	(0.721, 0.4976)	(2.4, 2.0)	(2.2, 1.3)	(2.2, 1.2)	(2.2, 1.1)	(2.1, 1.1)	(2.1, 1.1)
1.5	(0.399, 0.7493)	(1.3, 0.6)	(1.2, 0.5)	(1.2, 0.5)	(1.2, 0.5)	(1.2, 0.5)	(1.2, 0.5)
2.0	(0.140, 0.9999)	(1.0, 0.2)	(1.0, 0.2)	(1.0, 0.1)	(1.0, 0.1)	(1.0, 0.1)	(1.0, 0.1)

- Moreover, the practical result referred in Quesenberry (1993), that an overall sample of size $n \times m = 400$ enables to design control charts with estimated control limits with a similar performance to the corresponding chart with true limits does not hold in the case of CUSUM-$\tilde{X}$ charts (see, for instance, the $ARL_{1,m}$ and $SDRL_{1,m}$ values for small values of δ and, in particular, for $\delta = 0.1$).
- However, as the number of samples m increases the $ARL_{1,m}$ and $SDRL_{1,m}$ values converge to the values of the known parameters case, for each shift, although very slowly. In particular, when δ becomes large, the difference between the $ARL_{1,m}$ and $SDRL_{1,m}$ values in the known and estimated parameters cases tends to be non-significant. But the CUSUM charts are more attractive and efficient than the Shewhart charts for

detecting small changes, and thus, it is important to determine optimal parameters H' and K' in order to guarantee the desired performance even for m or δ small.

For completeness, in Table 3 we also present the in-control $ARL_{0,m}$ and $SDRL_{0,m}$ values for the same pairs (H, K) considered in Table 2. As in Table 2, we observe again that, as m increases, the $ARL_{0,m}$ and $SDRL_{0,m}$ values converge very slowly to the known parameters case values. As we can observe more than $m = 100$ samples are often needed to implement charts with known and estimated parameters with similar performance.

Table 3: $ARL_{0,m}$, $SDRL_{0,m}$, H, K values for several pairs of n and m when the process is in-control.

(H,K)	$n=3$				
	$m=5$	$m=10$	$m=20$	$m=50$	$m=100$
$(8.003, 0.0501)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(15540.8, >10^5)$	$(1429.9, 16918.4)$
$(6.003, 0.0999)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(2989.3, >10^5)$	$(865.0, 3768.4)$
$(4.813, 0.1497)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(70529.8, >10^5)$	$(1517.3, 17422.5)$	$(682.9, 1905.6)$
$(3.444, 0.2489)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(5501.3, >10^5)$	$(853.9, 3474.9)$	$(544.3, 1032.7)$
$(2.666, 0.3478)$	$(>10^5, >10^5)$	$(48309.8, >10^5)$	$(2109.7, 49864.4)$	$(663.5, 1754.4)$	$(488.1, 777.8)$
$(1.965, 0.4951)$	$(>10^5, >10^5)$	$(5812.4, >10^5)$	$(1114.4, 7639.5)$	$(548.4, 1066.1)$	$(447.5, 622.0)$
$(1.319, 0.7432)$	$(20921.5, >10^5)$	$(1593.5, 24654.9)$	$(704.3, 2093.0)$	$(470.9, 717.4)$	$(416.6, 515.1)$
$(0.934, 0.9963)$	$(3608.3, >10^5)$	$(952.1, 4648.1)$	$(572.8, 1184.7)$	$(437.6, 587.0)$	$(402.1, 466.6)$

