RELIABILITY ESTIMATION IN MULTISTAGE RANKED SET SAMPLING

Authors: M. Mahdizadeh
– Department of Statistics, Hakim Sabzevari University,
P.O. Box 397, Sabzevar, Iran
mahdizadeh.m@live.com

Ehsan Zamanzade
– Department of Statistics, University of Isfahan,
Isfahan, 81746-73441, Iran
e.zamanzade@sci.ui.ac.ir; ehsanzamanzadeh@yahoo.com

Abstract:

- A nonparametric reliability estimator based on multistage ranked set sampling is developed. It is shown that the estimator is unbiased and its efficiency relative to the simple random sampling rival is increasing in the number of stages. Numerical experiments are used to illustrate the theoretical findings. The suggested procedure is applied on a sport data set.

Key-Words:

- *Covariate information; Judgment ranking; Stress-strength model.*

AMS Subject Classification:

- 62N05, 62G30.

1. INTRODUCTION

Ranked set sampling (RSS) is a data collection technique which is advantageous in settings where precise measurement is difficult (i.e. time-consuming, expensive or destructive), but small sets of units can be accurately ranked without actual quantification. The ranking of the units is usually done by using expert opinion, concomitant variable, or a combination of them, and need not to be exact.

The RSS method was introduced by McIntyre [7] for estimating average yields in agriculture. In this setup, precise measurement entails harvesting the crops, and thus is expensive. An expert, however, can accurately rank the yields in a small set of adjacent fields by visual inspection. There has been a surge of research on RSS in the last two decades. The RSS has been applied in a variety of areas such as forestry, environmental science and medicine. For a book-length treatment of RSS and its applications, see Chen et el. [4].

The RSS design can be elucidated as follows:

1. Draw m random samples, each of size m, from the target population.
2. Apply judgement ordering, by any cheap method, on the elements of the ith ($i = 1, \ldots, m$) sample and identify the ith smallest unit.
3. Actually measure the m identified units in step 2.
4. Repeat steps 1-3, p times (cycles), if necessary, to obtain a ranked set sample of size $M = p\,m$.

Let X_{ik} be the ith judgement order statistic from the kth cycle. Then, the resulting ranked set sample is denoted by $\{X_{ik} : i = 1, \ldots, m\,; k = 1, \ldots, p\}$. The design parameter m is called set size.

A ranked set sample contains more information than a simple random sample of comparable size because it contains not only information carried by quantified observations but also information provided by the judgment ranking mechanism. Thus, statistical procedures based on RSS tend to be superior to their simple random sampling (SRS) analogs.

The success of RSS hinges on accuracy of the ranking process. To reduce possible errors, the set size m should be kept small in the basic version of RSS. Al-Saleh and Al-Kadiri [1] suggested double RSS (DRSS) that increases efficiency of the RSS mean estimator, given a fixed m. Al-Saleh and Al-Omari [2] generalized DRSS to multistage RSS (MSRSS), and showed that further gain in efficiency can be achieved in estimating the population mean. Al-Saleh and Samuh [3] investigated the distribution function and the median estimation based on MSRSS.

The MSRSS scheme can be summarized as follows:

1. Randomly identify m^{r+1} units from the population of interest, where r is the number of stages.
2. Allot the m^{r+1} units randomly into m^{r-1} sets of m^2 units each.
3. For each set in step 2, apply 1-2 of RSS procedure explained above, to get a (judgement) ranked set of size m. This step gives m^{r-1} (judgement) ranked sets, each of size m.
4. Without actual measuring of the ranked sets, apply step 3 on the m^{r-1} ranked set to gain m^{r-2} second stage (judgement) ranked sets, of size m each.
5. Repeat step 3, without any actual measurement, until an rth stage (judgement) ranked set of size m is acquired.
6. Actually measure the m identified units in step 5.
7. Repeat steps 1-6, p times (cycles), if necessary, to obtain an rth stage ranked set sample of size $M = p\,m$.

Similar to our previous notation, $\{X_{ik}^{(r)} : i = 1, \ldots, m\,; k = 1, \ldots, p\}$ denotes the rth stage ranked set sample. Clearly, the especial case of MSRSS with $r = 1$ corresponds to RSS. Also, DRSS is obtained by setting $r = 2$.

The estimation of system reliability has drawn much attention in the statistical literature. Reliability of a component with strength X which is subjected to stress Y is quantified by $\theta = P(X > Y)$. This approach is known as the stress-strength model. The estimation of θ has been extensively investigated in the literature when X and Y are independent random variables, and belong to the same family of distributions. A comprehensive account of this topic appear in Kotz et al. [5]. In this article, we study reliability estimation in MSRSS setup.

In Section 2, a nonparametric estimator is proposed and its properties are investigated in theory. Section 3 is given to a Monte Carlo analysis of the finite sample behavior of the estimator. A sport data set is analyzed in Section 4. The paper is concluded with a summary in Section 5.

2. ESTIMATION USING MSRSS

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random samples from two populations with density functions f and g, respectively. The corresponding distribution functions are denoted by F and G. The standard nonparametric estimator of θ is

$$\hat{\theta} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I(X_i > Y_j),$$

where $I(.)$ is the indicator function.

