TWO-FACTOR EXPERIMENTS WITH SPLIT UNITS CONSTRUCTED BY CYCLIC DESIGNS AND SQUARE LATTICE DESIGNS

Authors: KAZUHIRO OZAWA
– Department of Nursing, Gifu College of Nursing, Japan
ozawa@gifu-cn.ac.jp

IWONA MEJZA
– Department of Mathematical and Statistical Methods,
Poznań University of Life Sciences, Poland
iwona.mejza@up.poznan.pl

STANISLAW MEJZA
– Department of Mathematical and Statistical Methods,
Poznań University of Life Sciences, Poland
stanislaw.mejza@up.poznan.pl

SHINJI KURIKI
– Department of Mathematical Sciences, Graduate School of Engineering,
Osaka Prefecture University, Japan
kuriki@ms.osakafu-u.ac.jp

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Abstract:
• We consider nested row-column designs with split units for a two-factor experiment. The most optimal design in this case is that of using for the whole plots a Latin square while for the subplot treatments with a completely randomized design for each whole plot. Such a design, in fact optimal, utilizes many experimental units and quite a large space. Hence to construct new designs of reduced size of the experiment we use a cyclic design for the whole plot treatments and a square lattice design for the subplot treatments. The proposed designs are generally balanced and they allow for giving the stratum efficiency factors, especially useful to design of experiments.

Key-Words:
• cyclic design; general balance; square lattice design; stratum efficiency factor.

AMS Subject Classification:
• 62K15, 62K10, 05B05.
1. INTRODUCTION

In many biological and agricultural (field) experiments, a nested row-column design with split units is often used. The design is for a two-factor experiment of split-plot type with $b$ blocks. The first factor $A$ has $v_1$ levels $A_1, A_2, ..., A_{v_1}$ and the second factor $B$ has $v_2$ levels $B_1, B_2, ..., B_{v_2}$. Each block is divided into $k_1$ rows and $k_2$ columns and these $k_1k_2$ units are treated as whole plots. Moreover, each whole plot is divided into $k_3$ subplots. The levels of $A$ and $B$ are applied to the whole plots (called whole plot treatments) and the subplots (called subplot treatments), respectively. Such a design is called a nested row-column design with split units.

Kachlicka and Mejza (1996) considered a mixed linear model with fixed treatment effects and random block, row, column, whole plot and subplot effects for the nested row-column design with split units. The $h$th factorial treatment combination effect $\tau_h$ is defined by

$$
\tau_h = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}
$$

for $h = (i-1)v_2 + j$, $i = 1, 2, ..., v_1$ and $j = 1, 2, ..., v_2$, where $\mu$ is the general mean, $\alpha_i$ denotes the main effect of the $i$th whole plot treatment $A_i$, $\beta_j$ denotes the main effect of the $j$th subplot treatment $B_j$ and $(\alpha\beta)_{ij}$ denotes the interaction effect of $A_i$ and $B_j$. Here $\sum_{i=1}^{v_1} \alpha_i = 0$, $\sum_{j=1}^{v_2} \beta_j = 0$, $\sum_{i=1}^{v_1} (\alpha\beta)_{ij} = 0$ for $j = 1, 2, ..., v_2$ and $\sum_{j=1}^{v_2} (\alpha\beta)_{ij} = 0$ for $i = 1, 2, ..., v_1$. The mixed linear model results from a four-step randomization, i.e., the randomization of blocks, the randomization of rows within each block, the randomization of columns within each block and the randomization of subplots within each whole plot. This kind of randomization leads us to an experiment with orthogonal block structure as defined by Nelder (1965a, 1965b) and the multistratum analysis proposed by Nelder (1965a, 1965b) and Houtman and Speed (1983) can be applied to the analysis of data in the experiment. In this case, we have five strata, except zero stratum connected with the general mean only, (I) inter-block stratum, (II) inter-row stratum, (III) inter-column stratum, (IV) inter-whole plot stratum and (V) inter-subplot stratum. The statistical properties of the nested row-column design with split units are strictly connected with the eigenvalues and the eigenvectors of the stratum information matrices for the treatment combinations. The stratum information matrices $A_1, A_2, A_3, A_4$ and $A_5$ are given by

