REVSTAT – Statistical Journal
Volume 16, Number 4, October 2018, 475–486

HIGHLY D-EFFICIENT WEIGHING DESIGNS FOR AN EVEN NUMBER OF OBJECTS

Authors: Bronisław Ceranka
– Department of Mathematical and Statistical Methods, Poznań University of Life Sciences, Poznań, Poland
bronicer@up.poznan.pl

Małgorzata Graczyk
– Department of Mathematical and Statistical Methods, Poznań University of Life Sciences, Poznań, Poland
magra@up.poznan.pl

Received: April 2016 Revised: November 2016 Accepted: November 2016

Abstract:

- In this paper we formulate how to add $a = 1, 2, 3$ runs to a near D-optimal weighing design to get a highly D-efficient weighing design when the number of objects p is even.

Key-Words:

- *D-optimal design; efficiency; spring balance weighing design.*

AMS Subject Classification:

- 62K05, 05B20.

1. INTRODUCTION

We study a weighing experiment where observations follow the linear model $\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e}$, where $\mathbf{y} = (y_1, y_2, ..., y_n)^{'}$ is a $n \times 1$ random vector of observations, $\mathbf{X}$ is the model matrix identified by the weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$, where $\mathbf{\Phi}_{n\times p}\{0,1\}$ denotes the set of all $n \times p$ matrices with elements 0 or 1, $rank(\mathbf{X}) = p$, $\mathbf{w} = (w_1, w_2, ..., w_p)^{'}$ is a $p \times 1$ vector of true unknown parameters (weights) and $\mathbf{e} = (e_1, e_2, ..., e_n)^{'}$ is $n \times 1$ random vector of errors. We assume, $\mathrm{E}(\mathbf{e}) = \mathbf{0}_n$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{I}_n$, where $\mathbf{0}_n$ is the $n \times 1$ zero vector and $\mathbf{I}_n$ is the identity matrix of order n. The least squares estimator of $\mathbf{w}$ is of the form $\hat{\mathbf{w}} = (\mathbf{X}^{'}\mathbf{X})^{-1}\mathbf{X}^{'}\mathbf{y}$ and the variance matrix of $\hat{\mathbf{w}}$ is given by the formula $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2(\mathbf{X}^{'}\mathbf{X})^{-1}$ and $\mathbf{X}^{'}\mathbf{X}$ is called the information matrix for the design.
Our goal is to determine an optimal experimental plan $\mathbf{X}$ that minimizes the volume of the confidence region for $\mathbf{w}$ assuming that the errors are normally distributed. This is equivalent to the determining a design $\mathbf{X}$ such that $\det(\mathbf{X}^{'}\mathbf{X})$ is maximum. Such a design $\mathbf{X}$ is called D-optimal. D-optimality of weighing designs is studied in [3], [4], [6].

2. THE MAIN RESULT

Through the paper we assume that p is even. In [5], for even p it is shown that the maximum $\det(\mathbf{X}^{'}\mathbf{X})$ is attained if $\mathbf{X}^{'}\mathbf{X} = t\,(\mathbf{I}_p + \mathbf{J}_p)$ and each row of $\mathbf{X}$ contains k or $k+1$ ones, where $p = 2k$ and $\mathbf{J}$ is a matrix of all 1s. For the design $\mathbf{X}$ having k ones in each row and even p, an upper bound for $\det(\mathbf{X}^{'}\mathbf{X})$ is given in [1]. In [1], the following theorem was also proven.

Theorem 2.1. *For any* $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$,

$$\det(\mathbf{X}^{'}\mathbf{X}) = (p-1)\left(\frac{np}{4(p-1)}\right)^p \tag{2.1}$$

if and only if

$$\mathbf{X}^{'}\mathbf{X} = \frac{n}{4(p-1)}\left(p\mathbf{I}_p + (p-2)\mathbf{J}_p\right), \tag{2.2}$$

where $\frac{np}{4(p-1)}$ *and* $\frac{n(p-2)}{4(p-1)}$ *are integers.*

Here, we define $\mathrm{D}_{\text{eff}}(\mathbf{X})$ as

$$\mathrm{D}_{\text{eff}}(\mathbf{X}) = \left(\frac{\det(\mathbf{X}^{'}\mathbf{X})}{\det(\mathbf{Y}^{'}\mathbf{Y})}\right)^{\frac{1}{p}}, \tag{2.3}$$

where $\mathbf{Y}$ is a regular D-optimal spring balance weighing design having k or $k+1$ ones in each row $(p = 2k)$ and $\mathbf{Y}'\mathbf{Y} = \frac{(p+2)n}{4(p+1)}(\mathbf{I}_p + \mathbf{J}_p)$, see [5].