To construct an estimator under MSRSS, one needs two ranked set samples of sizes m and n from f and g. It is assumed that the samples are drawn using a single cycle. The results in the general setup are then easily followed. If $X_i^{(r)}$, $i = 1, \ldots, m$, and $Y_j^{(s)}$, $j = 1, \ldots, n$, are the two multistage ranked set samples, then

$$\hat{\theta}_{r,s} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I(X_i^{(r)} > Y_j^{(s)})$$

is a natural estimator of θ. The especial case of $r = s = 1$ was treated by Sengupta and Mukhuti [8].

Let $f_i^{(r)}$ and $F_i^{(r)}$ be the density and distribution function of $X_i^{(r)}$, respectively. The notation $g_j^{(s)}$ and $G_j^{(s)}$ will be used for similar functions associated with $Y_j^{(s)}$. Suppose the ith order statistic of an $(r-1)$th stage ranked set sample of size m from f, say $Z_1^{(r-1)}, \ldots, Z_m^{(r-1)}$, is denoted by $Z_{(i)}^{(r-1)}$. Under the assumption of no error in judgment ranking, we have $X_i^{(r)} \stackrel{d}{=} Z_{(i)}^{(r-1)}$.

In our mathematical development, the two identities

$$\frac{1}{m}\sum_{i=1}^{m} f_i^{(r)}(x) = f(x)$$

and

$$\frac{1}{n}\sum_{j=1}^{n} g_j^{(s)}(y) = g(y),$$

observed by Al-Saleh and Al-Omari [2], are repeatedly used. The above identities can be expressed in terms of distribution functions, as well.

It is straightforward to see that $\hat{\theta}$ is unbiased. The unbiasedness of $\hat{\theta}_{r,s}$ is verified in the following proposition.

Proposition 1 $\hat{\theta}_{r,s}$ *is an unbiased estimator of* θ*.*

Proof.

$$
\begin{aligned}
E\left\{\sum_{i=1}^{m}\sum_{j=1}^{n} I(X_i^{(r)} > Y_j^{(s)})\right\} &= \sum_{i=1}^{m}\sum_{j=1}^{n} P(X_i^{(r)} > Y_j^{(s)}) \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n} \int P(X_i^{(r)} > y) g_j^{(s)}(y)\,\mathrm{d}y \\
&= n\sum_{i=1}^{m} \int P(X_i^{(r)} > y) g(y)\,\mathrm{d}y \\
&= n\sum_{i=1}^{m} P(X_i^{(r)} > Y) \\
&= n\sum_{i=1}^{m} \int P(x > Y) f_i^{(r)}(x)\,\mathrm{d}x \\
&= mn \int P(x > Y) f(x)\,\mathrm{d}x \\
&= mnP(X > Y). \quad \square
\end{aligned}
$$

We now derive variance expressions of the two estimators.

Proposition 2 *The variances of $\hat{\theta}$ and $\hat{\theta}_{r,s}$ are given by*

$$
\begin{aligned}
m^2n^2Var(\hat{\theta}) \;=\; & m(m-1)n(n-1)\theta^2 + nm(m-1)E\left\{\bar{F}(Y)\right\}^2 \\
& + \; mn(n-1)E\left\{G(X)\right\}^2 + mn\theta - m^2n^2\theta^2,
\end{aligned}
\tag{2.1}
$$

and

$$
\begin{aligned}
m^2n^2Var(\hat{\theta}_{r,s}) \;=\; & E\left\{m^2\Big[\sum_{j=1}^{n}\bar{F}(Y_j^{(s)})\Big]^2 - \sum_{i=1}^{m}\Big[\sum_{j=1}^{n}\bar{F}_i^{(r)}(Y_j^{(s)})\Big]^2\right\} \\
& + \; mE\left\{n^2\Big[G(X)\Big]^2 - \sum_{j=1}^{n}\Big[G_j^{(s)}(X)\Big]^2\right\} \\
& + \; mn\theta - m^2n^2\theta^2.
\end{aligned}
\tag{2.2}
$$

Proof. It is easy to show that

$$
m^2n^2E(\hat{\theta}^2) = E(A_1 + A_2 + A_3 + A_4), \tag{2.3}
$$

where

$$
E(A_1) = E\left\{\sum_{i\neq i'=1}^{m}\sum_{j\neq j'=1}^{n} I(X_i > Y_j)I(X_{i'} > Y_{j'})\right\} = m(m-1)n(n-1)\theta^2,
$$

(2.4)

$$
\begin{aligned}
E(A_2) &= E\left\{ \sum_{j=1}^{n} \sum_{i \neq i'=1}^{m} I(X_i > Y_j) I(X_{i'} > Y_j) \right\} \\
&= \sum_{j=1}^{n} \sum_{i \neq i'=1}^{m} EE\left\{ I(X_i > Y_j) I(X_{i'} > Y_j) \Big| Y_j \right\} \\
&= \sum_{j=1}^{n} \sum_{i \neq i'=1}^{m} E\left\{ \bar{F}(Y) \right\}^2 = nm(m-1)E\left\{ \bar{F}(Y) \right\}^2, \qquad (2.5)
\end{aligned}
$$

$$
\begin{aligned}
E(A_3) &= E\left\{ \sum_{i=1}^{m} \sum_{j \neq j'=1}^{n} I(X_i > Y_j) I(X_i > Y_{j'}) \right\} \\
&= \sum_{i=1}^{m} \sum_{j \neq j'=1}^{n} EE\left\{ I(X_i > Y_j) I(X_i > Y_{j'}) \Big| X_i \right\} \\
&= \sum_{i=1}^{m} \sum_{j \neq j'=1}^{n} E\left\{ G(X) \right\}^2 = mn(n-1)E\left\{ G(X) \right\}^2, \qquad (2.6)
\end{aligned}
$$

and

$$
E(A_4) = E\left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} I(X_i > Y_j) \right\} = mn\theta. \qquad (2.7)
$$