\begin{align*}
A_1 &= \frac{1}{k_1k_2k_3} N_0 N_0' - \frac{r}{v} J_v, \\
A_2 &= \frac{1}{k_2k_3} N_1 N_1' - \frac{1}{k_1k_2k_3} N_0 N_0', \\
A_3 &= \frac{1}{k_1k_3} N_2 N_2' - \frac{1}{k_1k_2k_3} N_0 N_0', \\
A_4 &= \frac{1}{k_3} N_3 N_3' - \frac{1}{k_1k_3} N_2 N_2' - \frac{1}{k_2k_3} N_1 N_1' + \frac{1}{k_1k_2k_3} N_0 N_0'.
\end{align*}
and

\[(1.4)\quad A_5 = rI_v - \frac{1}{k_3}N_3N_3',\]

where \(v = v_1v_2\), \(N_0\), \(N_1\), \(N_2\) and \(N_3\) are the incidence matrices for the treatment combinations vs. blocks, rows, columns and whole plots, respectively, \(I_v\) is the identity matrix of order \(v\) and \(J_v\) is the \(v \times v\) matrix with every element unity. Here we assume that every treatment combination \(A_iB_j (i = 1, 2, \ldots, v_1, j = 1, 2, \ldots, v_2)\) occurs in precisely \(r\) blocks and the treatment combinations are ordered lexicographically.

A generally balanced design was firstly introduced by Nelder (1965a, 1965b), for which the stratum information matrices are spanned by a common set of eigenvectors. Let \(s_0, s_1, \ldots, s_{v-1}\) be the mutually orthonormal common eigenvectors of the stratum information matrices \(A_1, A_2, A_3, A_4\) and \(A_5\). Since \(A_fI_v = 0\) for \(f = 1, 2, 3, 4, 5\), \(\frac{1}{\sqrt{v}}1_v\) may be chosen as the first eigenvector \(s_0\), where \(1_v\) is the \(v \times 1\) vector of unit elements. Let \(\xi_{fh}\) be an eigenvalue of a matrix \(A_f^* = r^{-1}A_f\) corresponding to the eigenvector \(s_h\) for \(f = 1, 2, 3, 4, 5\) and \(h = 1, 2, \ldots, v - 1\). Then, a basic contrast of the treatment effects (see Pearce et al. (1974)) is defined by \(s_h^\tau\) for \(h = 1, 2, \ldots, v - 1\), where \(\tau\) is the \(v \times 1\) vector of the treatment effects. The eigenvalue \(\xi_{fh}\) can be identified as a stratum efficiency factor of the design concerning estimation of the \(h\)th basic contrast in the \(f\)th stratum for \(f = 1, 2, 3, 4, 5\) and \(h = 1, 2, \ldots, v - 1\) (see, Houtman and Speed (1983)).

Many experiments require a long time or a large space (units) often making it impossible to carry out a conventional, complete (orthogonal) design of the considered type. For example, in agricultural field experiments, because of soil fertility it is difficult to find units (plots) fulfilling restrictions concerning the homogeneity of blocks, rows, columns, whole plots or subplots. Then, to satisfy the main experimental principles it is necessary to design the experiment as an incomplete (non-orthogonal) one. Such an experiment usually utilizes smaller units, with respect to size and also utilizes smaller number of units (the experiment is cheaper). The problem is to find an incomplete design proper to experimental material structure and optimal with respect to statistical properties of the design.

Kuriki et al. (2009), Mejza et al. (2009) and Mejza and Kuriki (2013) constructed nested row-column designs with split units by the Kronecker product of the incidence matrices of two designs. They used a Youden square for the whole plot treatments and various proper designs for the subplot treatments. Mejza et al. (2014) have used a balanced incomplete block design with nested rows and columns instead of the Youden square to construct a nested row-column design with split units. The designs obtained by this way need usually a large number of units. In this paper, we construct a nested row-column design with split units by a modified Kronecker product (called a semi-Kronecker product) of the incidence
matrices of two designs. We use a cyclic design for the whole plot treatments and a square lattice design for the subplot treatments. We give the stratum efficiency factors for such a nested row-column design with split units, which has the general balance property.

These designs have smaller numbers of blocks than the conventional experiments. Therefore, they would be useful in practice, for example, the reduction of the experimental expenses and effort, and the easier implementation of the experiments by using the well-known cyclic designs and square lattice designs in the literature (see, John (1987), John and Williams (1995) and Raghavarao (1971), etc.).