(H,K)	$n=5$				
	$m=5$	$m=10$	$m=20$	$m=50$	$m=100$
$(5.903, 0.0500)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(7577.1, >10^5)$	$(1192.1, 9102.0)$
$(4.269, 0.0999)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(1883.9, 25530.6)$	$(747.4, 2380.3)$
$(3.349, 0.1496)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(11797.0, >10^5)$	$(1092.2, 6198.7)$	$(605.9, 1346.7)$
$(2.329, 0.2487)$	$(>10^5, >10^5)$	$(33318.5, >10^5)$	$(2239.1, 38019.5)$	$(691.1, 1873.1)$	$(498.4, 812.6)$
$(1.767, 0.3473)$	$(>10^5, >10^5)$	$(5584.4, >10^5)$	$(1183.7, 7401.6)$	$(565.8, 1128.0)$	$(454.9, 643.8)$
$(1.270, 0.4949)$	$(17364.4, >10^5)$	$(1774.2, 22295.1)$	$(756.6, 2351.7)$	$(485.6, 767.9)$	$(423.1, 534.6)$
$(0.812, 0.7467)$	$(2555.9, 62815.4)$	$(872.2, 3340.4)$	$(555.7, 1078.3)$	$(433.5, 571.3)$	$(400.4, 461.0)$
$(0.511, 0.9990)$	$(1497.0, 14155.3)$	$(696.7, 1890.8)$	$(500.9, 824.3)$	$(416.6, 511.7)$	$(392.6, 435.7)$

(H,K)	$n=7$				
	$m=5$	$m=10$	$m=20$	$m=50$	$m=100$
$(4.749, 0.0500)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(5134.7, >10^5)$	$(1060.8, 6361.6)$
$(3.348, 0.0999)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(33886.4, >10^5)$	$(1474.5, 12982.1)$	$(683.9, 1848.5)$
$(2.586, 0.1496)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(5777.0, >10^5)$	$(918.3, 3825.0)$	$(564.8, 1115.2)$
$(1.762, 0.2487)$	$(>10^5, >10^5)$	$(10278.2, >10^5)$	$(1538.4, 13243.4)$	$(619.3, 1401.7)$	$(474.6, 715.3)$
$(1.316, 0.3472)$	$(48309.2, >10^5)$	$(2766.0, 57099.5)$	$(921.1, 3705.8)$	$(521.7, 913.9)$	$(438.0, 583.3)$
$(0.926, 0.4963)$	$(4897.4, >10^5)$	$(1170.3, 6580.6)$	$(638.6, 1505.7)$	$(457.4, 656.4)$	$(411.1, 495.0)$
$(0.557, 0.7484)$	$(1671.5, 17793.4)$	$(734.8, 2135.8)$	$(514.2, 880.8)$	$(421.0, 526.9)$	$(394.7, 442.4)$
$(0.286, 0.9998)$	$(1282.1, 9130.3)$	$(652.2, 1599.9)$	$(485.7, 760.6)$	$(411.7, 494.9)$	$(390.3, 428.3)$

(H,K)	$n=9$				
	$m=5$	$m=10$	$m=20$	$m=50$	$m=100$
$(4.007, 0.0500)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(3935.5, >10^5)$	$(974.1, 4940.8)$
$(2.770, 0.0999)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(17455.0, >10^5)$	$(1253.9, 8409.7)$	$(642.6, 1558.6)$
$(2.114, 0.1496)$	$(>10^5, >10^5)$	$(>10^5, >10^5)$	$(3795.7, >10^5)$	$(819.6, 2826.8)$	$(538.3, 983.3)$
$(1.416, 0.2487)$	$(>10^5, >10^5)$	$(5549.9, >10^5)$	$(1230.8, 7364.1)$	$(576.9, 1169.2)$	$(459.4, 657.7)$
$(1.044, 0.3474)$	$(14888.5, >10^5)$	$(1880.1, 19722.5)$	$(790.3, 2495.0)$	$(494.9, 799.0)$	$(427.2, 546.5)$
$(0.721, 0.4976)$	$(2879.4, 59731.6)$	$(946.2, 3818.3)$	$(580.8, 1186.6)$	$(441.5, 598.3)$	$(404.1, 472.5)$
$(0.399, 0.7493)$	$(1428.2, 11824.2)$	$(685.3, 1800.5)$	$(497.4, 808.7)$	$(415.6, 508.2)$	$(392.2, 434.3)$
$(0.140, 0.9999)$	$(1233.9, 8245.2)$	$(641.1, 1533.9)$	$(481.7, 744.4)$	$(410.4, 490.3)$	$(389.7, 426.3)$