From (2.3)-(2.7) and unbiasedness of $\hat{\theta}$, the proof of the first part is complete. Similarly,

$$
m^2 n^2 E(\hat{\theta}_{r,s}^2) = E(B_1 + B_2 + B_3), \qquad (2.8)
$$

where

$$
\begin{aligned}
E(B_1) &= E\Bigg\{ \sum_{i\neq i'=1}^{m} \sum_{j\neq j'=1}^{n} I(X_i^{(r)} > Y_j^{(s)}) I(X_{i'}^{(r)} > Y_{j'}^{(s)}) \\
&+ \sum_{j=1}^{n} \sum_{i\neq i'=1}^{m} I(X_i^{(r)} > Y_j^{(s)}) I(X_{i'}^{(r)} > Y_j^{(s)}) \Bigg\} \\
&= \sum_{i\neq i'=1}^{m} \sum_{j\neq j'=1}^{n} EE\Big\{ I(X_i^{(r)} > Y_j^{(s)}) \Big| Y_j^{(s)} \Big\} EE\Big\{ I(X_{i'}^{(r)} > Y_{j'}^{(s)}) \Big| Y_{j'}^{(s)} \Big\} \\
&+ \sum_{j=1}^{n} \sum_{i\neq i'=1}^{m} EE\Big\{ I(X_i^{(r)} > Y_j^{(s)}) I(X_{i'}^{(r)} > Y_j^{(s)}) \Big| Y_j^{(s)} \Big\} \\
&= E\Bigg\{ \sum_{i\neq i'=1}^{m} \sum_{j\neq j'=1}^{n} \Big[\bar{F}_i^{(r)}(Y_j^{(s)})\Big]\Big[\bar{F}_{i'}^{(r)}(Y_{j'}^{(s)})\Big] \\
&+ \sum_{j=1}^{n} \sum_{i\neq i'=1}^{m} \Big[\bar{F}_i^{(r)}(Y_j^{(s)})\Big]\Big[\bar{F}_{i'}^{(r)}(Y_j^{(s)})\Big] \Bigg\} \\
&= E\Bigg\{ \Big[\sum_{i=1}^{m} \sum_{j=1}^{n} \bar{F}_i^{(r)}(Y_j^{(s)}) \Big]^2 - \sum_{i=1}^{m} \sum_{j=1}^{n} \Big[\bar{F}_i^{(r)}(Y_j^{(s)})\Big]^2 \\
&- \sum_{i=1}^{m} \sum_{j\neq j'=1}^{n} \Big[\bar{F}_i^{(r)}(Y_j^{(s)})\Big]\Big[\bar{F}_i^{(r)}(Y_{j'}^{(s)})\Big] \Bigg\} \\
&= E\Bigg\{ m^2 \Big[\sum_{j=1}^{n} \bar{F}(Y_j^{(s)}) \Big]^2 - \sum_{i=1}^{m} \Big[\sum_{j=1}^{n} \bar{F}_i^{(r)}(Y_j^{(s)}) \Big]^2 \Bigg\}, \qquad (2.9)
\end{aligned}
$$

$$
\begin{aligned}
E(B_2) &= E\Bigg\{ \sum_{i=1}^{m} \sum_{j\neq j'=1}^{n} I(X_i^{(r)} > Y_j^{(s)}) I(X_i^{(r)} > Y_{j'}^{(s)}) \Bigg\} \\
&= m \sum_{j\neq j'=1}^{n} E\Big\{ I(X > Y_j^{(s)}) I(X > Y_{j'}^{(s)}) \Big\} \\
&= m \sum_{j\neq j'=1}^{n} EE\Big\{ I(X > Y_j^{(s)}) I(X > Y_{j'}^{(s)}) \Big| X \Big\} \\
&= m \sum_{j\neq j'=1}^{n} E\Big\{ \Big[G_j^{(s)}(X)\Big]\Big[G_{j'}^{(s)}(X)\Big] \Big\} \\
&= mE\Bigg\{ n^2 \Big[G(X)\Big]^2 - \sum_{j=1}^{n} \Big[G_j^{(s)}(X)\Big]^2 \Bigg\}, \qquad (2.10)
\end{aligned}
$$

and

$$
E(B_3) = E\Bigg\{ \sum_{i=1}^{m} \sum_{j=1}^{n} I(X_i^{(r)} > Y_j^{(s)}) \Bigg\} = mn\theta. \qquad (2.11)
$$

Now the second part follows from (2.8)-(2.11) and unbiasedness of $\hat{\theta}_{r,s}$. □