Definition 2.1. Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$ for which p is even is said to be near D-optimal if $\det(\mathbf{X}'\mathbf{X}) = (p-1)\left(\frac{np}{4(p-1)}\right)^p$.

In [1], some construction methods for near D-optimal weighing designs for certain values of n and p were provided. However, construction methods are needed for general n and p. Given a near D-optimal design for p objects and $n-a$ measurements we describe how to add a measurements in such way that the resulting design is highly D-efficient.

2.1. Adding $a = 1$ measurements

Let $\mathbf{X}_1$ be a near D-optimal design in $\mathbf{\Psi}_{(n-1)\times p}\{0,1\}$. In order to locate highly D-efficient design in $\mathbf{\Phi}_{n\times p}\{0,1\}$, we add one measurement, i.e. $p\times 1$ vector $\mathbf{x}$ of 0's or 1's having property $\mathbf{x}'\mathbf{1}_p = t$. So, $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$ is given in the following form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}' \end{bmatrix}. \tag{2.4}$$

Thus for $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$ in (2.4), $\det(\mathbf{X}'\mathbf{X}) = \left(1 + \mathbf{x}'(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{x}\right)\cdot\det(\mathbf{X}_1'\mathbf{X}_1)$, by Theorem 18.1.1 in [2]. Then we have the following theorem.

Theorem 2.2. *For any* $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$ *given by* (2.4),

$$\det\left(\mathbf{X}'\mathbf{X}\right) \le (p-1)\left(\frac{(n-1)p}{4(p-1)}\right)^p\left(1 + \frac{p^3+8}{(n-1)p^2}\right). \tag{2.5}$$

Proof: By Theorem 2.1

$$\det(\mathbf{X}_1'\mathbf{X}_1) = (p-1)\left(\frac{(n-1)p}{4(p-1)}\right)^p \tag{2.6}$$

implies

$$\mathbf{X}_1'\mathbf{X}_1 = \frac{n-1}{4(p-1)}\left(p\mathbf{I}_p + (p-2)\mathbf{J}_p\right), \tag{2.7}$$

where $\frac{(n-1)p}{4(p-1)}$ and $\frac{(n-1)(p-2)}{4(p-1)}$ are integers. Apply the formula given in (2.6) to compute the determinant of the information matrix. So,

$$\det(\mathbf{X}'\mathbf{X}) = (p-1)\left(\frac{(n-1)p}{4(p-1)}\right)^p\left(1 + \mathbf{x}'(\mathbf{X}_1'\mathbf{X}_1)^{-1}\mathbf{x}\right).$$

Since $(\mathbf{X}_1'\mathbf{X}_1)^{-1} = \frac{4(p-1)}{(n-1)p}\left(\mathbf{I}_p - \frac{p-2}{p(p-1)}\mathbf{J}_p\right)$, we obtain

$$\text{(2.8)} \quad \det\left(\mathbf{X}'\mathbf{X}\right) = (p-1)\left(\frac{(n-1)p}{4(p-1)}\right)^p \left(1 + \frac{4(p-1)}{(n-1)p}\left(\mathbf{x}'\mathbf{x} - \frac{p-2}{p(p-1)}\,\mathbf{x}'\mathbf{J}_p\,\mathbf{x}\right)\right).$$

To maximise (2.8), we determine the maximum value of the function

$$\text{(2.9)} \qquad \eta(\mathbf{x}) = \mathbf{x}'\mathbf{x} - \frac{p-2}{p(p-1)}\mathbf{x}'\mathbf{J}_p\mathbf{x}.$$

Consequently, $\eta(\mathbf{x}) = t - \frac{p-2}{p(p-1)}t^2 \le \frac{p^3+8}{4p(p-1)}$ and the equality holds if and only if $t = 0.5(p+2)$. From the above and (2.8) we obtain (2.5). □

Corollary 2.1. *For a spring balance weighing design* $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$ *given by* (2.4), $\det\left(\mathbf{X}'\mathbf{X}\right) = (p-1)\left(\frac{(n-1)p}{4(p-1)}\right)^p\left(1 + \frac{p^3+8}{(n-1)p^2}\right)$ *provided that* (2.7) *holds and* $\mathbf{x}'\mathbf{1}_p = 0.5(p+2)$.

