PREDICTION INTERVALS FOR TIME SERIES AND THEIR APPLICATIONS TO PORTFOLIO SELECTION

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Abstract:

- This study considers prediction intervals for time series and applies the results to portfolio selection. The dynamics of the high and low underlying returns are depicted by time series models, which lead to a prediction interval of future returns. We propose an innovative criterion for portfolio selection based on the prediction interval. A new concept of coherent risk measures for the interval of returns is introduced. An empirical study is conducted with the stocks of the Dow Jones Industrial Average Index. A self-financing trading strategy is established by daily reallocating the holding positions via the proposed portfolio selection criterion. The numerical results indicate that the proposed prediction interval has promising coverage, efficiency and accuracy for prediction. The proposed portfolio selection criterion constructed from the prediction intervals is capable of suggesting an optimal portfolio according to the economic conditions.

Key-Words:

- *Coherent risk measure; Portfolio selection; Prediction interval.*

AMS Subject Classification:

- 37M10, 91G10.

1. Introduction

We propose to obtain prediction intervals of a time series by constructing interval-valued time series (ITS) models. The proposed method is used to integrate the information of the daily high, low and closing prices of a stock and is applied to the problem of portfolio selection. Optimal portfolio selection has been extensively discussed in the fields of financial investment and risk management. Marlowitz (1952, 1959) introduced a mean-variance portfolio optimization procedure by using the standard deviation of a portfolio as the measure of risk and assuming that the returns of the underlying assets are independent and identically distributed (i.i.d.). During the past decade, risk measures other than the standard deviation have been considered for selecting investment portfolios. For example, the value-at-risk (VaR), conditional VaR (CVaR) and spectral risk measure (SRM) are commonly used risk measures by market practitioners and analysts in the recent literature on portfolio selection (Rockafellar and Uryasev, 2000, 2002; Acerbi, 2002; Krokhmal et al., 2002; Adam et al., 2008). However, many empirical findings indicate that the return processes of the underlying assets in financial markets usually exhibit autocorrelation, negative skewness, kurtosis, conditional heteroscedasticity and tail dependence (Tsay, 2010). To reflect these features, time series models are used to depict the dynamics of the underlying asset returns for portfolio selection (Harris and Mazibas, 2013). However, the development of the above portfolio selection issue uses only information about the closing prices of the underlying assets. The daily high and low prices of a stock are public information and can be observed in the market. The main purpose of this study is to apply daily high and low price information to portfolio selection by ITS models.

One of the main techniques for analyzing ITS is to fit univariate time series models to the interval bounds (Teles and Brito, 2005). Maia et al. (2008) proposed fitting univariate ARIMA models to the midpoints and ranges of the observed interval process and used these models to forecast the interval bounds. Recently, many more complicated ITS models have been proposed and applied to solve problems in various fields. For example, He and Hu (2009) used the interval computing approach to forecast the annual and quarterly variability of the stock market. Arroyo et al. (2010, 2011) discussed financial applications based on forecasting with ITS data. García-Ascanio and Maté (2010) used vector autoregressive (VAR) models to forecast electric power demand. Yang et al. (2012) proposed autoregressive conditional interval-valued models with exogenous explanatory interval variables to forecast crude oil prices. Rodrigues and Salish (2015) used threshold models to analyze and forecast ITS and applied their model to a weekly sample of S&P500 index returns. Fischer et al. (2016) predicted stock return volatility using regression models for return intervals. The results of these studies showed that the interval forecasts obtained by ITS perform better than those obtained by the classic approach based on fitting a single time series model to closing prices.

Following Markowitz (1952, 1959)'s approach, the basic idea of various portfolio selection criteria is to determine asset allocations by maximizing the expected investment returns subject to a risk limit of the investment. In addition to daily high and low prices, we also consider the closing prices of a stock. Subsequently, the daily high (low) log returns should be defined as the differences between the logarithms of the daily high (low) price and the last closing price. Therefore, we propose fitting time series models to the daily high and low log returns rather than fitting ITS models directly to the interval bounds of stock prices. Furthermore, an innovative criterion for portfolio selection is proposed based on the predicted interval of the log returns. Specifically, we maximize the expected high log returns of a portfolio subject to a limitation on the predicted low log returns. We also introduce the concept of a coherent risk measure for the interval of returns, which extends the axioms of the coherent risk measure proposed by Artzner et al. (1999) for classic financial risk management. In the empirical investigation, we employ the stocks of the companies on the Dow Jones Industrial Average Index (DJIA Index) during the financial crisis period (from July 2, 2007 to June 24, 2009) and under improved market conditions (from July 1, 2014 to June 23, 2016). For each time period, the first 250 daily data are used to fit a time series model to determine the initial trading strategy. A self-financing trading strategy is constructed by daily reallocating the holding weights of the optimal portfolio via the proposed scheme, where a rolling scheme is employed and the time series model is updated with the previous 250 daily historical data. The numerical results indicate that the proposed interval estimation has promising coverage, efficiency and accuracy for predicting high and low prices. Moreover, the proposed portfolio suggests conservative investments during 2008-2009 but aggressive investments during 2015-2016.

The rest of this paper is organized as follows. Section 2 introduces the model assumptions and the prediction interval for ITS. The proposed criterion for portfolio selection using the prediction intervals is introduced in Section 3. Section 4 presents a study to compare the coverage, efficiency and accuracy of the proposed interval estimation for ITS data with those of various approaches in the literature. An empirical study to assess the performance of the self-financing trading strategy constructed by the proposed criterion of portfolio selection is presented in Section 5. Conclusions are given in Section 6, and technical proofs, figures and tables are included in the Appendix.

2. The proposed interval time series model

Let $P^C_{m,t}$ be the daily closing price of the mth underlying stock price at time t, and let $P^H_{m,t}$ and $P^L_{m,t}$ be the intraday high and low stock prices, respectively, $m = 1, \dots, p$. Denote the set of information up to time t by $\mathcal{F}_t$. To obtain a one-step-ahead prediction interval of the price of the mth underlying stock for a given $\mathcal{F}_t$, a classic approach is to fit a time series model for the historical closing prices, $P^C_{m,s}$, $s = 1, \ldots, t$, and then derive a 95% prediction interval, for example, for $P^C_{m,t+1}$, from the fitted model. Recently, many studies have proposed fitting ITS models for interval observations $[P^L_{m,s}, P^H_{m,s}]$, $s = 1, \ldots, t$, and then obtaining an interval estimation of $[P^L_{m,t+1}, P^H_{m,t+1}]$ from the fitted ITS model (see Arroyo et al., 2010, 2011; Teles and Brito, 2015 and the references therein).