Since the out-of-control $ARL_{1,m}$ values are clearly different in the known and in the estimated parameters case, it is important to determine the number m of Phase I samples we should consider to get approximately the same out-of-control $ARL_{1,m}$ values in both cases, using the same optimal control chart parameters H and K displayed in Table 2. Thus, with

$$\Delta = \frac{|ARL_{1,m} - ARL_{1,\infty}|}{ARL_{1,\infty}}$$

denoting the relative difference between the out-of-control $ARL_{1,m}$ (estimated parameter case) and $ARL_{1,\infty}$ (known parameter case) values, we computed the minimum value of m satisfying $\Delta \leq 0.05$ or $\Delta \leq 0.01$. The obtained minimum number m of Phase I samples is given in Table 4 for some values of n and δ. From this table we observe that:

- The value of m satisfying $\Delta < 0.05$ or $\Delta < 0.01$ can be very large, for instance, $m > 100$ if $\delta \leq 0.3$. In particular, for $\delta = 0.1$ and $n = 9$, to have $\Delta < 0.05$ we must consider at least 417 samples.
- For the most common subgroup sample size $n = 5$, the number of subgroups must be 569 for very small shifts (say, for $\delta = 0.1$), and consequently $n \times m$ will be 2845, much larger than 400, as suggested by Quesenberry (1993).
- The number of the required samples decreases with the increase shift size δ. We also observe that the number of subgroups m that are needed decreases with the sample size n, but the number of observations of the overall sample needed for the estimation, $n \times m$, also increases.

Table 4: Minimum number m of Phase I samples required to satisfy $\Delta = 0.05$ (left value) and $\Delta = 0.01$ (right value) when the process is out of control.

δ	$\Delta = (0.05, 0.01)$			
	$n = 3$	$n = 5$	$n = 7$	$n = 9$
0.1	$(711, 3339)$	$(569, 2669)$	$(479, 2241)$	$(417, 1945)$
0.2	$(319, 1485)$	$(237, 1101)$	$(191, 881)$	$(159, 737)$
0.3	$(181, 835)$	$(129, 597)$	$(101, 467)$	$(83, 385)$
0.5	$(81, 371)$	$(55, 257)$	$(43, 197)$	$(35, 159)$
0.7	$(45, 209)$	$(31, 141)$	$(23, 107)$	$(19, 87)$
1.0	$(25, 109)$	$(17, 73)$	$(13, 55)$	$(9, 45)$
1.5	$(11, 51)$	$(7, 35)$	$(5, 27)$	$(5, 21)$
2.0	$(7, 33)$	$(5, 21)$	$(3, 11)$	$(3, 5)$

As a conclusion, we observe that in most of the cases a very large number m of Phase I samples is needed so that the charts with known and estimated parameters have the same ARL performance. But this requirement is in general very hard to handle in practice for economical and logistic reasons. Therefore, for fixed values of m and n, the determination of adequate control chart parameters, taking into consideration the variability introduced by the parameters estimation is very challenging.

Table 5: Optimal values for H', K', ARL_1 and $SDRL_1$ subject to the constraint $ARL_0 = 370.4$.