Other variants of incomplete split plot designs are given, for example, by Mejza and Mejza (1996), Ozawa et al. (2004), Aastveit et al. (2009), Mejza et al. (2012) and Kuriki et al. (2012).

2. CONSTRUCTION BY A CYCLIC DESIGN AND A SQUARE LATTICE DESIGN

Firstly, we need the semi-Kronecker product (see, Khatri and Rao (1968) and Mejza, Kuriki and Mejza (2001)) of two matrices that will be used to construct nested row-column designs with split units. Suppose that two matrices $E$ and $F$ are divided into the same number of submatrices as follows:

$$E = (E_1 : E_2 : \cdots : E_m) \quad \text{and} \quad F = (F_1 : F_2 : \cdots : F_m).$$

Then, the semi-Kronecker product $E \tilde{\otimes} F$ is defined by

$$E \tilde{\otimes} F = (E_1 \otimes F_1 : E_2 \otimes F_2 : \cdots : E_m \otimes F_m),$$

where $\otimes$ denotes the usual Kronecker product.

Next, we need a cyclic design and a square lattice design. Let $V$ be a set of $v$ treatments and let $B$ be a collection of subsets (called blocks) of $V$. A design $(V, B)$ is denoted by $D(v, r, k)$ if every treatment occurs in precisely $r$ blocks and each block contains $k$ treatments. Let $Z_v$ be the additive group of integers modulo $v$ and let $(V, B)$ be a $D(v, r, k)$ with $V = Z_v$ for which if $\{a_1, a_2, \ldots, a_k\}$ is a block, then $\{a_1 + 1, a_2 + 1, \ldots, a_k + 1\}$ is also a block. A set of blocks $\{(a_1 + i, a_2 + i, \ldots, a_k + i) \mid i \in Z_v\}$ is called a cyclic class and a block taken arbitrarily from each cyclic class is called an initial block. If the collection $B$ of blocks is divided into some cyclic classes, then $(V, B)$ is said to be cyclic and it is denoted by $CD(v, r, k)$. Here we consider only a case where the number of blocks in each cyclic class is $v$. 

Let \((V, B)\) be a \(D(v, r, k)\). If the collection \(B\) of blocks can be grouped in such a way that every treatment occurs precisely once in every group (called a resolution class), then \((V, B)\) is said to be resolvable. A resolvable \(D(v, r, k)\) \((V, B)\) such that \(v = s^2\), \(r \leq s + 1\) and \(k = s\) for a positive integer \(s\) is called a square lattice design if any two blocks from different resolution classes contain just one common treatment, and it is denoted by \(SLD(s^2, r, s)\). If \(r = s + 1\), it is called a balanced square lattice design and it is well known that there exists a balanced square lattice design if \(s\) is a prime or a prime power (see, Raghavarao (1971)).

Now we construct a nested row-column design with split units. Let \((V_A, B_A)\) be a \(CD(v_A, r_A, k_A)\) with \(m = r_A/k_A\) initial blocks. Each cyclic class of \((V_A, B_A)\) is treated as a block with \(k_A\) rows and \(v_A\) columns such that the columns are blocks of \(B_A\) and that every treatment of \(V_A\) occurs precisely once in each row. Such a design is denoted by \(D_A\). An \(SLD(s^2, m, s)\) is denoted by \(D_B\). The whole plot treatments occur in \(D_A\) and the subplot treatments occur in \(D_B\). We construct a nested row-column design, say \(D\), with split units embedding each block of the \(i\)th resolution class of \(D_B\) in every whole plot of the \(i\)th block of \(D_A\) for \(i = 1, 2, ..., m\). The parameters of \(D\) are \(v_1 = v_A, v_2 = s^2, b = ms, r = mk_A = r_A, k_1 = k_A, k_2 = v_A\) and \(k_3 = s\).