We propose an alternative approach to obtain an estimate of $[P^L_{m,t+1}, P^H_{m,t+1}]$ conditional on $\mathcal{F}_t$ based on the following daily low and high log returns at time t:

$$X^{(CL)}_{m,t} = \log(P^L_{m,t}/P^C_{m,t-1}) \text{ and } X^{(CH)}_{m,t} = \log(P^H_{m,t}/P^C_{m,t-1}). \tag{2.1}$$

The definitions of $X^{(CL)}_{m,t}$ and $X^{(CH)}_{m,t}$ are similar to the classic daily log returns, $X_{m,t} = \log(P^C_{m,t}/P^C_{m,t-1})$ discussed widely in the literature of finance and statistics. $X^{(CL)}_{m,t}$ and $X^{(CH)}_{m,t}$ are capable of depicting realistic investment characteristics. Suppose that an investor buys a given stock on the previous day with closing price $P^C_{m,t-1}$ and sells it on day t. Then, the investor's return belongs to the interval $[X^{(CL)}_{m,t}, X^{(CH)}_{m,t}]$ depending on when he/she sells the stock during day t. According to the definitions of $X^{(CL)}_{m,t}$ and $X^{(CH)}_{m,t}$ in (2.1), we have the following inequality

$$X^{(CL)}_{m,t} \leq_{st} X^{(CH)}_{m,t} \tag{2.2}$$

since $P^L_{m,t} \leq_{st} P^H_{m,t}$, for all $t = 0, 1, \ldots$, and $m = 1, \ldots, p$, where the notation $A \leq_{st} B$ means that random variable A is stochastically less than or equal to random variable B. Hence, $\mathbf{X}^{(CI)}_{m,t} = [X^{(CL)}_{m,t}, X^{(CH)}_{m,t}]$, $t = 1, 2, \ldots$, also form an ITS, and the prediction interval of $[P^L_{m,t+1}, P^H_{m,t+1}]$ can be obtained. For example, let $[\widehat{P}^L_{m,t+1}, \widehat{P}^H_{m,t+1}]$ denote the prediction of $[P^L_{m,t+1}, P^H_{m,t+1}]$ conditional on $\mathcal{F}_t$. By using (2.1), our proposed scheme is to model the interval observations, $[X^{(CL)}_{m,s}, X^{(CH)}_{m,s}]$, $s = 1, \ldots, t$, and then estimate $[\widehat{P}^L_{m,t+1}, \widehat{P}^H_{m,t+1}]$ by

$$[P^C_{m,t} \exp\{\widehat{X}^{(CL)}_{m,t+1}\}, P^C_{m,t} \exp\{\widehat{X}^{(CH)}_{m,t+1}\}],$$

where $\widehat{X}^{(CL)}_{m,t+1}$ and $\widehat{X}^{(CH)}_{m,t+1}$ are the predictions of $X^{(CL)}_{m,t+1}$ and $X^{(CH)}_{m,t+1}$, respectively, which can be obtained from the time series models defined below. Traditionally, ITS data are formed by only the high and low prices (Arroyo et al., 2010, 2011; Maia et al., 2008). This study includes the closing prices in the model and investigates whether this additional information can improve the interval prediction.

To jointly model $X^{(h)}_{m,t}$, $h = CL, CH$, we need to capture the features inherent in the data. For example, $X^{(h)}_{m,t}$, $h = CL, CH$ could be conditionally heteroscedastic and auto- and cross-correlated. To characterize these features, a two-stage procedure is proposed to model the dynamics of $X^{(h)}_{m,t}$, $h = CL, CH$. The first stage is to adjust the conditional heteroscedasticity of $X^{(h)}_{m,t}$ marginally for $h = CL, CH$. The second stage is to simultaneously model the auto- and cross-correlation of the adjusted time series.

In the first stage, we propose to de-GARCH $X_{m,t}^{(h)}$ to obtain volatility-adjusted returns. De-GARCHing is a widely used technique for modeling multivariate time series. For example, Engle (2002, 2009) proposed a dynamic conditional correlation (DCC) model to capture time-varying correlations. The first step of their scheme is to de-GARCH the data. Härdle et al. (2015) also used de-GARCHing with a GARCH(1,1) model to analyze the multi-dimensional dependencies of time series data with a hidden Markov model for hierarchical Archimedean copulae. Grigoryeva et al. (2017) proposed a method based on various state space models to extract global stochastic (GST) financial trends from non-synchronous financial data. They mentioned that de-GARCHing is commonly used for GST. In this study, we propose to fit $X_{m,t}^{(h)}$ with a univariate ARMA-GARCH model and let

$$\tilde{X}_{m,t}^{(h)} = (X_{m,t}^{(h)} - \mu_m^{(h)})/\sigma_{m,t}^{(h)} \tag{2.3}$$

be the de-GARCHed process of $X_{m,t}^{(h)}$, $h = CL, CH$, where $\mu_m^{(h)}$ is the stationary (unconditional) mean of $X_{m,t}^{(h)}$ and $\sigma_{m,t}^{(h)}$ is the conditional standard deviation of $X_{m,t}^{(h)}$, which is estimated from the univariate GARCH-type model

$$\sigma_{m,t}^{(h)} = g_{m,t-1}^{(h)}(X_{m,s}^{(h)}, \sigma_{m,s}^{(h)}, s < t), \tag{2.4}$$

which is $\mathcal{F}_{t-1}$-measurable. This type of model (2.4) is capable of describing many features of financial data, for example, conditional heteroscedasticity, volatility clustering and asymmetry. It also includes various univariate financial time series models that are widely used by practitioners in economics, statistics and finance (see Engle, 1982; Bollerslev, 1986; Nelson, 1990; Tsay, 2010 and the references therein). In particular, we employ the stationary mean (not the conditional mean) to define the proposed de-GARCHed process in (2.3). The main reason for this design is to retain the autocorrelation in $\tilde{X}_{m,t}^{(h)}$, $h = CL, CH$ and to model the auto- and cross-correlation of $\tilde{X}_{m,t}^{(h)}$, $h = CL, CH$ simultaneously in the second stage of the proposed procedure.

In the second stage, we employ the following vector autoregressive-moving-average model of orders p and q, denoted by VARMA(p,q), to depict the dynamics of the two de-GARCHed processes, $\{\tilde{X}_{m,t}^{(h)}, t = 1, \ldots, T\}$, $h = CL, CH$,

$$\begin{aligned} \begin{pmatrix} \tilde{X}_{m,t}^{(CL)} \\ \tilde{X}_{m,t}^{(CH)} \end{pmatrix} &= \sum_{i=1}^{p} \begin{pmatrix} \phi_{m,i}^{LL} & \phi_{m,i}^{LH} \\ \phi_{m,i}^{HL} & \phi_{m,i}^{HH} \end{pmatrix} \begin{pmatrix} \tilde{X}_{m,t-i}^{(CL)} \\ \tilde{X}_{m,t-i}^{(CH)} \end{pmatrix} \\ &+ \begin{pmatrix} \varepsilon_{m,t}^{(CL)} \\ \varepsilon_{m,t}^{(CH)} \end{pmatrix} + \sum_{j=1}^{q} \begin{pmatrix} \theta_{m,j}^{LL} & \theta_{m,j}^{LH} \\ \theta_{m,j}^{HL} & \theta_{m,j}^{HH} \end{pmatrix} \begin{pmatrix} \varepsilon_{m,t-j}^{(CL)} \\ \varepsilon_{m,t-j}^{(CH)} \end{pmatrix}, \end{aligned} \tag{2.5}$$

for $m = 1, \ldots, p$, where $(\varepsilon_{m,t}^{(CL)}, \varepsilon_{m,t}^{(CH)})^\top$, $t = 1, \ldots, T$, are uncorrelated random vectors of a bivariate normal distribution with mean zero and covariance matrix Σ. In addition, $(\varepsilon_{m,t}^{(CL)}, \varepsilon_{m,t}^{(CH)})^\top$, $t = 1, \ldots, T$, are assumed to be independent of $(\tilde{X}_{m,s}^{(CL)}, \tilde{X}_{m,s}^{(CH)})^\top$, $s < t$.