δ	$n=3$				
	$m=5$	$m=10$	$m=20$	$m=50$	$m=100$
0.1	$(1.747, 0.01)$ $(121.6, > 10^5)$	$(2.739, 0.01)$ $(101.3, 32840.0)$	$(3.976, 0.01)$ $(88.0, 3910.3)$	$(5.886, 0.01)$ $(81.9, 444.6)$	$(7.379, 0.01)$ $(84.6, 151.3)$
0.2	$(1.747, 0.01)$ $(45.2, 55017.6)$	$(2.739, 0.01)$ $(35.7, 4544.5)$	$(3.976, 0.01)$ $(32.7, 441.4)$	$(5.886, 0.01)$ $(35.6, 55.1)$	$(7.024, 0.02)$ $(40.5, 32.8)$
0.3	$(1.747, 0.01)$ $(19.9, 11098.1)$	$(2.739, 0.01)$ $(17.1, 682.8)$	$(3.976, 0.01)$ $(18.0, 63.5)$	$(5.679, 0.02)$ $(22.4, 17.2)$	$(5.197, 0.09)$ $(25.0, 17.5)$
0.5	$(1.747, 0.01)$ $(6.9, 493.1)$	$(2.739, 0.01)$ $(7.7, 24.1)$	$(3.976, 0.01)$ $(9.7, 6.8)$	$(3.678, 0.16)$ $(11.9, 7.6)$	$(3.517, 0.21)$ $(12.6, 7.8)$
0.7	$(1.747, 0.01)$ $(4.0, 28.2)$	$(2.739, 0.01)$ $(5.2, 3.8)$	$(2.960, 0.15)$ $(6.5, 4.0)$	$(2.805, 0.27)$ $(7.4, 4.3)$	$(2.671, 0.32)$ $(7.7, 4.5)$
1.0	$(1.747, 0.01)$ $(2.7, 2.0)$	$(2.197, 0.16)$ $(3.5, 2.0)$	$(2.050, 0.35)$ $(4.1, 2.3)$	$(1.997, 0.44)$ $(4.4, 2.4)$	$(1.938, 0.48)$ $(4.5, 2.4)$
1.5	$(1.476, 0.18)$ $(1.8, 0.9)$	$(1.406, 0.49)$ $(2.1, 1.1)$	$(1.331, 0.64)$ $(2.3, 1.2)$	$(1.329, 0.70)$ $(2.4, 1.2)$	$(1.346, 0.71)$ $(2.4, 1.2)$
2.0	$(1.122, 0.46)$ $(1.3, 0.6)$	$(1.040, 0.74)$ $(1.5, 0.7)$	$(0.977, 0.88)$ $(1.5, 0.7)$	$(0.952, 0.95)$ $(1.6, 0.7)$	$(0.947, 0.97)$ $(1.6, 0.8)$

δ	$n=5$				
	$m=5$	$m=10$	$m=20$	$m=50$	$m=100$
0.1	$(1.536, 0.01)$ $(104.2, 46570.9)$	$(2.294, 0.01)$ $(82.1, 7398.7)$	$(3.230, 0.01)$ $(68.8, 1439.2)$	$(4.684, 0.01)$ $(64.1, 232.1)$	$(5.827, 0.01)$ $(67.4, 92.9)$
0.2	$(1.536, 0.01)$ $(34.4, 8053.3)$	$(2.294, 0.01)$ $(26.1, 916.8)$	$(3.230, 0.01)$ $(24.3, 136.6)$	$(4.684, 0.01)$ $(27.7, 27.7)$	$(4.642, 0.05)$ $(31.4, 23.8)$
0.3	$(1.536, 0.01)$ $(14.4, 1446.5)$	$(2.294, 0.01)$ $(12.6, 128.4)$	$(3.230, 0.01)$ $(13.8, 20.3)$	$(3.791, 0.06)$ $(17.3, 12.0)$	$(3.511, 0.11)$ $(18.8, 12.4)$
0.5	$(1.536, 0.01)$ $(5.3, 57.0)$	$(2.294, 0.01)$ $(6.1, 6.3)$	$(2.671, 0.08)$ $(7.6, 4.8)$	$(2.429, 0.19)$ $(8.8, 5.3)$	$(2.378, 0.22)$ $(9.1, 5.3)$
0.7	$(1.536, 0.01)$ $(3.3, 4.3)$	$(2.044, 0.07)$ $(4.2, 2.5)$	$(1.898, 0.22)$ $(4.9, 2.9)$	$(1.811, 0.30)$ $(5.4, 3.0)$	$(1.806, 0.32)$ $(5.5, 3.0)$
1.0	$(1.457, 0.05)$ $(2.3, 1.2)$	$(1.328, 0.31)$ $(2.8, 1.5)$	$(1.314, 0.40)$ $(3.0, 1.6)$	$(1.282, 0.46)$ $(3.2, 1.6)$	$(1.295, 0.47)$ $(3.2, 1.6)$
1.5	$(0.948, 0.38)$ $(1.5, 0.7)$	$(0.858, 0.59)$ $(1.6, 0.8)$	$(0.835, 0.67)$ $(1.7, 0.8)$	$(0.816, 0.72)$ $(1.7, 0.8)$	$(0.805, 0.74)$ $(1.7, 0.9)$
2.0	$(0.680, 0.62)$ $(1.1, 0.4)$	$(0.613, 0.80)$ $(1.2, 0.4)$	$(0.563, 0.90)$ $(1.2, 0.4)$	$(0.553, 0.94)$ $(1.2, 0.4)$	$(0.510, 0.99)$ $(1.2, 0.4)$