The variances of $\hat{\theta}$ and $\hat{\theta}_{r,s}$ are compared in the next proposition.
Proposition 3 *For any $m, n \geq 2$ and $r, s \geq 1$, $Var(\hat{\theta}_{r,s}) \leq Var(\hat{\theta})$.*
Proof. Using equations (2.1) and (2.2), it can be shown

$$m^2n^2\Big[Var(\hat{\theta}) - Var(\hat{\theta}_{r,s})\Big] = C_1 + C_2 + C_3,$$

where

$$\begin{aligned} C_1 &= E\Bigg\{\sum_{i=1}^{m}\Big[\sum_{j=1}^{n}\bar{F}_i^{(r)}(Y_j^{(s)})\Big]^2 - m\Big[\sum_{j=1}^{n}\bar{F}(Y_j^{(s)})\Big]^2\Bigg\} \\ &= E\Bigg\{\sum_{i=1}^{m}\Big(\sum_{j=1}^{n}\Big[\bar{F}_i^{(r)}(Y_j^{(s)}) - \bar{F}(Y_j^{(s)})\Big]\Big)^2\Bigg\}, \end{aligned}$$

$$\begin{aligned} C_2 &= mn(n-1)E\Big\{G(X)\Big\}^2 - mE\Big\{n^2\Big[G(X)\Big]^2 - \sum_{j=1}^{n}\Big[G_j^{(s)}(X)\Big]^2\Big\} \\ &= mE\Big\{\sum_{j=1}^{n}\Big[G_j^{(s)}(X)\Big]^2 - n\Big[G(X)\Big]^2\Big\} \\ &= mE\Big\{\sum_{j=1}^{n}\Big[G_j^{(s)}(X) - G(X)\Big]^2\Big\}, \end{aligned}$$

and

$$\begin{aligned} C_3 &= m(m-1)n(n-1)\theta^2 + nm(m-1)E\Big\{\bar{F}(Y)\Big\}^2 \\ &- m(m-1)E\Big\{\Big[\sum_{j=1}^{n}\bar{F}(Y_j^{(s)})\Big]^2\Big\} \\ &= m(m-1)\Bigg[(1-\frac{1}{n})\Big(\sum_{j=1}^{n}E\Big\{\bar{F}(Y_j^{(s)})\Big\}\Big)^2 \\ &- \sum_{j\neq j'=1}^{n}E\Big\{\bar{F}(Y_j^{(s)})\Big\}E\Big\{\bar{F}(Y_{j'}^{(s)})\Big\}\Bigg] \\ &= m(m-1)\Bigg[\sum_{j=1}^{n}E^2\Big\{\bar{F}(Y_j^{(s)})\Big\} - \frac{1}{n}\Big(\sum_{j=1}^{n}E\Big\{\bar{F}(Y_j^{(s)})\Big\}\Big)^2\Bigg] \\ &= m(m-1)\sum_{j=1}^{n}E^2\Big\{\bar{F}(Y_j^{(s)}) - \bar{F}(Y)\Big\}. \end{aligned}$$

Clearly, $C_i \geq 0$, $i = 1, 2, 3$, as was asserted. □

As mentioned earlier, increasing the number of stages leads to improvement in the context of mean and distribution function estimation based on MSRSS. So, it is natural to observe similar trend in the case of reliability estimation. The next result attends to this problem.

Proposition 4 *For fixed m and n, $Var(\hat{\theta}_{r,s})$ is decreasing in r and s.*

Proof. It suffices to show that $Var(\hat{\theta}_{r,s}) \leq Var(\hat{\theta}_{r-1,s})$ and $Var(\hat{\theta}_{r,s}) \leq Var(\hat{\theta}_{r,s-1})$. From the beginning of proof for the second part of Proposition 2, one can write

$$\begin{aligned} m^2n^2E(\hat{\theta}^2_{r,s}) &= E\Bigg\{ \sum_{i\neq i'=1}^{m} \sum_{j\neq j'=1}^{n} I(X_i^{(r)} > Y_j^{(s)})I(X_{i'}^{(r)} > Y_{j'}^{(s)}) \\ &+ \sum_{i=1}^{m} \sum_{j\neq j'=1}^{n} I(X_i^{(r)} > Y_j^{(s)})I(X_i^{(r)} > Y_{j'}^{(s)}) \\ &+ \sum_{j=1}^{n} \sum_{i\neq i'=1}^{m} I(X_i^{(r)} > Y_j^{(s)})I(X_{i'}^{(r)} > Y_j^{(s)}) \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{n} I(X_i^{(r)} > Y_j^{(s)}) \Bigg\}. \end{aligned} \tag{2.12}$$

We now establish some equalities and inequalities regarding the four expectation terms on the right-hand side of the above equation. Let $W_{(i)}^{(r-1)}$ be the ith order statistic of an $(r-1)$th stage ranked set sample of size m from f. As to the first term, we have

$$\begin{aligned} E\Big\{I(X_i^{(r)} > Y_j^{(s)})I(X_{i'}^{(r)} > Y_{j'}^{(s)})\Big\} &= EE\Big\{I(X_i^{(r)} > Y_j^{(s)})I(X_{i'}^{(r)} > Y_{j'}^{(s)})\Big|Y_j^{(s)}, Y_{j'}^{(s)}\Big\} \\ &= E\Big[E\Big\{I(X_i^{(r)} > Y_j^{(s)})\Big|Y_j^{(s)}, Y_{j'}^{(s)}\Big\} \\ &\times E\Big\{I(X_{i'}^{(r)} > Y_{j'}^{(s)})\Big|Y_j^{(s)}, Y_{j'}^{(s)}\Big\}\Big] \\ &= E\Big[E\Big\{I(W_{(i)}^{(r-1)} > Y_j^{(s)})\Big|Y_j^{(s)}, Y_{j'}^{(s)}\Big\} \\ &\times E\Big\{I(W_{(i')}^{(r-1)} > Y_{j'}^{(s)})\Big|Y_j^{(s)}, Y_{j'}^{(s)}\Big\}\Big] \\ &\leq EE\Big\{I(W_{(i)}^{(r-1)} > Y_j^{(s)})I(W_{(i')}^{(r-1)} > Y_{j'}^{(s)})\Big|Y_j^{(s)}, Y_{j'}^{(s)}\Big\} \\ &= E\Big\{I(W_{(i)}^{(r-1)} > Y_j^{(s)})I(W_{(i')}^{(r-1)} > Y_{j'}^{(s)})\Big\}, \end{aligned} \tag{2.13}$$

where the inequality holds owing to the positive covariance between any pair of order statistics in a sample (see Lehmann [6]).