Example 2.1. We use an \(A\)-efficient cyclic design \(CD(6, 6, 3)\) with initial blocks \(\{0, 1, 2\}\) and \(\{0, 1, 3\}\) given by John (1987). From two cyclic classes of this design, we have the following two blocks with 3 rows and 6 columns of \(D_A\):

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 0 \\
2 & 3 & 4 & 5 & 0 & 1
\end{array}
\quad\text{and}\quad
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 0 \\
3 & 4 & 5 & 0 & 1 & 2
\end{array}
\]

We also use a square lattice design \(SLD(9, 2, 3)\) \(D_B = (V_B, B_B)\) with \(V_B = \{1, 2, ..., 9\}\). The following columns are 6 blocks of \(D_B\):

\[
\begin{array}{cccc}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9
\end{array}
\quad\text{and}\quad
\begin{array}{cccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{array}
\]

where the first resolution class is constituted by the first 3 blocks and the second one is constituted by the remaining blocks. We construct a nested row-column design \(D\) with split units embedding each block of the first (second) resolution class of \(D_B\) in every whole plot of the first (second) block of \(D_A\), replacing the treatments \(0, 1, 2, 3, 4, 5\) of \(D_A\) with \(A_1, A_2, A_3, A_4, A_5, A_6\) and the treatments \(1, 2, ..., 9\) of \(D_B\) with \(B_1, B_2, ..., B_9\). The design \(D\) has 6 blocks as follows:
Two-Factor Experiments with Split Units

<table>
<thead>
<tr>
<th></th>
<th>A_1</th>
<th>A_2</th>
<th>A_3</th>
<th>A_4</th>
<th>A_5</th>
<th>A_6</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B_1</td>
<td>B_2</td>
<td>B_3</td>
<td>B_1</td>
<td>B_2</td>
<td>B_3</td>
</tr>
<tr>
<td></td>
<td>A_2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B_1</td>
<td>B_2</td>
<td>B_3</td>
<td>B_1</td>
<td>B_2</td>
<td>B_3</td>
</tr>
<tr>
<td></td>
<td>A_3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>B_1</td>
<td>B_2</td>
<td>B_3</td>
<td>B_1</td>
<td>B_2</td>
<td>B_3</td>
</tr>
</tbody>
</table>

We note that if the nested row-column design with split units is constructed
by the usual Kronecker product of the incidence matrices (see, Mejza et al.
(2014)), then the number of blocks becomes \( m^2 s = 12 \). Generally, the number of
blocks of a nested row-column design with split units by the Kronecker product
is \( m \) times larger than those of a nested row-column design with split units by
the semi-Kronecker product.

Let

\[
N_A = (N_{A1} : N_{A2} : \cdots : N_{Am}) \quad \text{and} \quad N_B = (N_{B1} : N_{B2} : \cdots : N_{Bm})
\]
be the incidence matrices of the cyclic design $CD(v_A, r_A, k_A)$ and the square lattice design $SLD(s^2, m, s)$, where $N_{Ai}$ and $N_{Bi}$ correspond to the $i$th cyclic and resolution classes, respectively. By the definition of the square lattice design,

$$N'_{Bi}N_{Bi} = sI_s \quad \text{and} \quad N'_{Bi}N_{Bj} = J_s$$

hold for $i, j = 1, 2, \ldots, m$, $i \neq j$. Then, the incidence matrix $N_2$ of the nested row-column design $\mathcal{D}$ with split units is given by the semi-Kronecker product of $N_A$ and $N_B$, i.e.,

$$N_2 = N_A \tilde{\otimes} N_B = (N_{A1} \otimes N_{B1} : N_{A2} \otimes N_{B2} : \cdots : N_{Am} \otimes N_{Bm})$$

in a suitable order of columns of $\mathcal{D}$, and the concurrence matrices $N_0', N_1', N_2', N_3'$ of $\mathcal{D}$ are given by

$$N_0N_0' = \sum_{i=1}^{m} (k_A^2 J_{v_A} \otimes N_{Bi}N'_{Bi}) = k_A^2 J_{v_A} \otimes N_BN'_B,$$

$$N_1N_1' = \sum_{i=1}^{m} (k_A J_{v_A} \otimes N_{Bi}N'_{Bi}) = k_A J_{v_A} \otimes N_BN'_B,$$

$$N_2N'_2 = \sum_{i=1}^{m} (N_{Ai}N'_{Ai} \otimes N_{Bi}N'_{Bi})$$

and

$$N_3N'_3 = \sum_{i=1}^{m} (k_A I_{v_A} \otimes N_{Bi}N'_{Bi}) = k_A I_{v_A} \otimes N_BN'_B.$$