Table 6: (Cont'd) Optimal values for H', K', ARL_1 and $SDRL_1$ subject to the constraint $ARL_0 = 370.4$.

δ	$n=7$				
	$m=5$	$m=10$	$m=20$	$m=50$	$m=100$
0.1	$(1.355, 0.01)$ $(88.9, 19453.2)$	$(1.988, 0.01)$ $(67.5, 3754.2)$	$(2.771, 0.01)$ $(56.1, 844.7)$	$(3.987, 0.01)$ $(53.4, 147.8)$	$(4.940, 0.01)$ $(57.2, 65.7)$
0.2	$(1.355, 0.01)$ $(26.4, 2921.9)$	$(1.988, 0.01)$ $(20.2, 389.8)$	$(2.771, 0.01)$ $(19.6, 65.3)$	$(3.987, 0.01)$ $(23.3, 18.3)$	$(3.597, 0.06)$ $(26.0, 18.5)$
0.3	$(1.355, 0.01)$ $(10.8, 464.4)$	$(1.988, 0.01)$ $(10.1, 48.2)$	$(2.771, 0.01)$ $(11.6, 10.9)$	$(2.862, 0.08)$ $(14.2, 9.4)$	$(2.671, 0.12)$ $(15.2, 9.7)$
0.5	$(1.355, 0.01)$ $(4.3, 16.0)$	$(1.988, 0.01)$ $(5.2, 3.5)$	$(1.917, 0.13)$ $(6.3, 3.9)$	$(1.842, 0.20)$ $(7.0, 4.1)$	$(1.829, 0.22)$ $(7.3, 4.1)$
0.7	$(1.355, 0.01)$ $(2.9, 2.0)$	$(1.488, 0.14)$ $(3.6, 2.0)$	$(1.397, 0.25)$ $(4.0, 2.2)$	$(1.346, 0.31)$ $(4.3, 2.3)$	$(1.326, 0.33)$ $(4.4, 2.3)$
1.0	$(1.038, 0.18)$ $(2.0, 1.1)$	$(0.997, 0.34)$ $(2.3, 1.2)$	$(0.943, 0.43)$ $(2.4, 1.3)$	$(0.933, 0.47)$ $(2.5, 1.3)$	$(0.936, 0.48)$ $(2.5, 1.3)$
1.5	$(0.668, 0.46)$ $(1.3, 0.5)$	$(0.600, 0.62)$ $(1.3, 0.6)$	$(0.573, 0.69)$ $(1.4, 0.6)$	$(0.581, 0.71)$ $(1.4, 0.6)$	$(0.568, 0.73)$ $(1.4, 0.6)$
2.0	$(0.429, 0.69)$ $(1.0, 0.2)$	$(0.386, 0.82)$ $(1.1, 0.2)$	$(0.316, 0.93)$ $(1.1, 0.2)$	$(0.320, 0.95)$ $(1.1, 0.2)$	$(0.318, 0.96)$ $(1.1, 0.2)$