Similarly, it follows that

$$
\begin{aligned}
E\left\{ I(X_i^{(r)} > Y_j^{(s)}) I(X_{i'}^{(r)} > Y_j^{(s)}) \right\} &= EE\left\{ I(X_i^{(r)} > Y_j^{(s)}) I(X_{i'}^{(r)} > Y_j^{(s)}) \middle| Y_j^{(s)} \right\} \\
&= E\Bigg[E\left\{ I(X_i^{(r)} > Y_j^{(s)}) \middle| Y_j^{(s)} \right\} \\
&\quad \times E\left\{ I(X_{i'}^{(r)} > Y_j^{(s)}) \middle| Y_j^{(s)} \right\} \Bigg] \\
&= E\Bigg[E\left\{ I(W_{(i)}^{(r-1)} > Y_j^{(s)}) \middle| Y_j^{(s)} \right\} \\
&\quad \times E\left\{ I(W_{(i')}^{(r-1)} > Y_j^{(s)}) \middle| Y_j^{(s)} \right\} \Bigg] \\
&\le EE\left\{ I(W_{(i)}^{(r-1)} > Y_j^{(s)}) I(W_{(i')}^{(r-1)} > Y_j^{(s)}) \middle| Y_j^{(s)} \right\} \\
&= E\left\{ I(W_{(i)}^{(r-1)} > Y_j^{(s)}) I(W_{(i')}^{(r-1)} > Y_j^{(s)}) \right\}. \qquad (2.14)
\end{aligned}
$$

In addition,

$$
\begin{aligned}
E\left\{ I(X_i^{(r)} > Y_j^{(s)}) I(X_i^{(r)} > Y_{j'}^{(s)}) \right\} &= EE\left\{ I(X_i^{(r)} > Y_j^{(s)}) I(X_i^{(r)} > Y_{j'}^{(s)}) \middle| Y_j^{(s)}, Y_{j'}^{(s)} \right\} \\
&= EE\left\{ I(W_{(i)}^{(r-1)} > Y_j^{(s)}) I(W_{(i)}^{(r-1)} > Y_{j'}^{(s)}) \middle| Y_j^{(s)}, Y_{j'}^{(s)} \right\} \\
&= E\left\{ I(W_{(i)}^{(r-1)} > Y_j^{(s)}) I(W_{(i)}^{(r-1)} > Y_{j'}^{(s)}) \right\}, \qquad (2.15)
\end{aligned}
$$

and

$$
\begin{aligned}
E\left\{ I(X_i^{(r)} > Y_j^{(s)}) \right\} &= EE\left\{ I(X_i^{(r)} > Y_j^{(s)}) \middle| Y_j^{(s)} \right\} \\
&= EE\left\{ I(W_{(i)}^{(r-1)} > Y_j^{(s)}) \middle| Y_j^{(s)} \right\} \\
&= E\left\{ I(W_{(i)}^{(r-1)} > Y_j^{(s)}) \right\}. \qquad (2.16)
\end{aligned}
$$

Putting (2.12)-(2.16) together, we get

$$\begin{aligned} m^2n^2E(\hat{\theta}^2_{r,s}) \leq E\Bigg\{ & \sum_{i\neq i'=1}^{m}\sum_{j\neq j'=1}^{n} I(W_{(i)}^{(r-1)} > Y_j^{(s)})I(W_{(i')}^{(r-1)} > Y_{j'}^{(s)}) \\ & + \sum_{i=1}^{m}\sum_{j\neq j'=1}^{n} I(W_{(i)}^{(r-1)} > Y_j^{(s)})I(W_{(i)}^{(r-1)} > Y_{j'}^{(s)}) \\ & + \sum_{j=1}^{n}\sum_{i\neq i'=1}^{m} I(W_{(i)}^{(r-1)} > Y_j^{(s)})I(W_{(i')}^{(r-1)} > Y_j^{(s)}) \\ & + \sum_{i=1}^{m}\sum_{j=1}^{n} I(W_{(i)}^{(r-1)} > Y_j^{(s)})\Bigg\} = m^2n^2E(\hat{\theta}^2_{r-1,s}). \end{aligned}$$

This implies that $Var(\hat{\theta}_{r,s}) \leq Var(\hat{\theta}_{r-1,s})$ because $\hat{\theta}_{r,s}$ is unbiased for any $r, s \geq 1$. A similar argument proves the second part. □

The above theoretical development assumes perfect rankings. It is possible to obtain some results in the imperfect ranking situation. Suppose the ranking mechanism is such that

$$\frac{1}{m}\sum_{i=1}^{m} \tilde{f}_i^{(r)}(x) = f(x),$$

and

$$\frac{1}{n}\sum_{j=1}^{n} \tilde{g}_j^{(s)}(y) = g(y),$$

where $\tilde{f}_i^{(r)}$ and $\tilde{g}_j^{(s)}$ are the density functions of the multistage judgment order statistics drawn from the two populations. Then one can simply verify that Propositions 1 and 3 still hold. However, it may not be an easy job to prove Proposition 4 in this setup. In the next section, effect of the ranking errors is assessed using Monte Carlo simulations.