3. STRATUM EFFICIENCY FACTORS FOR $\mathcal{D}$

In this section, we give the stratum efficiency factors for the nested row-column design $\mathcal{D}$ with split units constructed in Section 2. To find the stratum efficiency factors, it is necessary to find the eigenvalues of the stratum information matrices $A_1, A_2, A_3, A_4$ and $A_5$ of $\mathcal{D}$. It is easy to find these eigenvalues if $A_1, A_2, A_3, A_4$ and $A_5$ have the common eigenvectors, i.e., if $\mathcal{D}$ is generally balanced. It follows, from (2.1), that

$$N_{Bi}N'_{Bi}N_{Bj}N'_{Bj} = J_{s^2}$$

holds for $i, j = 1, 2, \ldots, m$, $i \neq j$. From (3.1), it is easily verified that the concurrence matrices $N_0N'_0, N_1N'_1, N_2N'_2$ and $N_3N'_3$ given in (2.2)–(2.5) are mutually commutative. Thus, by use of (1.1)–(1.4), the stratum information matrices $A_1$, ...
A_2, A_3, A_4 and A_5 are mutually commutative, which means that A_1, A_2, A_3, A_4 and A_5 have the common eigenvectors. Therefore, D is generally balanced.

In order to find the common eigenvectors of the stratum information matrices A_1, A_2, A_3, A_4 and A_5, i.e., those of the concurrence matrices N_0N_0', N_1N_1', N_2N_2' and N_3N_3', we consider the eigenvectors of N_{Ai}N_{Ai}' for the i-th cyclic class of the cyclic design CD(v_A, r_A, k_A) and those of N_{Bi}N_{Bi}' for the i-th resolution class of the square lattice design SLD(s^2, m, s) for i = 1, 2, ..., m. For the incidence matrix N_A of the CD(v_A, r_A, k_A), since N_{A1}N_{A1}', N_{A2}N_{A2}', ..., N_{Am}N_{Am}' are symmetric circulant matrices, these matrices have the mutually orthonormal common eigenvectors, which are denoted by x_0, x_1, ..., x_{v_A} - 1 with x_0 = \frac{1}{\sqrt{v_A}} 1_{v_A}.

The corresponding eigenvalues of N_{Ai}N_{Ai}' are given by

$$\theta_j^{(i)} = \sum_{h=0}^{v_A-1} \lambda_h^{(i)} \cos \left( \frac{2\pi jh}{v_A} \right)$$

for i = 1, 2,..., m and j = 0, 1, ..., v_A - 1, where \lambda_h^{(i)} (h \neq 0) denotes the number of blocks containing two treatments 0 and h in the i-th cyclic class of the CD(v_A, r_A, k_A) and \lambda_h^{(i)} = k_A. In particular, \theta_0^{(i)} = k_A^2 and the corresponding eigenvector is x_0 = \frac{1}{\sqrt{v_A}} 1_{v_A} (see, John (1987) and John and Williams (1995)). These eigenvalues and common eigenvectors are summarized in the following table:

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Common eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>k_A^2</td>
<td>\frac{1}{\sqrt{v_A}} 1_{v_A}</td>
</tr>
<tr>
<td>\theta_j^{(i)}</td>
<td>x_j (j = 1, 2, ..., v_A - 1)</td>
</tr>
</tbody>
</table>

Similarly, for the incidence matrix N_B of the SLD(s^2, m, s), from (2.1), N_{Bi}N_{Bi}' has the eigenvalues s and 0 with multiplicities s and s(s - 1) for each i = 1, 2, ..., m. From (3.1), N_{Bi}N_{B1}', N_{B2}N_{B2}', ..., N_{Bm}N_{Bm}' are mutually commutative, so these matrices have the common eigenvectors. Let Q = (q_0, q_1, ..., q_{s-1}) be an orthogonal matrix of order s with q_0 = \frac{1}{\sqrt{s}} 1_s. For each i = 1, 2, ..., m, from (2.1), the mutually orthonormal eigenvectors of N_{Bi}N_{Bi}' corresponding to the eigenvalue s are given by

$$z_{ip} = \frac{1}{\sqrt{s}} N_{Bi} q_p$$

for p = 0, 1, ..., s - 1. In particular, z_{i0} = \frac{1}{\sqrt{s}} 1_{s^2}. The eigenvectors z_{ip} are also the eigenvectors of N_{Bh}N_{Bh}' (h \neq i) for any other resolution class, and the eigenvalues
of \( N_{Bh} N'_{Bh} \) corresponding to \( z_{i0} \) and \( z_{ip} \) \((p \neq 0)\) are \( s \) and \( 0 \), respectively. Furthermore, the mutually orthonormal common eigenvectors of \( N_{B1} N'_{B1}, N_{B2} N'_{B2}, \ldots, N_{Bm} N'_{Bm} \) corresponding to the eigenvalue 0 are denoted by \( z_q^* \) for \( q = 1, 2, \ldots, s^2 - m(s - 1) - 1 \). These eigenvalues and common eigenvectors are summarized in Table 2.