2.2. Adding $a = 2$ measurements

Let $\mathbf{X}_1 \in \mathbf{\Phi}_{(n-2)\times p}\{0,1\}$ be near D-optimal. Let $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$ be in the following form

$$\text{(2.10)} \qquad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}' \\ \mathbf{y}' \end{bmatrix},$$

where $\mathbf{x}$ and $\mathbf{y}$ are vectors of 0's and 1's and $\mathbf{x}'\mathbf{1}_p = t$, $\mathbf{y}'\mathbf{1}_p = u$, $\mathbf{x}'\mathbf{y} = m$, $0 \le m \le \min(t,u)$.

Theorem 2.3. *For any* $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$ *given by* (2.10)

$$\det\left(\mathbf{X}'\mathbf{X}\right) \le \begin{cases} Q(n,p)R(n,p) & \text{if } p = 0 \bmod 4 \\ Q(n,p)L(n,p) & \text{if } p+2 = 0 \bmod 4, \end{cases}$$

where

$$\text{(2.11)} \qquad \begin{aligned} Q(n,p) &= (p-1)\left(\frac{(n-2)p}{4(p-1)}\right)^p, \\ R(n,p) &= \left(1 + \frac{p^3+p^2+16}{(n-2)p^2}\right)\left(1 + \frac{p-1}{n-2}\right), \\ L(n,p) &= \left(1 + \frac{(p-1)(p+2)}{(n-2)p}\right)\left(1 + \frac{(p+2)(p^2-3p+8)}{(n-2)p^2}\right). \end{aligned}$$

Proof: By Theorem 2.1

$$\det\left(\mathbf{X}_1'\mathbf{X}_1\right) = (p-1)\left(\frac{(n-2)p}{4(p-1)}\right)^p \tag{2.12}$$

implies

$$\mathbf{X}_1'\mathbf{X}_1 = \frac{n-2}{4(p-1)}\left(p\mathbf{I}_p + (p-2)\mathbf{J}_p\right), \tag{2.13}$$

where $\frac{(n-2)p}{4(p-1)}$ and $\frac{(n-2)(p-2)}{4(p-1)}$ are integers. By Theorem 18.1.1 in [2]

$$\det(\mathbf{X}'\mathbf{X}) = \det(\mathbf{X}_1'\mathbf{X}_1)\det\left(\mathbf{I}_2 + \begin{bmatrix}\mathbf{x}'\\ \mathbf{y}'\end{bmatrix}\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\begin{bmatrix}\mathbf{x} & \mathbf{y}\end{bmatrix}\right)$$

and

$$\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1} = \frac{4(p-1)}{(n-2)p}\left(\mathbf{I}_p - \frac{p-2}{p(p-1)}\mathbf{J}_p\right).$$

Next, by the formula given in (2.12) we have

$$\det(\mathbf{X}'\mathbf{X}) = (p-1)\left(\frac{(n-2)p}{4(p-1)}\right)^p \cdot \det(\mathbf{\Omega}), \tag{2.14}$$

where

$$\mathbf{\Omega} = \begin{bmatrix} 1 + \frac{4(p-1)}{(n-2)p}\left(t - \frac{p-2}{p(p-1)}t^2\right) & \frac{4(p-1)}{(n-2)p}\left(m - \frac{p-2}{p(p-1)}tu\right) \\ \frac{4(p-1)}{(n-2)p}\left(m - \frac{p-2}{p(p-1)}tu\right) & 1 + \frac{4(p-1)}{(n-2)p}\left(u - \frac{p-2}{p(p-1)}u^2\right) \end{bmatrix}.$$