3. NUMERICAL RESULTS

This section reports results of simulation studies carried out to compare the performances of $\hat{\theta}$ and $\hat{\theta}_{r,s}$. It is assumed that both populations follow normal, exponential or uniform distribution. Suppose X and $Y-\mu$ are standard normal random variables. Then, it is simply shown that

$$\theta = \Phi\left(\frac{-\mu}{\sqrt{2}}\right),$$

where $\Phi(.)$ is the distribution function of X. Similarly, for standard exponential random variables X and Y/α, we have

$$\theta = \frac{1}{1+\alpha}.$$

Table 1: Parameter values corresponding to case A, B and C.

Parameter	A	B	C
μ	0.95387	0	-0.95387
α	3	1	1/3
β	2	1	1/2

Finally, let X and Y/β be uniformly distributed on the unit interval. Then, it follows that

$$\theta = \begin{cases} 1-\beta/2 & 0<\beta<1 \\ 1/(2\beta) & \beta \geq 1 \end{cases}.$$

Under each parent distribution, three values were assigned to the associated parameter so as to produce $\theta = 0.25, 0.5, 0.75$ which are referred to as case A, B and C, respectively. The appropriate parameter values are given in Table 1. Also, sample sizes $(m,n) \in \{(3,3),(4,4),(5,5)\}$ and stage numbers $(r,s) \in \{(1,1),(2,2),(2,4),(3,3),(4,4),(4,6),(5,5)\}$ were selected.

We assume that the ranking the variables of interest X and Y are done based on concomitant variables $\mathcal{X}$ and $\mathcal{Y}$ which are related according to equations

$$\mathcal{X} = \rho_1 \left(\frac{X-\mu_x}{\sigma_x}\right) + \sqrt{1-\rho_1^2} Z_1,$$

and

$$\mathcal{Y} = \rho_2 \left(\frac{Y-\mu_y}{\sigma_y}\right) + \sqrt{1-\rho_2^2} Z_2,$$

where $\rho_i \in [0,1]$ $(i=1,2)$, and Z_1 (Z_2) is a standard normal random variable independent from X (Y). Moreover, Z_1 and Z_2 are independent. The quality of rankings are controlled by the parameter ρ_i's. It is easy to see that $Corr(X,\mathcal{X}) = \rho_1$ and $Corr(Y,\mathcal{Y}) = \rho_2$. The chosen values of (ρ_1,ρ_2) are $(1,1)$ for perfect rankings of X and Y, $(1,0.8)$ for perfect ranking of X and fairly accurate ranking of Y, and $(0.8,0.8)$ for fairly accurate rankings of X and Y.

For each combination of distribution, sample sizes and correlations, 5,000 pairs of samples were generated in SRS and MSRSS (with the aforesaid stage numbers). The two estimators were computed from each pair of samples, and their variances were determined. The relative efficiency (RE) is defined as the ratio of $\widehat{Var}(\hat{\theta})$ to $\widehat{Var}(\hat{\theta}_{r,s})$. The RE values larger than one indicate that $\hat{\theta}_{r,s}$ is more efficient than $\hat{\theta}$. Tables 2-4 display the results.

It is observed that that MSRSS based estimator outperforms its SRS contender in all situations considered. Moreover, for any (m,n), the RE is increasing in both r and s, when the other factors are fixed. For example, compare entries for $m=n=3$. In general, no comparison can be made between REs in two setups that one stage number is increased, and the other one is decreased. The efficiency gain could be substantial if the set sizes and stage numbers are large, e.g. when $m=n=r=s=5$, the parent distribution is uniform, and the

Table 2: Estimated REs for different sample sizes and stage numbers under normal distribution.

		$(\rho_1,\rho_2)=(1,1)$			$(\rho_1,\rho_2)=(1,0.8)$			$(\rho_1,\rho_2)=(0.8,0.8)$		
(m,n)	(r,s)	A	B	C	A	B	C	A	B	C
(3,3)	(1,1)	1.719	1.860	1.737	1.613	1.546	1.408	1.425	1.402	1.262
	(2,2)	2.281	2.547	2.255	1.963	1.778	1.482	1.582	1.469	1.311
	(2,4)	2.623	3.284	2.599	2.127	1.906	1.460	1.654	1.563	1.284
	(3,3)	2.662	3.323	2.724	2.391	1.996	1.527	1.766	1.584	1.355
	(4,4)	3.078	4.034	3.052	2.535	2.149	1.697	1.880	1.685	1.410
	(4,6)	3.295	4.325	3.155	2.468	2.223	1.664	1.778	1.692	1.470
	(5,5)	3.291	4.435	3.304	2.641	2.224	1.634	1.834	1.652	1.423
(4,4)	(1,1)	2.141	2.334	2.118	1.847	1.721	1.493	1.526	1.461	1.302
	(2,2)	3.006	3.760	2.959	2.421	2.152	1.703	1.755	1.678	1.442
	(2,4)	3.626	4.559	3.534	2.598	2.275	1.766	1.910	1.738	1.502
	(3,3)	3.911	5.064	3.952	2.757	2.401	1.787	1.934	1.790	1.517
	(4,4)	4.440	6.059	4.389	3.125	2.669	1.948	2.046	1.818	1.596
	(4,6)	4.698	6.641	4.638	3.128	2.677	1.881	2.067	1.892	1.510
	(5,5)	4.625	6.685	4.666	3.259	2.793	2.038	2.019	1.877	1.594
(5,5)	(1,1)	2.458	2.813	2.501	2.083	1.840	1.591	1.645	1.551	1.368
	(2,2)	3.904	4.994	3.948	2.674	2.325	1.879	1.860	1.749	1.530
	(2,4)	4.902	6.412	4.825	3.145	2.683	1.944	2.102	1.886	1.604
	(3,3)	5.061	6.916	5.019	3.258	2.741	1.942	2.067	1.882	1.536
	(4,4)	6.071	8.783	6.079	3.415	2.925	2.080	2.156	2.014	1.680
	(4,6)	6.435	9.502	6.405	3.379	2.942	2.054	2.111	1.978	1.589
	(5,5)	6.627	10.050	6.726	3.768	3.162	2.166	2.261	2.042	1.641