Denote the 1-step-ahead predictions of $X^{(h)}_{m,t+1}$ conditional on $\mathcal{F}_t$ by $\widehat{X}^{(h)}_{m,t}(1) = E_t(X^{(h)}_{m,t+1})$, $h = CL, CH$, where $E_t(X)$ denotes the conditional expectation of X given $\mathcal{F}_t$. From (2.3)-(2.5), we have

$$
\begin{aligned}
\widehat{X}^{(CL)}_{m,t}(1) &= E_t(X^{(CL)}_{m,t+1}) = \mu^{(CL)}_m + \sigma^{(CL)}_{m,t+1} E_t(\tilde{X}^{(CL)}_{m,t+1}) \\
&= \mu^{(CL)}_m + \sigma^{(CL)}_{m,t+1} \Big\{ \sum_{i=1}^{p} \Big(\phi^{LL}_{m,i} \tilde{X}^{(CL)}_{m,s-i} + \phi^{LH}_{m,i} \tilde{X}^{(CH)}_{m,s-i} \Big) \\
&\qquad + \sum_{j=1}^{q} \Big(\theta^{LL}_{m,j} \varepsilon^{(CL)}_{m,s-j} + \theta^{LH}_{m,j} \varepsilon^{(CH)}_{m,s-j} \Big) \Big\}.
\end{aligned} \tag{2.6}
$$

and

$$
\begin{aligned}
\widehat{X}^{(CH)}_{m,t}(1) &= E_t(X^{(CH)}_{m,t+1}) = \mu^{(CH)}_m + \sigma^{(CH)}_{m,t+1} E_t(\tilde{X}^{(CH)}_{m,t+1}) \\
&= \mu^{(CH)}_m + \sigma^{(CH)}_{m,t+1} \Big\{ \sum_{i=1}^{p} \Big(\phi^{HL}_{m,i} \tilde{X}^{(CL)}_{m,s-i} + \phi^{HH}_{m,i} \tilde{X}^{(CH)}_{m,s-i} \Big) \\
&\qquad + \sum_{j=1}^{q} \Big(\theta^{HL}_{m,j} \varepsilon^{(CL)}_{m,s-j} + \theta^{HH}_{m,j} \varepsilon^{(CH)}_{m,s-j} \Big) \Big\}
\end{aligned} \tag{2.7}
$$

To guarantee the mathematical coherence $\widehat{X}^{(CL)}_{m,t+1} \leq_{st} \widehat{X}^{(CH)}_{m,t+1}$ in their predictions, let

$$\widehat{X}^{(CL)}_{m,t+1} = \min\{\widehat{X}^{(CL)}_{m,t}(1), \widehat{X}^{(CH)}_{m,t}(1)\}$$

and

$$\widehat{X}^{(CH)}_{m,t+1} = \max\{\widehat{X}^{(CL)}_{m,t}(1), \widehat{X}^{(CH)}_{m,t}(1)\},$$

and $[\widehat{X}^{(CL)}_{m,t+1}, \widehat{X}^{(CH)}_{m,t+1}]$ forms a prediction interval of X_{t+1} conditional on $\mathcal{F}_t$. In our empirical study, there are 250(days) $\times$ 30(companies) $\times$ 2(time periods) $= 15,000$ prediction intervals, and the situation of $\widehat{X}^{(CL)}_{m,t}(1) > \widehat{X}^{(CH)}_{m,t}(1)$ occurs only 8 times. The numerical results indicate that the proposed scheme is capable of guaranteeing $\widehat{X}^{(CL)}_{m,t+1} \leq_{st} \widehat{X}^{(CH)}_{m,t+1}$ in most cases.

3. Application of the ITS prediction to portfolio selection

In this section, we propose an innovative portfolio selection scheme on the basis of the ITS prediction with models (2.6) and (2.7). The literature contains many different models from models (2.6) and (2.7) for analyzing ITS. Nevertheless, the proposed portfolio selection scheme is not restricted to our considered model.

The classic portfolio optimization problem is represented as follows:

$$\max_{\mathbf{c}_t} E_t\Big(\sum_{m=1}^{p} c_{m,t} X_{m,t+1} \Big) \quad \text{subject to} \quad \mathbf{c}_t \geq 0, \ \sum_{m=1}^{p} c_{m,t} \leq 1 \text{ and } \rho_t \leq L, \tag{3.1}$$

where $\mathbf{c}_t = (c_{1,t}, \ldots, c_{p,t})^\top$, $c_{m,t}$ denotes the holding position of $X_{m,t}$ at time t, $\mathbf{c}_t \geq 0$ is the no short-selling constraint, $\sum_{m=1}^{p} c_{m,t} \leq 1$ is the budget constraint, ρ_t is the value of a predetermined risk measure at time t, and L is a pre-specified upper bound of the investment risk. The main objective is to select the holding positions $\mathbf{c}_t$ at time t. In the portfolio selection literature, when $X_{m,t}$, $t = 1, 2, \ldots$, are assumed to be i.i.d. for each $m = 1, \ldots, p$, Markowitz (1952, 1959) used the standard deviation of a portfolio, Rockafellar and Uryasev (2000, 2002) and Krokhmal et al. (2002) employed the CVaR, and Adam et al. (2008) considered the SRM as the risk measure to determine $\mathbf{c}_t$. Recently, Harris and Mazibas (2013) and Huang et al. (2017) further considered fitting time series models for the underlying asset returns, $X_{m,t}$, $m = 1, \ldots, p$, $t = 1, 2, \ldots$, with the CVaR and SRM to solve (3.1).