**Table 2**: Eigenvalues and common eigenvectors of \( N_{Bi} N'_{Bi} \) in the SLD\((s^2, m, s)\).

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Common eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_{B1} N'_{B1} )</td>
<td>( 1_{s^2} )</td>
</tr>
<tr>
<td>( N_{B2} N'_{B2} )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>( N_{Bm} N'_{Bm} )</td>
<td>( z_{ip} ) ((p = 1, 2, \ldots, s - 1))</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( z_{2p} ) ((p = 1, 2, \ldots, s - 1))</td>
</tr>
<tr>
<td>( \ldots )</td>
<td>( \ldots )</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( z_{mp} ) ((p = 1, 2, \ldots, s - 1))</td>
</tr>
<tr>
<td>( 0 )</td>
<td>( z_q^* ) ((q = 1, 2, \ldots, s^2 - m(s - 1) - 1))</td>
</tr>
</tbody>
</table>

Combining the eigenvectors of Table 1 and Table 2, we consider the following 6 sets of vectors:

\[
\begin{align*}
(1) & \quad \frac{1}{\sqrt{v_A}} 1_{v_A} \otimes \frac{1}{s^2} 1_{s^2}, \\
(2) & \quad x_j \otimes \frac{1}{s} 1_{s^2}, \\
(3) & \quad \frac{1}{\sqrt{v_A}} 1_{v_A} \otimes z_{ip}, \\
(4) & \quad \frac{1}{\sqrt{v_A}} 1_{v_A} \otimes z_q^*, \\
(5) & \quad x_j \otimes z_{ip}, \\
(6) & \quad x_j \otimes z_q^*
\end{align*}
\]

for \( i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, v_A - 1, \quad p = 1, 2, \ldots, s - 1 \) and \( q = 1, 2, \ldots, s^2 - m(s - 1) - 1 \). The vectors of (1)–(6) are mutually orthonormal and the total number of the vectors is \( v_A s^2 \). We show that the vectors of (1)–(6) are the common eigenvectors of \( N_0 N_0', N_1 N_1', N_2 N_2', \) and \( N_3 N_3' \), and we find the corresponding eigenvalues of \( N_0 N_0', N_1 N_1', N_2 N_2', \) and \( N_3 N_3' \).

Firstly, we take into account the matrix \( N_0 N_0' \). For (1), we have, from (2.2), Table 1 and Table 2,

\[
\begin{align*}
N_0 N_0' \left( \frac{1}{\sqrt{v_A}} 1_{v_A} \otimes \frac{1}{s^2} 1_{s^2} \right) &= \left( k_A^2 J_{v_A} \otimes N_B N'_{B} \right) \left( \frac{1}{\sqrt{v_A}} 1_{v_A} \otimes \frac{1}{s} 1_{s^2} \right) \\
&= \left( k_A^2 J_{v_A} \frac{1}{\sqrt{v_A}} 1_{v_A} \right) \otimes \left( N_B N'_{B} \frac{1}{s} 1_{s^2} \right) = \left( v_A k_A^2 \frac{1}{\sqrt{v_A}} 1_{v_A} \right) \otimes \left( ms \frac{1}{s} 1_{s^2} \right) \\
&= m v_A k_A^2 s \left( \frac{1}{\sqrt{v_A}} 1_{v_A} \otimes \frac{1}{s} 1_{s^2} \right).
\end{align*}
\]
The corresponding eigenvalue is $mv_Ak_A^2s$.

For (2), we have

$$N_0N_0' \left( x_j \otimes \frac{1}{s}1_{s^2} \right) = (k_A^2J_{v_A}1_{v_A}) \otimes \left( N_BN_B'1_{s^2} \right) = 0.$$  

The corresponding eigenvalue is zero for each $i=1,2,...,m$ and $j=1,2,...,v_A-1$.