As we want to maximise (2.14), we determine the maximum values of

$$t - \frac{p-2}{p(p-1)}t^2 \text{ and } u - \frac{p-2}{p(p-1)}u^2 \tag{2.15}$$

and concomitantly the minimum value of

$$\left(m - \frac{p-2}{p(p-1)}tu\right)^2. \tag{2.16}$$

The maximum values in (2.15) each as a function of p is attained if and only if $t = u = 0.5(p+2)$. If $p = 0 \bmod 4$, then the minimum value of (2.16) is equal to $\frac{(p^2+8)^2}{16p^2(p-1)^2}$ when $m = 0.25(p+4)$. Hence $\det(\mathbf{\Omega}) \leq \left(1 + \frac{p^3+p^2+16}{(n-2)p^2}\right)\left(1 + \frac{p-1}{n-2}\right)$ and

$$\det(\mathbf{X}'\mathbf{X}) \leq (p-1)\left(1 + \frac{p^3+p^2+16}{(n-2)p^2}\right)\left(1 + \frac{p-1}{n-2}\right)\left(\frac{(n-2)p}{4(p-1)}\right)^p. \tag{2.17}$$

The equality in (2.17) holds if and only if $t = u = 0.5(p+2)$ and $m = 0.25(p+4)$.

If $p+2 = 0 \bmod 4$, then the minimum value of (2.16) is equal to $\frac{(p+2)^2(p-4)^2}{16p^2(p-1)^2}$ when $m = 0.25(p+2)$. Therefore, $\det(\boldsymbol{\Omega}) \leq \left(1 + \frac{(p-1)(p+2)}{(n-2)p}\right)\left(1 + \frac{(p+2)(p^2-3p+8)}{(n-2)p^2}\right)$ and

$$\det(\mathbf{X}'\mathbf{X}) \leq (p-1)\left(1 + \frac{(p-1)(p+2)}{(n-2)p}\right) \times \left(1 + \frac{(p+2)(p^2-3p+8)}{(n-2)p^2}\right)\left(\frac{(n-2)p}{4(p-1)}\right)^p . \tag{2.18}$$

The equality in (2.18) holds if and only if $t = u = 0.5(p+2)$ and $m = 0.25(p+2)$. □

Corollary 2.2. *Let $Q(n,p)$, $R(n,p)$, $L(n,p)$ be of the form* (2.11) *and p be even. Then for a spring balance weighing design $\mathbf{X} \in \boldsymbol{\Phi}_{n\times p}\{0,1\}$ given by* (2.10),

$$\det\left(\mathbf{X}'\mathbf{X}\right) = \begin{cases} Q(n,p)R(n,p) & \text{if } p = 0 \bmod 4 \\ Q(n,p)L(n,p) & \text{if } p+2 = 0 \bmod 4, \end{cases}$$

provided (2.13) *holds and*

$$\begin{cases} \mathbf{x}'\mathbf{1}_p = \mathbf{y}'\mathbf{1}_p = 0.5(p+2) & \\ \text{and} & \\ \mathbf{x}'\mathbf{y} = 0.25(p+4) & \text{if } p = 0 \bmod 4, \\ \mathbf{x}'\mathbf{y} = 0.25(p+2) & \text{if } p+2 = 0 \bmod 4. \end{cases}$$

2.3. Adding $a = 3$ measurements

Next, we assume that there exists a near D-optimal spring balance weighing design $\mathbf{X}_1$ for p objects and $n-3$ measurements in the class $\boldsymbol{\Phi}_{(n-3)\times p}\{0,1\}$. So, $\mathbf{X} \in \boldsymbol{\Phi}_{n\times p}\{0,1\}$ is given in the form

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{x}' \\ \mathbf{y}' \\ \mathbf{z}' \end{bmatrix}, \tag{2.19}$$

where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are vectors of 0's and 1's and

$$\begin{cases} \mathbf{x}'\mathbf{1}_p = t, \ \mathbf{x}'\mathbf{y} = m, \ 0 \leq m \leq \min(t,u) \\ \mathbf{y}'\mathbf{1}_p = u, \ \mathbf{x}'\mathbf{z} = q, \ 0 \leq q \leq \min(t,w) \\ \mathbf{z}'\mathbf{1}_p = w, \ \mathbf{y}'\mathbf{z} = h, \ 0 \leq h \leq \min(u,w). \end{cases} \tag{2.20}$$