δ	$n=9$				
	$m=5$	$m=10$	$m=20$	$m=50$	$m=100$
0.1	$(1.220, 0.01)$ $(76.9, 11291.0)$	$(1.774, 0.01)$ $(57.1, 2426.2)$	$(2.459, 0.01)$ $(47.5, 568.6)$	$(3.521, 0.01)$ $(46.4, 103.4)$	$(4.350, 0.01)$ $(50.3, 50.5)$
0.2	$(1.220, 0.01)$ $(21.0, 1482.2)$	$(1.774, 0.01)$ $(16.5, 210.2)$	$(2.459, 0.01)$ $(16.7, 37.5)$	$(3.143, 0.03)$ $(20.3, 15.3)$	$(2.889, 0.07)$ $(22.4, 15.5)$
0.3	$(1.220, 0.01)$ $(8.6, 208.6)$	$(1.774, 0.01)$ $(8.6, 23.5)$	$(2.459, 0.01)$ $(10.1, 7.4)$	$(2.226, 0.10)$ $(12.2, 8.0)$	$(2.224, 0.12)$ $(12.9, 7.8)$
0.5	$(1.220, 0.01)$ $(3.8, 6.7)$	$(1.685, 0.03)$ $(4.6, 2.7)$	$(1.532, 0.15)$ $(5.4, 3.2)$	$(1.465, 0.21)$ $(5.9, 3.3)$	$(1.438, 0.23)$ $(6.1, 3.4)$
0.7	$(1.220, 0.01)$ $(2.6, 1.4)$	$(1.164, 0.18)$ $(3.1, 1.7)$	$(1.095, 0.27)$ $(3.4, 1.8)$	$(1.056, 0.32)$ $(3.6, 1.9)$	$(1.061, 0.33)$ $(3.6, 1.9)$
1.0	$(0.825, 0.23)$ $(1.8, 0.9)$	$(0.758, 0.38)$ $(1.9, 1.0)$	$(0.724, 0.45)$ $(2.0, 1.1)$	$(0.736, 0.47)$ $(2.1, 1.1)$	$(0.735, 0.48)$ $(2.1, 1.1)$
1.5	$(0.519, 0.48)$ $(1.1, 0.4)$	$(0.446, 0.63)$ $(1.2, 0.4)$	$(0.401, 0.71)$ $(1.2, 0.5)$	$(0.382, 0.75)$ $(1.2, 0.5)$	$(0.379, 0.76)$ $(1.2, 0.5)$
2.0	$(0.315, 0.68)$ $(1.0, 0.1)$	$(0.250, 0.82)$ $(1.0, 0.1)$	$(0.205, 0.90)$ $(1.0, 0.1)$	$(0.206, 0.92)$ $(1.0, 0.1)$	$(0.213, 0.92)$ $(1.0, 0.1)$

In this paper we computed, for fixed values of m and n, new chart parameters denoted as (H', K'), in order to achieve the desired in-control performance, i.e. such that, for fixed values of m and n, we have $ARL(m, n, H', K', \delta = 0) = 370.4$ and, for a fixed value of δ, $ARL(m, n, H', K', \delta)$ is the smallest out-of-control $ARL_{1,m}$. These new pairs of constants are given in Tables 5 and 6 for various combinations of n, m and δ, and might be used as the chart parameters in the CUSUM-$\tilde{X}$ charts defined in (3.8) and (3.7), i.e., we might choose $h_g^- = h_g^+ = H'$ and $k_g^- = k_g^+ = K'$. In each cell of Tables 5 and 6, the two numbers in the first row are new chart parameters (H', K'), and the two numbers in the second row are the $ARL_{1,m}$ and $SDRL_{1,m}$ values. As we can observe, with these constants (H', K') determined with the unconditional run length distribution, we can guarantee the same performance of the corresponding chart implemented with known process parameters, or even a better performance, except in the cases of $m = 5, 10$ and $\delta = 0.1$. For the shift size δ that must be quickly detected, the values presented in Tables 5 and 6 allow the practitioners to easily implement the most efficient median CUSUM control chart. For instance, if $n = 5$ and $m = 20$, the optimal CUSUM-$\tilde{X}$ chart to detect a shift of size $\delta = 1$ must be designed with the constants $H' = 1.314$ and $K' = 0.40$. With these chart parameters we get the values $ARL_{1,m} = 3$ and $SDRL_{1,m} = 1.6$.