For (3), we have

$$N_0N_0' \left( \frac{1}{v_A^{1/2}}1_{v_A} \otimes z_{ip} \right) = \left( k_A^2J_{v_A} \frac{1}{v_A^{1/2}}1_{v_A} \right) \otimes \left( N_BN_B'z_{ip} \right) = \left( v_Ak_A^2 \frac{1}{v_A^{1/2}}1_{v_A} \otimes (sz_{ip}) \right) = v_Ak_A^2s \left( \frac{1}{v_A^{1/2}}1_{v_A} \otimes z_{ip} \right).$$

The corresponding eigenvalue is $v_Ak_A^2s$ for each $i=1,2,...,m$ and $p=1,2,...,s-1$.

For (4), we have

$$N_0N_0' \left( \frac{1}{v_A^{1/2}}1_{v_A} \otimes z_{q^*} \right) = \left( k_A^2J_{v_A} \frac{1}{v_A^{1/2}}1_{v_A} \right) \otimes \left( N_BN_B'z_{q^*} \right) = \left( v_Ak_A^2 \frac{1}{v_A^{1/2}}1_{v_A} \otimes \left( \sum_{i=1}^{m} N_B1_{v_A} \right) = 0. \right)$$

The corresponding eigenvalue is zero for $q=1,2,...,s^2 - m(s-1) - 1$. Moreover, for (5) and (6), the eigenvalue is also zero.

Similarly, from (2.3)–(2.5), we can show that the vectors of (1)–(6) are also the eigenvectors of $N_0N_0'$, $N_1N_1'$, $N_2N_2'$ and $N_3N_3'$. The corresponding eigenvalues of $N_0N_0'$, $N_1N_1'$, $N_2N_2'$ and $N_3N_3'$ are summarized in the table below:

**Table 3:** Eigenvalues and common eigenvectors of $N_0N_0'$, $N_1N_1'$, $N_2N_2'$ and $N_3N_3'$.

<table>
<thead>
<tr>
<th>Eigenvalues</th>
<th>Common eigenvectors</th>
</tr>
</thead>
<tbody>
<tr>
<td>$mv_Ak_A^2s$</td>
<td>$mv_Ak_A^2s$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$v_Ak_A^2s$</td>
<td>$v_Ak_A^2s$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Here $i=1,2,...,m$ and $j=1,2,...,v_A-1$. 
The vectors (1)–(6) are also the common eigenvectors of the stratum information matrices $A_1$, $A_2$, $A_3$, $A_4$ and $A_5$. By use of (1.1)–(1.4) and Table 3, the stratum efficiency factors for $D$ can be calculated as in the following table:

**Table 4:** Stratum efficiency factors for $D$.

<table>
<thead>
<tr>
<th>Type of contrasts</th>
<th>Number of contrasts</th>
<th>Strata</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>I</td>
</tr>
<tr>
<td>$A$</td>
<td>$v_A - 1$</td>
<td>0</td>
</tr>
<tr>
<td>$B$</td>
<td>$m(s - 1)$</td>
<td>$1/m$</td>
</tr>
<tr>
<td>$A \times B$</td>
<td>$m(v_A - 1)(s - 1)$</td>
<td>$1/m - \xi_{ij}$</td>
</tr>
<tr>
<td></td>
<td>$(v_A - 1)(s^2 - m(s - 1) - 1)$</td>
<td>0</td>
</tr>
</tbody>
</table>

for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, v_A - 1$, where $A$ and $B$ denote the basic contrasts among the main effects of whole plot and subplot treatments, respectively, $A \times B$ denotes the basic contrasts among the interaction effects, $\xi_{ij} = \theta_{ij}^{(1)}/(mk^2_A)$ and $\omega_j = \sum_{i=1}^{m} \xi_{ij}$. The eigenvectors of (2), (3)–(4) and (5)–(6) define the basic contrasts $A$, $B$ and $A \times B$, respectively. We use Table 4 in order to improve the estimators for the basic contrasts of the treatment effects combining the estimators obtained from the strata I, III, IV and V. This procedure was proposed by Nelder (1965a, 1965b) and Houtman and Speed (1983). Especially, we see that some basic contrasts of $B$ and $A \times B$ are estimable with full efficiency.