By Theorem 2.1

$$\det(\mathbf{X}_1'\mathbf{X}_1) = (p-1)\left(\frac{(n-3)p}{4(p-1)}\right)^p, \tag{2.21}$$

implies

$$\mathbf{X}_1'\mathbf{X}_1 = \frac{n-3}{4(p-1)}\left(p\mathbf{I}_p + (p-2)\mathbf{J}_p\right), \tag{2.22}$$

where $\frac{n-3}{4(p-1)}$ and $\frac{(n-3)(p-2)}{4(p-1)}$ are integers. By using the formula given in (2.21) and Theorem 18.1.1 in [2], we obtain

$$\det(\mathbf{X}'\mathbf{X}) = (p-1)\left(\frac{(n-3)p}{4(p-1)}\right)^p \det\left(\mathbf{I}_3 + \begin{bmatrix}\mathbf{x}'\\ \mathbf{y}'\\ \mathbf{z}'\end{bmatrix}\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1}\left[\,\mathbf{x}\ \mathbf{y}\ \mathbf{z}\,\right]\right).$$

Because $\left(\mathbf{X}_1'\mathbf{X}_1\right)^{-1} = \frac{4(p-1)}{(n-3)p}\left(\mathbf{I}_p - \frac{p-2}{p(p-1)}\mathbf{J}_p\right)$, we have

$$\det(\mathbf{X}'\mathbf{X}) = (p-1)\left(\frac{(n-3)p}{4(p-1)}\right)^p \det(\mathbf{T}), \tag{2.23}$$

where $\mathbf{T} = \mathbf{I}_3 + \frac{4(p-1)}{(n-3)p}\begin{bmatrix}\mathbf{x}'\\ \mathbf{y}'\\ \mathbf{z}'\end{bmatrix}\left(\mathbf{I}_p - \frac{p-2}{p(p-1)}\mathbf{J}_p\right)\left[\,\mathbf{x}\ \mathbf{y}\ \mathbf{z}\,\right]$. By (2.20),

$$\begin{aligned}\det(\mathbf{T}) = & \left(1+\frac{4(p-1)}{(n-3)p}\left(t-\frac{p-2}{p(p-1)}t^2\right)\right)\left(1+\frac{4(p-1)}{(n-3)p}\left(u-\frac{p-2}{p(p-1)}u^2\right)\right)\\ & \cdot\left(1+\frac{4(p-1)}{(n-3)p}\left(w-\frac{p-2}{p(p-1)}w^2\right)\right)\\ & + 2\left(\frac{4(p-1)}{(n-3)p}\right)^3\left(m-\frac{p-2}{p(p-1)}tu\right)\left(q-\frac{p-2}{p(p-1)}tw\right)\left(h-\frac{p-2}{p(p-1)}uw\right)\\ & -\left(1+\frac{4(p-1)}{(n-3)p}\left(t-\frac{p-2}{p(p-1)}t^2\right)\right)\left(\frac{4(p-1)}{(n-3)p}\right)^2\left(h-\frac{p-2}{p(p-1)}uw\right)^2\\ & -\left(1+\frac{4(p-1)}{(n-3)p}\left(u-\frac{p-2}{p(p-1)}u^2\right)\right)\left(\frac{4(p-1)}{(n-3)p}\right)^2\left(q-\frac{p-2}{p(p-1)}tw\right)^2\\ & -\left(1+\frac{4(p-1)}{(n-3)p}\left(w-\frac{p-2}{p(p-1)}w^2\right)\right)\left(\frac{4(p-1)}{(n-3)p}\right)^2\left(m-\frac{p-2}{p(p-1)}tu\right)^2.\end{aligned}$$

As we want to maximise (2.23), we simultaneously determine the maximum values of

$$t-\frac{p-2}{p(p-1)}t^2,\ u-\frac{p-2}{p(p-1)}u^2 \text{ and } w-\frac{p-2}{p(p-1)}w^2 \tag{2.24}$$

and the minimum values of

$$\left(h-\frac{p-2}{p(p-1)}uw\right)^2,\ \left(q-\frac{p-2}{p(p-1)}tw\right)^2 \text{ and } \left(m-\frac{p-2}{p(p-1)}tu\right)^2. \tag{2.25}$$