In this study, we determine the allocations of the underlying assets with the following criterion:

$$
\begin{aligned}
&\max_{\mathbf{c}_t} E_t\Big(\sum_{m=1}^{p} c_{m,t} X_{m,t+1}^{(CH)}\Big) \\
&\text{subject to} \quad \mathbf{c}_t \geq 0, \quad \sum_{m=1}^{p} c_{m,t} \leq 1 \\
&\text{and} \quad -\sum_{m=1}^{p} c_{m,t} E_t\Big(X_{m,t+1}^{(CL)} \;\Big|\; X_{m,t+1}^{(CL)} \leq q_{\alpha,m,t+1}\Big) \leq L,
\end{aligned}
\tag{3.2}
$$

where $X_{m,t+1}^{(h)} = \mu_m^{(h)} + \sigma_{m,t+1}^{(h)} \tilde{X}_{m,t+1}^{(h)}$, $h = CH, CL$, follows models (2.3)-(2.5), and $q_{\alpha,m,t+1}$ is the αth quantile of $X_{m,t+1}^{(CL)}$ conditional on $\mathcal{F}_t$. In practice, since the expected values of daily stock returns are usually very close to 0, one can select a sufficiently small α such that $q_{\alpha,m,t+1} < 0$. The main concept behind (3.2) is to maximize the potential high portfolio returns subject to a predetermined limitation, L, on the corresponding potential low and nonpositive returns. In contrast to (3.1), we use $E_t(\sum_{m=1}^{p} c_{m,t} X_{m,t+1}^{(CH)})$ to replace $E_t(\sum_{m=1}^{p} c_{m,t} X_{m,t+1})$ and use

$$
-\sum_{m=1}^{p} c_{m,t} E_t\Big(X_{m,t+1}^{(CL)} \;\Big|\; X_{m,t+1}^{(CL)} \leq q_{\alpha,m,t+1}\Big)
\tag{3.3}
$$

as the risk measure ρ_t in (3.1). In addition, the values of $E_t(X_{m,t+1}^{(CH)})$ and $E_t(X_{m,t+1}^{(CL)})$, $m = 1, \ldots, p$, are estimated by the models defined in (2.5). Moreover, the optimal allocations $c_{m,t}$, $m = 1, \ldots, p$, are in linear forms in the objective function and constraints in (3.2). Consequently, the optimal allocations in (3.2) can be obtained by linear programming, which is a popular technique for various portfolio selection criteria (Markowitz, 1952, 1959; Rockafellar and Uryasev, 2000, 2002; Adam et al., 2008; Huang, et al., 2017).

In the following, we introduce the concept of a coherent risk measure for the intervals of returns, which provides economic and financial reasons to use

(3.3) as a risk constraint in (3.2). In financial risk management, Artzner et al. (1999) introduced the following concept of the coherent risk measure for classic portfolio selection. Let $\mathcal{G}$ be the set of random portfolio returns, ρ be a risk measure, which is a mapping from $\mathcal{G}$ into $\mathbb{R}$, and X denote the return of an asset. A risk measure is called coherent if it satisfies the following properties:

(A1) Translation invariance: If A is a deterministic portfolio with guaranteed return α, then for all $X \in \mathcal{G}$, we have $\rho(X + A) = \rho(X) - \alpha$.

(A2) Subadditivity: For all X and $Y \in \mathcal{G}$, $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

(A3) Positive homogeneity: For all $\lambda \geq 0$ and all $X \in \mathcal{G}$, $\rho(\lambda X) = \lambda\rho(X)$.

(A4) Monotonicity: For all X and $Y \in \mathcal{G}$ with $X \leq Y$, we have $\rho(Y) \leq \rho(X)$.

The economic explanations of these four properties are as follows. Translation invariance implies that the addition of a definite amount of capital reduces the risk by the same amount. Subadditivity implies that diversification is beneficial. Positive homogeneity implies that the risk of a position is proportional to its size. Monotonicity implies that a portfolio with greater future returns has less risk.

In this study, we consider an interval of returns denoted by $\mathbf{X}^I = [X^L, X^H]$, where X^L and X^H are the low and high returns of an asset, respectively. To extend the concepts of (A1)-(A4) from random variables to random intervals, we propose the following properties for a risk measure of the interval of returns. Let $\mathcal{G}_1$ be the set of random intervals of portfolio returns and $\rho_I : \mathcal{G}_1 \to \mathbb{R}$ be a corresponding risk measure.

(A1') Translation invariance for the interval of returns: If A is a deterministic portfolio with guaranteed return α, then for all $\mathbf{X}^I \in \mathcal{G}_1$, we have $\rho_I(\mathbf{X}^I + A) = \rho_I(\mathbf{X}^I) - \alpha$, where we use $\mathbf{X}^I + A$ to denote $[X^L + A, X^H + A]$.

(A2') Subadditivity for the interval of returns: For all $\mathbf{X}^I$ and $\mathbf{Y}^I \in \mathcal{G}_1$, $\rho_I(\mathbf{X}^I + \mathbf{Y}^I) \leq \rho_I(\mathbf{X}^I) + \rho_I(\mathbf{Y}^I)$, where $\mathbf{X}^I + \mathbf{Y}^I = [X^L + Y^L, X^H + Y^H]$. In addition, one can also use the Cartesian join of $\mathbf{X}^I$ and $\mathbf{Y}^I$, denoted by $\mathbf{X}^I \oplus \mathbf{Y}^I = [\min(X^L, Y^L), \max(X^H, Y^H)]$, to define the subadditivity, that is, $\rho_I(\mathbf{X}^I \oplus \mathbf{Y}^I) \leq \rho_I(\mathbf{X}^I) + \rho_I(\mathbf{Y}^I)$.

(A3') Positive homogeneity for the interval of returns: For all $\lambda \geq 0$ and all $\mathbf{X}^I \in \mathcal{G}_1$, $\rho_I(\lambda\mathbf{X}^I) = \lambda\rho_I(\mathbf{X}^I)$.

(A4') Monotonicity for the interval of returns: For all $\mathbf{X}^I$ and $\mathbf{Y}^I \in \mathcal{G}_1$ with $\mathbf{X}^I \leq \mathbf{Y}^I$, where $\mathbf{X}^I \leq \mathbf{Y}^I$ if and only if $X^L \leq Y^L$ and $X^H \leq Y^H$, we have $\rho_I(\mathbf{Y}^I) \leq \rho_I(\mathbf{X}^I)$.

The economic explanations of (A1')-(A4') are similar to those of (A1)-(A4). Specifically, the monotonicity for the interval of returns (A4') implies only that a portfolio with greater future interval of returns has less risk. For the case of

$\mathbf{X}^I \subset \mathbf{Y}^I$, the relationship between $\rho_I(\mathbf{Y}^I)$ and $\rho_I(\mathbf{X}^I)$ is not clear. If a risk measure for the interval of returns satisfies (A1')-(A4'), we call it a coherent risk measure for the interval of returns. In the following proposition, a coherent risk measure for the interval of returns is proposed, and the proof is given in the Appendix.

Proposition 3.1. *Let $\mathbf{X}^I = [X^L, X^H]$ be an interval of returns, and let*

$$\rho_I(\mathbf{X}^I) = -E(X^{(L)} \mid X^{(L)} \le q_\alpha),$$

where q_α is the αth quantile of $X^{(L)}$. Then, $\rho_I(\cdot)$ is a coherent risk measure for the interval of returns.