rankings are perfect. It is to be mentioned that when $(\rho_1,\rho_2)=(1,1)$, the REs for cases A and C are in good agreement (and smaller than that of case B) for all distributions and sample sizes, particularly when $r=s$. As expected, the REs diminish in the presence of ranking errors. The smallest values are obtained for $(\rho_1,\rho_2)=(0.8,0.8)$.

4. APPLICATION TO REAL DATA

The MSRSS can be very efficient if the variable of interest is highly correlated to a concomitant variable. In this case, if the second variable can be measured with negligible cost, then we may use it in judgment ranking process (see Stokes [9] for more details). In doing so, in step 2 of the RSS procedure, the elements of the ith sample are ordered according to the concomitant variable, and then study variable is actually measured for unit ranked ith smallest. The MSRSS case is treated similarly.

Table 3: Estimated REs for different sample sizes and stage numbers under exponential distribution.

		$(\rho_1,\rho_2)=(1,1)$			$(\rho_1,\rho_2)=(1,0.8)$			$(\rho_1,\rho_2)=(0.8,0.8)$		
(m,n)	(r,s)	A	B	C	A	B	C	A	B	C
(3,3)	(1,1)	1.699	1.894	1.727	1.570	1.559	1.372	1.247	1.312	1.264
	(2,2)	2.310	2.724	2.279	1.798	1.853	1.544	1.267	1.421	1.323
	(2,4)	2.762	3.141	2.479	1.972	1.948	1.553	1.352	1.450	1.350
	(3,3)	2.733	3.420	2.667	2.067	2.120	1.643	1.313	1.504	1.379
	(4,4)	3.074	3.996	3.138	2.331	2.179	1.673	1.344	1.471	1.380
	(4,6)	3.418	4.262	3.191	2.296	2.214	1.667	1.428	1.548	1.394
	(5,5)	3.347	4.358	3.364	2.472	2.430	1.735	1.402	1.624	1.425
(4,4)	(1,1)	2.145	2.322	2.065	1.806	1.733	1.434	1.326	1.385	1.267
	(2,2)	3.102	3.811	3.101	2.371	2.152	1.711	1.407	1.536	1.396
	(2,4)	3.973	4.597	3.494	2.644	2.485	1.750	1.471	1.661	1.442
	(3,3)	3.832	5.034	3.880	2.589	2.476	1.810	1.393	1.557	1.443
	(4,4)	4.507	6.178	4.567	3.063	2.774	1.943	1.423	1.624	1.554
	(4,6)	5.282	7.028	4.795	2.991	2.789	1.983	1.469	1.673	1.601
	(5,5)	4.903	7.039	5.043	3.185	2.764	1.869	1.490	1.640	1.471
(5,5)	(1,1)	2.486	2.756	2.454	1.974	1.871	1.532	1.303	1.416	1.338
	(2,2)	3.961	4.954	4.081	2.896	2.606	1.829	1.513	1.712	1.518
	(2,4)	5.338	6.308	4.510	2.979	2.604	1.742	1.490	1.622	1.411
	(3,3)	5.468	7.232	5.453	3.347	2.765	1.780	1.545	1.716	1.466
	(4,4)	6.069	8.652	6.104	3.604	2.848	1.881	1.600	1.722	1.520
	(4,6)	7.227	9.814	6.458	3.607	3.057	1.960	1.541	1.747	1.545
	(5,5)	7.156	10.408	7.145	4.048	3.162	1.996	1.645	1.811	1.570

In this section, we illustrate the proposed procedure using a data set collected at the Australian Institute of Sport. It is made up of thirteen measured variables on 102 male and 100 female athletes[1]. We will consider lean body mass (LBM) and body mass index (BMI) for each athlete. The LBM is a component of body composition, calculated by subtracting body fat weight from total body weight. Exact measurement of the LBM is done using various technologies such as dual energy X-ray absorptiometry (DEXA) which is costly. On the other hand, the BMI is a well-accepted measure of obesity which is easy to calculate and readily accessible. A BMI value is is simply weight (in kg) divided by square of height (in m). The correlation coefficient between the two variables is 0.71. So, the BMI can serve as a concomitant variable.

Let X and Y be the LBM variable for the male and female populations, respectively. It is of interest to estimate $\theta = P(X > Y)$. For $m = n = 4$, 50,000 samples were drawn from the two hypothetical populations based on SRS and MSRSS (with $r = s = 1, 2$) designs. The sampling is done with replacement

[1]The data set can be found at http://www.statsci.org/data/oz/ais.html

Table 4: Estimated REs for different sample sizes and stage numbers under uniform distribution.