The maximum values in (2.24) are all attained if and only if $t=u=w=0.5(p+2)$. If $p=0 \bmod 4$, then the minimum values in (2.25) are equal to $\frac{(p^2+8)^2}{16p^2(p-1)^2}$ when $m=q=h=0.25(p+4)$. Then

$$\begin{aligned}\det(\mathbf{T}) &\le \left(1+\frac{p^3+8}{(n-3)p^2}\right)^3+2\left(\frac{p^2+8}{(n-3)p^2}\right)^3-3\left(1+\frac{p^3+8}{(n-3)p^2}\right)\left(\frac{p^2+8}{(n-3)p^2}\right)^2\\ &= \left(1-\frac{p-1}{n-3}\right)\left(\left(1+\frac{p^3+8}{(n-3)p^2}\right)\left(1+\frac{p^3+p^2+16}{(n-3)p^2}\right)-2\left(\frac{p^2+8}{(n-3)p^2}\right)^2\right)\end{aligned}$$

and

$$\begin{aligned}\det(\mathbf{X}'\mathbf{X}) &\le (p-1)\left(\frac{(n-3)p}{4(p-1)}\right)^p\left(1+\frac{p-1}{n-3}\right)\\ &\quad\cdot\left(\left(1+\frac{p^3+8}{(n-3)p^2}\right)\left(1+\frac{p^3+p^2+16}{(n-3)p^2}\right)-2\left(\frac{p^2+8}{(n-3)p^2}\right)^2\right).\end{aligned} \tag{2.26}$$

The equality in (2.26) holds if and only if $t=u=w=0.5(p+2)$ and $m=q=h=0.25(p+4)$.

If $p+2=0 \bmod 4$, then the minimum values in (2.25) are all equal to $\frac{(p+2)^2(p-4)^2}{16p^2(p-1)^2}$ when $m=q=h=0.25(p+2)$. An easy computation shows that

$$\begin{aligned}\det(\mathbf{T}) &\le \left(1+\tfrac{p^3+8}{(n-3)p^2}\right)^3-2\left(\tfrac{(p+2)(p-4)}{(n-3)p^2}\right)^3-3\left(1+\tfrac{p^3+8}{(n-3)p^2}\right)\left(\tfrac{(p+2)(p-4)}{(n-3)p^2}\right)^2\\ &= \left(1+\tfrac{(p-1)(p+2)}{(n-3)p}\right)\left(\left(1+\tfrac{p^3+8}{(n-3)p^2}\right)\left(1+\tfrac{(p+2)(p^2-3p+8)}{(n-3)p^2}\right)-2\left(\tfrac{(p+2)(p-4)}{(n-3)p^2}\right)^2\right)\end{aligned}$$

and consequently

$$\begin{aligned}\det(\mathbf{X}'\mathbf{X}) &\le (p-1)\left(\tfrac{(n-3)p}{4(p-1)}\right)^p\left(1+\tfrac{(p-1)(p+2)}{(n-3)p}\right)\\ &\quad\cdot\left(\left(1+\tfrac{p^3+8}{(n-3)p^2}\right)\left(1+\tfrac{(p+2)(p^2-3p+8)}{(n-3)p^2}\right)-2\left(\tfrac{(p+2)(p-4)}{(n-3)p^2}\right)^2\right).\end{aligned} \tag{2.27}$$

The equality in (2.27) holds if and only if $t=u=w=0.5(p+2)$ and $m=q=h=0.25(p+2)$. So, the following theorem is obtained.

Theorem 2.4. *For any* $\mathbf{X}\in\mathbf{\Phi}_{n\times p}\{0,1\}$ *given by* (2.19)

$$\det\left(\mathbf{X}'\mathbf{X}\right)\le\begin{cases}W(n,p)S(n,p) & \text{if } p=0 \bmod 4\\ W(n,p)Q(n,p) & \text{if } p+2=0 \bmod 4,\end{cases} \tag{2.28}$$

where

(2.29)

$$\begin{aligned}W(n,p)&=(p-1)\left(\tfrac{(n-3)p}{4(p-1)}\right)^p,\\ S(n,p)&=\left(1+\tfrac{p-1}{n-3}\right)\left[\left(1+\tfrac{p^3+8}{(n-3)p^2}\right)\left(1+\tfrac{p^3+p^2+16}{(n-3)p^2}\right)-2\left(\tfrac{p^2+8}{(n-3)p^2}\right)^2\right],\\ Q(n,p)&=\left(1+\tfrac{(p-1)(p+2)}{(n-3)p}\right)\left[\left(1+\tfrac{p^3+8}{(n-3)p^2}\right)\left(1+\tfrac{(p+2)(p^2-3p+8)}{(n-3)p^2}\right)-2\left(\tfrac{(p+2)(p-4)}{(n-3)p^2}\right)^2\right].\end{aligned}$$