By Proposition 3.1, the measurement defined in (3.3) can be rewritten as

$$\sum_{m=1}^{p} c_{m,t}\ \rho_I(\mathbf{X}_{m,t+1}^{(CI)} \mid \mathcal{F}_t),$$

which is a linear combination of coherent risk measures for the interval of returns, where

$$\rho_I(\mathbf{X}_{m,t+1}^{(CI)} \mid \mathcal{F}_t) = -E_t(X_{m,t+1}^{(CL)} \mid X_{m,t+1}^{(CL)} \le q_{\alpha,m,t+1}). \tag{3.4}$$

Due to the convexity of the coherent risk measure, we have

$$\rho_I\Big(\sum_{m=1}^{p} c_{m,t}\mathbf{X}_{m,t+1}^{(CI)} \mid \mathcal{F}_t\Big) \le \sum_{m=1}^{p} c_{m,t}\rho_I(\mathbf{X}_{m,t+1}^{(CI)} \mid \mathcal{F}_t). \tag{3.5}$$

For a portfolio with allocations $c_{m,t}$, $m = 1,\dots,p$, set up at time t, the left side of (3.5) represents the risk of the worst case occurring at time $t+1$ since each underlying return reaches the bottom of the corresponding prediction interval. However, if a limitation is set on $\rho_I(\sum_{m=1}^{p} c_{m,t}\mathbf{X}_{m,t+1}^{(CI)} \mid \mathcal{F}_t)$ in the portfolio selection criterion (3.2), the optimal allocations $c_{m,t}$, $m = 1,\dots,p$, are difficult to obtain directly using linear programming since $\rho_I(\cdot|\mathcal{F}_t)$ is a nonlinear function of $c_{m,t}$, $m = 1,\dots,p$. A similar situation is encountered in the classic portfolio selection problem shown in (3.1) when using the expected shortfall as the risk measure. Rockafellar and Uryasev (2000, 2002) proposed a method to overcome this difficulty by considering more latent variables, but the computational cost also increased. Therefore, we set a limitation on the right side of (3.5), and the optimal allocations can be obtained directly using linear programming.

In the following sections, we consider several scenarios to investigate the coverage, efficiency and accuracy of the proposed interval estimation and the performance of the proposed criterion for portfolio selection.

4. Evaluation of the proposed interval estimation method

Let $Y_t = [P_t^L, P_t^H]$ denote the realized ITS of the stock prices and $\hat{Y}_t$ be an estimation of Y_t, $t = 1,\dots,T$. In this section, we use the four measures to evaluate the performance of the proposed interval estimation (He and Hu, 2009; Rodrigues and Salish, 2015; Xiong et al., 2015). The first measure is the coverage rate

$$R_C = \frac{1}{T}\sum_{t=1}^{T}\frac{w(Y_t \cap \hat{Y}_t)}{w(Y_t)},$$

where $w(\cdot)$ denotes the width of the interval, R_C indicates what part of the realized ITS of the stock prices is covered by its forecast.

The second measure is the efficiency rate

$$R_E = \frac{1}{T}\sum_{t=1}^{T}\frac{w(Y_t \cap \hat{Y}_t)}{w(\hat{Y}_t)},$$

which provides information about what part of the forecast covers the realized ITS. It should be noted that R_C and R_E must be considered simultaneously; otherwise, incorrect conclusions may be drawn. For example, if Y_t is a subinterval of $\hat{Y}_t$, then R_C will be 1, but R_E might be much less than 1, which indicates that the predicted interval is much wider than the realized ITS. Therefore, we only conclude that the forecast is satisfactory when R_C and R_E are reasonably high and the difference between them is small.

The third measure is the accuracy ratio

$$R_A = \frac{1}{T}\sum_{t=1}^{T}\frac{w(Y_t \cap \hat{Y}_t)}{w(Y_t \cup \hat{Y}_t)}.$$

A prediction with a larger R_A performs better than a prediction with a smaller one.

The fourth measure is the U_I criterion

$$U_I = \sqrt{\frac{\sum_{t=1}^{T}(P_t^H - \hat{P}_t^H)^2 + \sum_{t=1}^{T}(P_t^L - \hat{P}_t^L)^2}{\sum_{t=1}^{T}(P_t^H - P_{t-1}^H)^2 + \sum_{t=1}^{T}(P_t^L - P_{t-1}^L)^2}},$$

which is derived from Theil's U statistic and compares the performance of an estimated method with a naïve estimate $[P_{t-1}^L, P_{t-1}^H]$ of $[P_t^L, P_t^H]$. The U_I statistic is less than one if the predictor performs better than the naïve predictor.

In addition to the proposed interval estimation $\hat{Y}_t^{(p)}$, three commonly used interval predictors are considered in our comparison studies. One is fitting time series models to the log return process $X_t = \log(P_t^C/P_{t-1}^C)$ and then deriving the corresponding 95% confidence interval of P_{t+1}^C. We denote this estimation of Y_t by $\hat{Y}_t^{(1)}$.

The second estimation of Y_t is the popular center-range prediction interval, which is obtained by separately fitting time series models to the processes of the center, $P_t^M = (P_t^H + P_t^L)/2$, and the range, $P_t^R = (P_t^H - P_t^L)/2$, of the price intervals and then deriving an interval estimation of $[P_{t+1}^L, P_{t+1}^H]$ conditional on $\mathcal{F}_t$. We denote the second estimation by $\hat{Y}_t^{(2)}$.

The third alternative estimation of Y_t is derived from a linear interval-data model motivated from Fischer et al. (2016). The center-range-representation of interval data can also be expressed as the following regression model

$$Y_t = \boldsymbol{\beta_0} + \beta_1 Y_{t-1}^C + \beta_2 Y_{t-1}^R + \delta_t, \tag{4.1}$$

where $Y_t^C = [P_t^M, P_t^M]$, $Y_t^R = [-P_t^R, P_t^R]$, δ_t is an interval-valued random error, and $\boldsymbol{\beta_0} = [a, a]$ and (a, β_1, β_2) are unknown parameters. Blanco-Fernández et al. (2011) derived the estimation procedures for (4.1), and the obtained predictor is denoted as $\hat{Y}_t^{(3)}$.

We conduct the comparison study using the stock prices of the 30 companies of the DJIA Index during the financial crisis period (from July 2, 2007 to June 24, 2009) and under improved market conditions (from July 1, 2014 to June 23, 2016). The 1-step-ahead prediction intervals during the two time periods (from June 27, 2008 to June 24, 2009 and from June 29, 2015 to June 23, 2016) are obtained with the previous 250 daily historical high and low returns. We adopt an ARMA(p,q)-GARCH(p_0,q_0) model, where p, $q \in \{0, 1, 2, 3, 4, 5\}$ and p_0, $q_0 \in \{0, 1\}$, to obtain the de-GARCHed process defined in (2.3) for $h = CL$ and CH, separately. The multivariate portmanteau test (Tsay, 2010, Chapter 8) is used for testing the auto- and cross-correlation in $\{(\tilde{X}_{m,t}^{(CL)}, \tilde{X}_{m,t}^{(CH)}),\ t = 1, \ldots, T\}$. If the de-GARCHed processes have significant auto- and cross-correlation, we model the vector time series $(\tilde{X}_{m,s}^{(CL)},\ \tilde{X}_{m,t}^{(CH)})^\top$ with VARMA(p_1, q_1) defined in (2.5), where (p_1, q_1) are selected from $\{(1, 0), (0, 1),$ and $(u, v),\ u, v = 1, 2, 3\}$ based on the Bayesian information criterion (BIC). Table 1 summarizes the p-values of the multivariate portmanteau test for the de-GARCHed processes and the residual processes $\{(\varepsilon_{m,t}^{(CL)}, \varepsilon_{m,t}^{(CH)}),\ t = 1, \ldots, T\}$. In Table 1, all the de-GARCHed processes have significant auto- and cross-correlation during 2008-2009, and most (around 96.2%) of the de-GARCHed processes have significant auto- and cross-correlation during 2015-2016. More than 99.4% of the p-values of the fitted residual processes during the two time periods are greater than 0.01, which indicates that the above scheme is capable of removing most of the auto- and cross-correlation of the de-GARCHed processes. Figure 1 summarizes the proportions of selected orders (p_1, q_1) in the two time periods, where the 3.8% de-GARCHed processes without significant auto- and cross-correlation during 2015-2016 are denoted by VARMA(0,0). VARMA(1,1) is the most commonly selected model during the financial crisis period, whereas VARMA(1,0) and VARMA(1,1) are frequently selected under improved market conditions.