		$(\rho_1, \rho_2) = (1, 1)$			$(\rho_1, \rho_2) = (1, 0.8)$			$(\rho_1, \rho_2) = (0.8, 0.8)$		
(m, n)	(r, s)	A	B	C	A	B	C	A	B	C
(3,3)	(1,1)	1.718	1.813	1.684	1.638	1.665	1.429	1.401	1.489	1.371
	(2,2)	2.369	2.656	2.301	2.182	2.160	1.749	1.645	1.755	1.647
	(2,4)	2.973	3.210	2.529	2.117	2.344	1.914	1.592	1.846	1.725
	(3,3)	2.927	3.370	2.866	2.429	2.341	1.848	1.725	1.795	1.707
	(4,4)	3.272	3.913	3.350	2.869	2.678	1.926	1.808	1.984	1.713
	(4,6)	3.788	4.311	3.463	2.720	2.592	1.922	1.688	1.907	1.691
	(5,5)	3.625	4.348	3.690	2.942	2.764	2.045	1.793	1.994	1.817
(4,4)	(1,1)	2.024	2.298	2.030	1.903	1.916	1.653	1.422	1.613	1.524
	(2,2)	3.142	3.726	3.283	2.763	2.593	2.004	1.766	1.915	1.796
	(2,4)	4.435	4.670	3.410	2.839	2.808	2.199	1.734	2.037	1.993
	(3,3)	4.209	5.158	4.263	3.377	2.900	2.019	1.853	2.040	1.792
	(4,4)	4.832	5.916	4.810	4.204	3.368	2.216	2.068	2.209	1.906
	(4,6)	5.515	6.499	5.035	3.824	3.462	2.383	1.935	2.220	2.055
	(5,5)	5.351	6.774	5.462	4.097	3.378	2.281	1.958	2.240	1.959
(5,5)	(1,1)	2.375	2.806	2.328	2.317	2.196	1.722	1.649	1.805	1.577
	(2,2)	4.166	5.092	4.147	3.350	2.908	2.052	1.874	2.072	1.840
	(2,4)	6.162	6.384	4.527	3.504	3.238	2.251	1.910	2.189	1.928
	(3,3)	5.593	6.732	5.547	4.096	3.395	2.235	1.995	2.244	1.894
	(4,4)	6.934	8.888	6.965	4.683	3.794	2.365	1.962	2.305	2.011
	(4,6)	8.384	9.715	7.319	4.943	3.732	2.452	2.125	2.391	2.077
	(5,5)	8.206	10.226	8.064	5.590	4.256	2.535	2.200	2.508	2.144

to ensure that the measured units are independent of each other. From each sample, the corresponding estimator was computed, and its variance was finally determined. The efficiencies of $\hat{\theta}_{1,1}$ and $\hat{\theta}_{2,2}$ relative to $\hat{\theta}$ are estimated as 1.193 and 1.275, respectively. As expected, the SRS estimator is outperformed by its RSS and DRSS versions. It is to be noted that the RE values are not much bigger than unity. This may root in the relatively low correlation of 0.71 between the variable of interest and the concomitant variable.

5. CONCLUSION

The RSS design is known to be a viable alternate to the usual SRS in situations that cost-efficiency is of high importance. It employs auxiliary information to direct attention toward the actual measurement of more representative units in the population under study. The success of RSS largely depends on the quality of ranking process. Since judgment ranking on large sets of units is prone to errors, the set size is chosen small in practice. The MSRSS allows to construct more efficient procedures by increasing the number of stages rather that the set size.

This article deals with reliability estimation for the stress-strength model using MSRSS. A nonparametric estimator is presented, and shown to be unbiased with smaller variance as compared with the usual estimator in SRS. It is further proved that the estimator becomes more efficient by increasing the number of stages for ranked set samples drawn from the two populations. Results of simulation studies support the mathematical findings. An application to a real data set clarifies how judgment ranking can be implemented using a concomitant variable.

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REFERENCES

[1] Al-Saleh, M.F., and Al-Kadiri, M. (2000). Double-ranked set sampling. Statistics and Probability Letters 48, 205-212.

[2] Al-Saleh, M.F., and Al-Omari, A.I. (2002). Multistage ranked set sampling. Journal of Statistical Planning and Inference 102, 273-286.

[3] Al-Saleh, M.F., and Samuh, M.J. (2008). On multistage ranked set sampling for distribution and median estimation. Computational Statistics and Data Analysis 52, 2066-2078.

[4] Chen, Z., Bai, Z., and Sinha, B.K. (2004). Ranked set sampling: Theory and Applications. Springer, New York.

[5] Kotz, S., Lumelskii, Y., and Pensky, M. (2003). The stress-strength model and its generalizations. Theory and applications. World Scientific, Singapore.

[6] Lehmann, E.L. (1966). Some concepts of dependence. The Annals of Mathematical Statistics 37, 1137-1153.

[7] McIntyre, G.A. (1952). A method of unbiased selective sampling using ranked sets. Australian Journal of Agricultural Research 3, 385-390.

[8] Sengupta, S., and Mukhuti, S. (2008). Unbiased estimation of $P(X > Y)$ using ranked set sample data. Statistics 42, 223-230.

[9] Stokes, S.L. (1977). Ranked set sampling with concomitant variables. Communications in Statistics: Theory and Methods 6, 1207-1211.