Corollary 2.3. *Let* $W(n,p)$, $S(n,p)$, $Q(n,p)$ *be of the form* (2.29) *and* $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$ *by* (2.19). *Then*

$$\det\left(\mathbf{X}'\mathbf{X}\right) = \begin{cases} W(n,p)S(n,p) & \text{if } p = 0 \bmod 4 \\ W(n,p)Q(n,p) & \text{if } p+2 = 0 \bmod 4 \end{cases}$$

provided that (2.22) *holds and*

$$\begin{cases} \mathbf{x}'\mathbf{1}_p = \mathbf{y}'\mathbf{1}_p = \mathbf{z}'\mathbf{1}_p = 0.25(p+2) & \\ \textit{and} & \\ \mathbf{x}'\mathbf{y} = \mathbf{x}'\mathbf{z} = \mathbf{y}'\mathbf{z} = 0.25(p+4) & \textit{if } \ p = 0 \bmod 4 \\ \mathbf{x}'\mathbf{y} = \mathbf{x}'\mathbf{z} = \mathbf{y}'\mathbf{z} = 0.25(p+2) & \textit{if } \ p+2 = 0 \bmod 4. \end{cases}$$

Some construction methods of $\mathbf{X}_1$ satisfying 2.2 are based on the incidence matrix of a balanced incomplete block design, see [1], Theorem 4. Such a matrix $\mathbf{X}_1$ exists only for certain values of p and n. Hence, if $\mathbf{X}_1$ does not exist in $\mathbf{\Phi}_{n\times p}\{0,1\}$ but exists among $\mathbf{\Phi}_{n-1\times p}\{0,1\}$, $\mathbf{\Phi}_{n-2\times p}\{0,1\}$ or $\mathbf{\Phi}_{n-3\times p}\{0,1\}$, then we can construct a highly D-efficient spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{n\times p}\{0,1\}$. This construction is based on corollaries 2.2, 2.3 and 2.4.

3. EXAMPLES

Example 3.1. Consider the problem of weighing $p = 4$ objects in $n = 7$ measurements. Since $\frac{np}{4(p-1)} = \frac{7}{3}$ and $\frac{n(p-2)}{4(p-1)} = \frac{7}{6}$ are not integers, the matrix $\mathbf{X} \in \mathbf{\Phi}_{7\times 4}\{0,1\}$ for which (2.2) is satisfied does not exist. Now, let $\mathbf{X}_1$ be a matrix for $p = 4$ objects and $n-1 = 6$ measurements. Then $\frac{(n-1)p}{4(p-1)} = 2$, $\frac{(n-1)(p-2)}{4(p-1)} = 1$ and for

$$\mathbf{X}_1 = \begin{bmatrix} 1&1&0&0 \\ 1&0&1&0 \\ 1&0&0&1 \\ 0&1&1&0 \\ 0&1&0&1 \\ 0&0&1&1 \end{bmatrix} \tag{3.1}$$

the condition (2.2) is fulfilled. By Corollary 2.1, the design $\mathbf{X} \in \mathbf{\Phi}_{7\times 4}\{0,1\}$ of the form $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ 1\,1\,1\,0 \end{bmatrix}$ is highly D-efficient.

Example 3.2. By Corollary 2.2, $\mathbf{X} \in \mathbf{\Phi}_{8\times 4}\{0,1\}$ such that $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ 1\,1\,1\,0 \\ 1\,1\,0\,1 \end{bmatrix}$, where $\mathbf{X}_1$ is given in (3.1), is highly D-efficient for weighing 4 objects in 8 measurements.

Example 3.3. In order to weigh 4 objects in $n = 9$ measurements, let $\mathbf{X} \in \mathbf{\Phi}_{9\times 4}\{0,1\}$ be of the form $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ 1\,1\,1\,0 \\ 1\,1\,0\,1 \\ 1\,0\,1\,1 \end{bmatrix}$, where $\mathbf{X}_1$ is of the form (3.1). Hence $\mathbf{X}$ is highly D-efficient.