Table 2 presents the average values of R_C, R_E, R_A and U_I of $\hat{Y}_t^{(p)}$ and $\hat{Y}_t^{(k)}$, $k = 1, 2, 3$, in the top panel. In the bottom panel, we present the improvement of $\hat{Y}_t^{(p)}$ for each $\hat{Y}_t^{(i)}$, $i = 1, 2, 3$, by calculating $(\hat{Y}_t^{(p)} - \hat{Y}_t^{(i)})/\hat{Y}_t^{(i)}$ for R_C, R_E, and R_A and $(\hat{Y}_t^{(i)} - \hat{Y}_t^{(p)})/\hat{Y}_t^{(i)}$ for U_I. The numerical results indicate that $\hat{Y}_t^{(p)}$ performs better than $\hat{Y}_t^{(i)}$, $i = 1, 2, 3$, in terms of R_E, R_A and U_I. Although

$\hat{Y}_t^{(2)}$ has a larger R_C than $\hat{Y}_t^{(p)}$, the improvement of $\hat{Y}_t^{(p)}$ in R_E is much greater than the loss of $\hat{Y}_t^{(p)}$ in R_C. In particular, the popular center-range prediction interval $\hat{Y}_t^{(2)}$ has U_I greater than 1, which indicates that the proposed prediction interval $\hat{Y}_t^{(p)}$ is more reliable than $\hat{Y}_t^{(2)}$. By contrast, $\hat{Y}_t^{(p)}$ outperforms $\hat{Y}_t^{(1)}$ and $\hat{Y}_t^{(3)}$, especially in 2015-2016, with an improvement in the 4 measures of at least 9.0%. Figure 2 presents the average values of R_C, R_E, R_A and U_I for the four prediction intervals for the 30 companies of the DJIA Index from June 27, 2008 to June 24, 2009. The results of the time period from June 29, 2015 to June 23, 2016 are given in Figure 3. These figures reveal similar findings as those in Table 2. The proposed prediction interval $\hat{Y}_t^{(p)}$ has the best performance with respect to R_A and U_I and performs robustly in R_C and R_E, especially in 2015-2016. The main reason for the good performance of $\hat{Y}_t^{(p)}$ is that $\hat{Y}_t^{(p)}$ uses more information than the other predictors. All the other predictors involve (traditional) ITS, that is, they are formed exclusively with the high and low returns and do not consider past closing prices. Therefore, the predictors $\hat{Y}_t^{(k)}$, $k = 1, 2, 3$, have a clear disadvantage relative to $\hat{Y}_t^{(p)}$; consequently, the latter should show much better performance.

5. Empirical study

In this section, an empirical study is designed to investigate the performance of the proposed criterion for selecting the optimal portfolio using the stock prices of the companies of the DJIA Index. The DJIA Index was launched on October 29, 2002. This Index covers the top 30 companies by total market capitalization and is reviewed quarterly in January, April, July and October every year. Suppose that a self-financing trading strategy, which daily reallocates the holding weights of the portfolio, is employed from the beginning of each period. The proposed criterion is used to reallocate the optimal portfolios daily during the financial crisis period and under improved market conditions by fitting the time series models defined in (2.5) with the previous 250 daily historical high and low returns for each underlying asset. Then, the corresponding 250 one-day-ahead returns of the optimal portfolios are computed and compared with the DJIA Index. In the following, we illustrate the details of the construction of the self-financing trading strategy during the financial crisis period:

1. Let DJ_t be the value of the DJIA Index at time t, where $t = 0$ stands for the date of June 27, 2008.

2. Let V_t denote the value of the self-financing portfolio at time t. Further, let V_0 be the value of the DJIA Index on June 27, 2008. The initial allocations of the underling assets, $c_{m,0}$, are obtained by solving (3.2), where the high and low return processes of each underlying asset are fitted by model (2.5) based on $X_{m,t}^{CH}$ and $X_{m,t}^{CL}$ for $t = -250, \ldots, -1$, $m = 1, \ldots, p$ and $p = 30$.

Moreover, since $\sum_{m=1}^{p} c_{m,0}$ can be less than 1, the amount $V_0 \sum_{m=1}^{p} c_{m,0}$ is invested in risky assets and the rest of the portfolio value, denoted by $C_0 = V_0(1 - \sum_{m=1}^{p} c_{m,0})$, is invested in the risk-free market.

3. At time $t = 1$, the value of the portfolio is

$$V_{1^-} = b^{(0)} \sum_{m=1}^{p} c_{m,0} P_{m,1}^{C} + e^{r_d} C_0,$$

prior to the adjustment of the holding portfolio, where

$$b^{(0)} = \frac{V_0 \sum_{m=1}^{p} c_{m,0}}{\sum_{m=1}^{p} c_{m,0} P_{m,0}^{C}}$$

and r_d is the daily risk-free interest rate. We reestimate the dynamic models of each return process using the data $P_{m,t}$, $t = -249, \ldots, 0$, and compute the updated optimal allocations, which are proportional to $c_{m,1}$ obtained by solving (3.2), where the value of the updated portfolio, denoted by V_1, is the same as V_{1^-} to satisfy self-financing. That is,

$$V_1 \;=\; V_{1^-} \;=\; b^{(1)} \sum_{m=1}^{p} c_{m,1} P_{m,1}^{C} + C_1,$$

where

$$b^{(1)} = \frac{V_{1^-} \sum_{m=1}^{p} c_{m,1}}{\sum_{m=1}^{p} c_{m,1} P_{m,1}^{C}}$$

and $C_1 = V_{1^-}(1 - \sum_{m=1}^{p} c_{m,1})$ denotes the amount invested in the risk-free market after reallocation.