5. AN ILLUSTRATIVE EXAMPLE

In order to illustrate the use of the CUSUM-$\tilde{X}$ chart when the parameters are estimated, let us consider the same example as the one in Castagliola and Figueiredo (2013), i.e. a 125g yogurt cup filling process for which the quality characteristic Y is the weight of each yogurt cup. The Phase I dataset used in this example consists of $m = 10$ subgroups of size $n = 5$ plotted in the left part of Figure 1 with "$\circ$". From this Phase I dataset, using (3.3) and (3.4), we obtain $\hat{\mu}_0' = 125.02$ and $\hat{\sigma}_0' = 0.864$. According to the quality practitioner in charge of this process, a shift of $0.5\sigma_0$ (i.e., $\delta = 0.5$) in the process position should be interpreted as a signal that something is going wrong in the production. For $m = 10$, $n = 5$ and $\delta = 0.5$, Table 5 suggests to use $K' = 2.294$ and $H' = 0.01$.

The Phase II dataset used in this example consists of $m = 30$ subgroups of size $n = 5$ plotted in the right part of Figure 1 with "$\bullet$". The first 15 subgroups are supposed to be in-control while the last 15 subgroups are supposed to have a smaller yogurt weight, and thus, to be out-of-control. In Figure 2, we plotted the statistics G_i^- and G_i^+ corresponding to (3.5) and (3.6). This figure shows that the 7th first subgroups are in-control but, from subgroups #8 to #15, the process experiences a light out-of-control situation (increase) as the points "$\bullet$" corresponding to the G_i^+'s are above the upper limit $K' = 2.294$. During subgroups #16 and #17, the process returns to the in-control state but, as expected, suddenly experiences a new strong out-of-control situation (decrease) as the points "$\circ$" corresponding to the G_i^-'s are now below the lower limit $-K' = -2.294$.

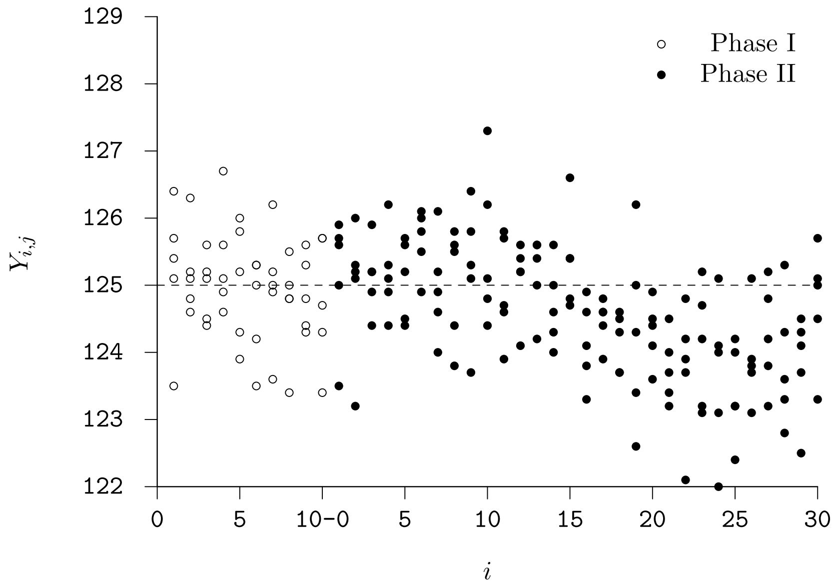

Figure 1: Phase I and Phase II samples corresponding to the 125g yogurt cup filling process.