**Example 3.1.** For the nested row-column design $D$ with split units given in Example 2.1, $m = 2$, $v_A = 6$, $k_A = 3$, $s = 3$, $\theta_1^{(1)} = 4$, $\theta_2^{(1)} = 0$, $\theta_3^{(1)} = 1$, $\theta_4^{(1)} = 0$, $\theta_5^{(1)} = 4$, $\theta_2^{(2)} = 1$, $\theta_3^{(2)} = 3$, $\theta_4^{(2)} = 3$, $\theta_5^{(2)} = 1$. Thus, by use of Table 4, the stratum efficiency factors can be calculated as in the following table:

**Table 5:** Stratum efficiency factors for $D$ given in Example 2.1.

<table>
<thead>
<tr>
<th>Type of contrasts</th>
<th>Number of contrasts</th>
<th>Strata</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>I</td>
</tr>
<tr>
<td>$A$</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$B$</td>
<td>4</td>
<td>1/2</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$A \times B$</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>0</td>
</tr>
</tbody>
</table>
4. REMARKS

In the design of experiments at least a few aspects play crucial roles. The first one concerns proper use of available structure of experimental units. The general rule, for example, in field agricultural experiments constitutes that smaller units better satisfy requirements concerning homogeneity of stratum units. In addition, usually smaller errors are associated after randomizations with these units.

The second aspect concerns statistical properties of designs. Using complete, orthogonal designs leads to the best unbiased estimators of the estimable functions of linear model parameters. In this work, we use a randomization-derived linear model (random block effect describing structure of units) with treatment (combination) effects being fixed. The structure of units and randomization performed lead to a design which possesses orthogonal block structure. In a complete case, the estimators of all estimable treatment effect functions are BLUEs. This means that the design is optimal from a point of view of statistical properties. Such a design can be used for our experiment if it is possible. However, many times there exist some limitations in available structure of experimental units (material). Then in our experiment some incomplete design can be applied only.

The new problem concerns how to choose an incomplete design that fits to the structure of experimental units, is optimal for the most interesting treatment effect functions, and is not so expensive (utilizes small as possible number of units of proper size). In the worse case we can use any incomplete design. Then it is difficult to describe the statistical properties of the proposed design.

The experimenter usually makes a ranking of linear functions of treatment effects (contrasts) with respect to scientific interest and an aim of the experiment. It would be helpful to have a design with known efficiencies of all estimable treatment effect functions. This property has a generally balanced design (see, for example, Mejza (1992) and Bailey (1994)). General balance aids interpretation; the design which is generally balanced with respect to meaningful contrasts may be superior to a technically optimal design. For generally balanced designs, we can identify the meaning of the treatment effect contrasts and their efficiency factors (cf. Table 2, Table 3 and Table 4). Hence we restrict our searching in the class of generally balanced designs.

Those considered here (nested row-column designs with split units) can be characterized by a few component block designs. We are looking for methods allowing for generation of new row-column designs with split units by using some known incomplete block designs instead of component designs. The Kronecker product of the component incomplete block designs is often used for constructing
new designs with split units. The final design possesses optimal properties, but it utilizes many experimental units (high cost of the experiment). To overcome this problem (size of the experiment) we proposed to use of the semi-Kronecker product as defined in Section 2 instead of the ordinary Kronecker product. The final design is much smaller and also possesses desirable statistical properties (see Example 2.1). Moreover using the semi-Kronecker product to generate new designs leads to much smaller number of units and smaller size. In the Example 2.1, one block of the complete design will have 6 rows and 6 columns while the whole plot consists of 9 units. For example, in agricultural field experiments (where such designs are very often used) it would be difficult to find so many homogeneous plots. In these cases the use of an incomplete design is recommended. In this paper, we construct a nested row-column design with split units by the semi-Kronecker product of the incidence matrices of a cyclic design for the whole plot treatments and a square lattice design for the subplot treatments. We give the stratum efficiency factors for such a nested row-column design with split units having the general balance property.

Although we proposed the new method for constructing the design in the class of nested row-column designs with split units, we still need new methods for constructing designs in the considered class which will lead to general balanced designs with desirable statistical properties and will have reasonable size. Naturally, in the future work for construction optimal row-column designs with nested structures someone can look for new methods and for another class of incomplete block designs as considered in the paper.

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