4. Repeat Step 3 until June 24, 2009.

In addition to adjusting the allocations of the above self-financing trading strategy daily, we proposed dynamic adjustment of the risk limitation L in (3.2) by considering

$$L = \frac{k}{p} \sum_{m=1}^{p} \rho_I(\mathbf{X}_{m,t+1}^{(CI)} \mid \mathcal{F}_t) \tag{5.1}$$

at time t, where k is a positive constant and $\rho_I(\mathbf{X}_{m,t+1}^{(CI)} \mid \mathcal{F}_t)$ is defined in (3.4). The L defined in (5.1) is a special case of (3.3) with $c_{m,t} = 1/p$, for $m = 1, \ldots, p$, multiplied by k. In other words, we set the limitation of the investment risk in (3.2) by considering the trading strategy of an equally weighted portfolio. Moreover, conditional on $\mathcal{F}_t$ and by (2.3)-(2.5), $X_{m,t+1}^{(CL)} = \mu_m^{(CL)} + \sigma_{m,t+1}^{(CL)} \tilde{X}_{m,t+1}^{(CL)}$ is normally distributed with conditional mean $\hat{X}_{m,t}^{(CL)}(1)$ defined in (2.6) and conditional standard deviation $\sigma_{m,t+1}^{(CL)}$. Consequently, (3.4) yields

$$\begin{aligned} \rho_I(\mathbf{X}_{m,t+1}^{(CI)} \mid \mathcal{F}_t) &= -E_t(X_{m,t+1}^{(CL)} \mid X_{m,t+1}^{(CL)} \le q_{\alpha,m,t+1}) \\ &= -\hat{X}_{m,t}^{(CL)}(1) + \sigma_{m,t+1}^{(CL)} \phi\Big((q_{\alpha,m,t+1} - \hat{X}_{m,t}^{(CL)}(1))/\sigma_{m,t+1}^{(CL)}\Big)\Big/\alpha, \end{aligned}$$

where $\phi(\cdot)$ is the density function of the standard normal distribution.

The numerical results are presented in Figures 4 and 5 with $\alpha = 0.05$, 0.20 and 0.35 and $r_d = 0$. Figure 6 presents the values of L in 2008-2009 and 2015-2016 with different settings of α and $k = 1$. Figure 6 shows that a portfolio constructed by (3.2) with a large α is more conservative than a portfolio constructed with a small α since the values of L with $\alpha = 0.35$ are smaller than their counterparts. In Figure 4, the solid, dashed and dash-dotted lines denote the ratios of the capitals of the proposed trading strategy with $k = 0.75$, 1 and 1.25, respectively, to the DJIA Index in 2008-2009, and the results for 2015-2016 are presented in Figure 5. For a fixed α, a portfolio with a small k is more conservative than one with a large k. In Figure 4, the proposed portfolio selection criterion (3.2) with L defined in (5.1) suggests a conservative portfolio during the financial crisis in 2008-2009 since the case with $k = 0.75$ performs better than the others for each α. In particular, the portfolio with $(\alpha, k) = (0.35, 0.75)$ has the best performance among all scenarios. For 2015-2016, compared with the portfolios selected in 2008-2009, the results presented in Figure 5 indicate that (3.2) suggests aggressive portfolios, decreasing α from 0.35 to 0.05 or 0.20 with $k = 0.75$ or increasing k from 0.75 to 1.00 with $\alpha = 0.35$. In view of the results in Figures 4 and 5, the proposed portfolio selection criterion (3.2) is capable of adjusting its suggestions according to the economic conditions.

6. Conclusion

In this study, we propose a prediction interval for future stock prices by fitting time series models to the high and low return processes. The proposed interval estimator is shown to have promising coverage, efficiency and accuracy. In particular, the numerical results of the U_I index indicate that the proposed interval estimator reduces the prediction error of the naïve interval predictor more remarkably than three popular interval estimators discussed in the literature. Consequently, an innovative criterion for portfolio selection is proposed on the basis of our interval estimator. The allocations of the underlying assets in the proposed optimal criterion are determined by maximizing the potential high portfolio returns subject to a predetermined limitation on the corresponding potential low and nonpositive returns. An empirical study is conducted to investigate the investment returns of the proposed optimal portfolio. A dynamic self-financing trading strategy is established by investing in the stocks of the 30 companies of the DJIA Index and adjusting the asset allocations by the proposed method daily during the financial crisis period and a period with improved market conditions. The numerical results indicate that the proposed portfolio selection criterion constructed from the prediction intervals is capable of suggesting an optimal portfolio according to the economic conditions.

This study demonstrates that ITS data, including daily closing, high, and low prices, are capable of improving the performance of investment decisions and risk management by means of the proposed scheme. Additionally, better prediction performance is expected if intra-daily ITS data are available. This is an interesting direction for future studies.

Acknowledgements

Huang's research was supported by the grant MOST 104-2118-M-390-003-MY2 from the Ministry of Science and Technology of Taiwan. Hsu's research was supported by the Ministry of Science and Technology of Taiwan, Grant No. MOST 105-2118-M-390-004.

A. Proof

Proof of Proposition 3.1.

Note that for a random variable X^L, $-E(X^L \mid X^L \leq q_\alpha)$ is the so-called expected shortfall, which is a coherent risk measure. Therefore, it is straightforward to obtain that $\rho_I(\mathbf{X}^I) = -E(X^L \mid X^L \leq q_\alpha)$ satisfies (A1'), (A2'), (A3'), and (A4'), where the interval addition in (A2') is defined by the usual way $\mathbf{X}^I + \mathbf{Y}^I = [X^L + Y^L, X^H + Y^H]$. In the following, we prove that $\rho_I(\mathbf{X}^I)$ also satisfies $\rho_I(\mathbf{X}^I \oplus \mathbf{Y}^I) \leq \rho_I(\mathbf{X}^I) + \rho_I(\mathbf{Y}^I)$.

Let $q_{\alpha,0}$, $q_{\alpha,X}$ and $q_{\alpha,Y}$ be the αth quantile of $\min(X^L, Y^L)$, X^L and Y^L, respectively. Apparently, $q_{\alpha,0} \leq \min(q_{\alpha,X}, q_{\alpha,Y})$ for any $\alpha \in (0,1)$. Let α be small enough such that $\max(q_{\alpha,X}, q_{\alpha,Y}) < 0$. Consequently, for all $\mathbf{X}^I$ and $\mathbf{Y}^I \in \mathcal{G}_1$, we have

$$\begin{aligned}
\rho_I(\mathbf{X}^I \oplus \mathbf{Y}^I) &= -E[\min(X^L, Y^L) \mid \min(X^L, Y^L) \leq q_{\alpha,0}] \\
&= -\frac{1}{\alpha} E[\min(X^L, Y^L) I(\min(X^L, Y^L) \leq q_{\alpha,0})] \\
&\leq -\frac{1}{\alpha}\{E[X^L I(X^L \leq q_{\alpha,X})] + E[Y^L I(Y^L \leq q_{\alpha,Y})]\} \\
&= -\{E(X^L \mid X^L \leq q_{\alpha,X}) + E(Y^L \mid Y^L \leq q_{\alpha,Y})\} \\
&= \rho_I(\mathbf{X}^I) + \rho_I(\mathbf{Y}^I),
\end{aligned}$$

where $I(\cdot)$ is an indicator function and the inequality holds by using the facts that

$$\begin{aligned}
&-\min(X^L, Y^L) I(\min(X^L, Y^L) \leq q_{\alpha,0}) \\
&\leq -X^L I(X^L \leq q_{\alpha,X}) - Y^L I(Y^L \leq q_{\alpha,Y}),
\end{aligned}$$

almost surely, for $q_{\alpha,0} \leq \min(q_{\alpha,X}, q_{\alpha,Y}) \leq \max(q_{\alpha,X}, q_{\alpha,Y}) < 0$. Thus, (A2') with the Cartesian join also holds and the proof is complete. $\square$

Table 1: The proportions of the p-values of the multivariate portmanteau test for testing auto- and cross-correlation in the de-GARCHed processes $\{(\tilde{X}_{m,t}^{(CL)}, \tilde{X}_{m,t}^{(CH)}),\ t = 1, \ldots, T\}$ (shown in rows) and the residual processes $\{(\varepsilon_{m,t}^{(CL)}, \varepsilon_{m,t}^{(CH)}),\ t = 1, \ldots, T\}$ (shown in columns).