Example 3.4. Consider the problem of measuring 6 objects in $n = 11$ measurements. Since $\frac{np}{4(p-1)} = \frac{33}{10}$ is not an integer, the matrix $\mathbf{X} \in \mathbf{\Phi}_{11\times 6}\{0,1\}$ for which (2.2) is satisfied does not exist. Now, let $\mathbf{X}_2$ be a matrix for $p = 6$ objects and $n - 1 = 10$ measurements. In this case $\frac{(n-1)p}{4(p-1)} = 3$ and $\frac{(n-1)(p-2)}{4(p-1)} = 2$ and for the matrix

$$\mathbf{X}_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \tag{3.2}$$

the condition (2.2) is fulfilled. By Corollary 2.1, the design $\mathbf{X} \in \mathbf{\Phi}_{11\times 6}\{0,1\}$ of the form $\mathbf{X} = \begin{bmatrix} \mathbf{X}_2 \\ 1\,1\,1\,1\,0\,0 \end{bmatrix}$ is highly D-efficient.

Example 3.5. For weighing $p = 6$ objects using $n = 12$ measurements the design $\mathbf{X} \in \mathbf{\Phi}_{12\times 6}\{0,1\}$ of the form $\mathbf{X} = \begin{bmatrix} \mathbf{X}_2 \\ 1\,1\,1\,1\,0\,0 \\ 1\,1\,0\,0\,1\,1 \end{bmatrix}$ is highly D-efficient, by Corollary 2.2.

Example 3.6. For weighing $p = 6$ objects in $n = 13$ measurements $\mathbf{X} \in \mathbf{\Phi}_{13\times 4}\{0,1\}$ of the form $\mathbf{X} = \begin{bmatrix} \mathbf{X}_2 \\ 1\,1\,1\,1\,0\,0 \\ 1\,1\,0\,0\,1\,1 \\ 0\,0\,1\,1\,1\,1 \end{bmatrix}$, where $\mathbf{X}_1$ is given in (3.2), is highly D-efficient, by Corollary 2.3.

4. DISCUSSION

For each p and n, the resulting D_{eff} based on the provided designs in Theorem 2.2, 2.3 and 2.4 are summarized in Table 1.

Table 1: $D_{\text{eff}}(\mathbf{X})$ of the design $\mathbf{X}$ for each p and n.

$p = 4$					
n	6	7	8	9	10
$D_{\text{eff}}(\mathbf{X})$	0.9779	0.9641	0.9652	0.9779	1
$p = 6$					
n	10	11	12	13	14
$D_{\text{eff}}(\mathbf{X})$	0.9927	0.9783	0.9719	0.9723	1
$p = 8$					
n	14	15	16	17	18
$D_{\text{eff}}(\mathbf{X})$	0.9968	0.9849	0.9776	0.9701	1

ACKNOWLEDGMENTS

The authors wish to express their gratitude to an anonymous Reviewer for many helpful suggestions and comments that improved the readability of the manuscript.

REFERENCES

[1] Ceranka, B. and Graczyk, M. (2016). Recent developments in D-optimal spring balance weighing designs, *Brazilian Journal of Probability and Statistics*, to appear.

[2] Harville, D.A. (1997). *Matrix Algebra from Statistician's Perspective*, Springer-Verlag, New York, Inc.

[3] Hudelson, M.; Klee, V. and Larman, D. (1996). Largest j-simplices in d-cubes: Some relatives to the Hadamard determinant problem, *Linear Algebra and its Applications*, **24**, 519–598.

[4] Jacroux, M. and Notz, W. (1983). On the optimality of spring balance weighing designs, *The Annals of Statistics*, **11**, 970–978.

[5] Neubauer, M.G.; Watkins, W. and Zeitlin, J. (1997). Maximal j-simpplices in the real d-dimensional unit cube, *Journal of Combinatorial Theory, Ser. A*, **80**, 1–12.

[6] Neubauer, M.G.; Watkins, W. and Zeitlin, J. (1998). Notes on D-optimal designs, *Linear Algebra and its Applications*, **280**, 109–127.