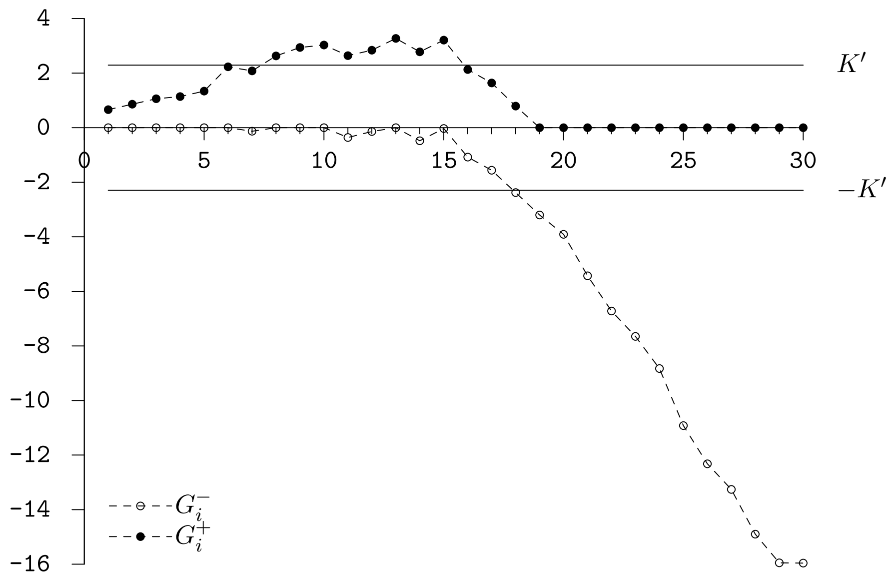

Figure 2: CUSUM-$\tilde{X}$ chart corresponding to the Phase II sample of Figure 1.

6. CONCLUSIONS

Although the CUSUM-$\tilde{X}$ chart has already been proposed by Yang *et al.* (2010) and Nazir *et al.* (2013a), in both of these papers the authors have only investigated the performance of this chart through simulations and compared its performance with other charts in terms of robustness. Moreover, in the implementation of the chart Yang *et al.* (2010) assumed the process parameters known, and Nazir *et al.* (2013a) also considered them fixed and known, after a prior estimation of such parameters through the use of different location and scale estimators. But they really did not analyze the effect of the parameters estimation in the performance of the chart in comparison with the performance of the corresponding chart implemented with true parameters, the main objective of our paper. We used a Markov chain methodology to compute the run length distribution and the moments of the CUSUM-$\tilde{X}$ chart in order to study its performance when the parameters are known and estimated. In this paper we present several tables that allow us to observe that the chart implemented with estimated parameters exhibits a completely different performance in comparison to the one of the chart implemented with known parameters. We also provide modified chart parameters that allow the practitioners to implement the CUSUM-$\tilde{X}$ chart with estimated control limits with a given desired in-control performance. More specifically, the main conclusions are: a) if the shift size δ or the number of samples m used in the estimation is small, there is a large difference between the $ARL_{1,m}$ and the $SDRL_{1,m}$ values obtained in the known and estimated parameters cases, b) for δ small, even if m is relatively large, the $ARL_{1,m}$ values are larger than the ones obtained in the case of known parameters, c) the $ARL_{1,m}$ and $SDRL_{1,m}$ values converge to the values of the known parameters case as the number of samples m increases, d) the number of subgroups m to have a relative difference between the out-of-control ARL values in the known and estimated parameters cases less than 5% or 1% can be very large, and depends on the value of δ, e) it is possible to obtain new chart parameters in order to achieve a desired in-control performance. As a general conclusion, the CUSUM-$\tilde{X}$ chart can be a valuable alternative chart for practitioners since it is simpler than the CUSUM-$\bar{X}$ chart. The fact that it is robust against outliers, contamination or small deviations from normality is another advantage.

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