(a) 2008-2009

de-GARCHed \ residual	p-value< 0.01	p-value≥ 0.01
p-value< 0.01	0.001	0.999
p-value≥ 0.01	0.000	0.000

(b) 2015-2016

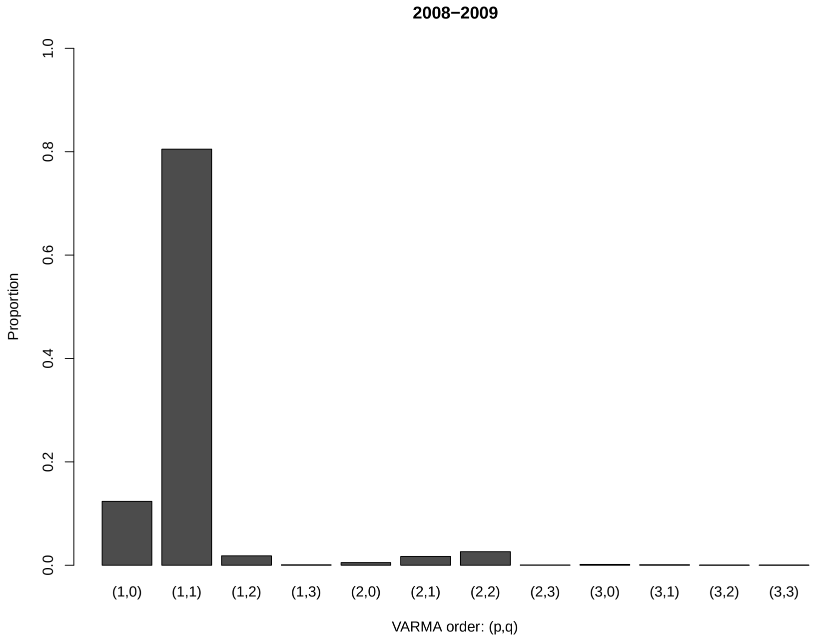
de-GARCHed \ residual	p-value< 0.01	p-value≥ 0.01
p-value< 0.01	0.011	0.951
p-value≥ 0.01	0.000	0.038

Table 2: The average values of R_C, R_E, R_A and U_I of $\hat{Y}_t^{(1)}$, $\hat{Y}_t^{(2)}$, $\hat{Y}_t^{(3)}$ and $\hat{Y}_t^{(p)}$ in June 27, 2008 - June 24, 2009 and June 29, 2015 - June 23, 2016, in the top panel. The bottom panel presents the improvement of $\hat{Y}_t^{(p)}$ for each $\hat{Y}_t^{(i)}$, $i = 1, 2, 3$, by calculating $(\hat{Y}_t^{(p)} - \hat{Y}_t^{(i)})/\hat{Y}_t^{(i)}$ for R_C, R_E, and R_A, and $(\hat{Y}_t^{(i)} - \hat{Y}_t^{(p)})/\hat{Y}_t^{(i)}$ for U_I.

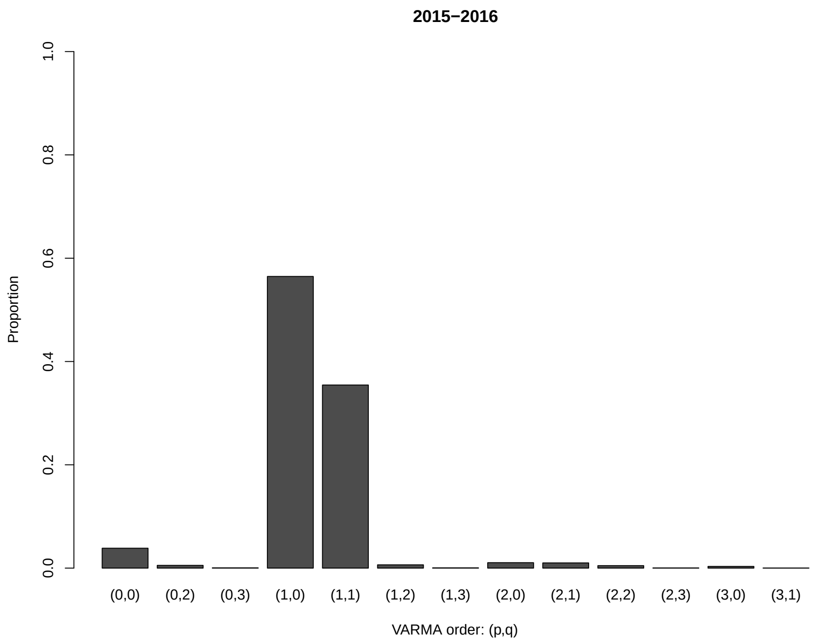
	2008 - 2009				2015 - 2016			
	$\hat{Y}_t^{(1)}$	$\hat{Y}_t^{(2)}$	$\hat{Y}_t^{(3)}$	$\hat{Y}_t^{(p)}$	$\hat{Y}_t^{(1)}$	$\hat{Y}_t^{(2)}$	$\hat{Y}_t^{(3)}$	$\hat{Y}_t^{(p)}$
R_C	0.57	0.96	0.61	0.63	0.53	0.96	0.55	0.60
R_E	0.55	0.34	0.53	0.60	0.51	0.30	0.51	0.56
R_A	0.42	0.33	0.41	0.46	0.39	0.30	0.39	0.44
U_I	0.98	1.70	0.99	0.87	0.99	1.82	0.97	0.88

The improvement of $\hat{Y}_t^{(p)}$ for each $\hat{Y}_t^{(i)}$, $i = 1, 2, 3$.

	2008 - 2009			2015 - 2016		
	$\hat{Y}_t^{(1)}$	$\hat{Y}_t^{(2)}$	$\hat{Y}_t^{(3)}$	$\hat{Y}_t^{(1)}$	$\hat{Y}_t^{(2)}$	$\hat{Y}_t^{(3)}$
R_C	10.4%	-35.0%	1.7%	14.6%	-37.3%	9.4%
R_E	8.5%	75.5%	13.0%	9.0%	85.6%	10.5%
R_A	9.8%	38.5%	12.3%	12.4%	47.3%	10.9%
U_I	9.8%	47.9%	10.99%	10.5%	51.4%	9.2%



(a) 2008-2009



(b) 2015-2016

Figure 1: Summaries of the selected orders of VARMA for 15,000 prediction intervals in the two time periods (June 27, 2008 to June 24, 2009 and June 29, 2015 to June 23, 2016).

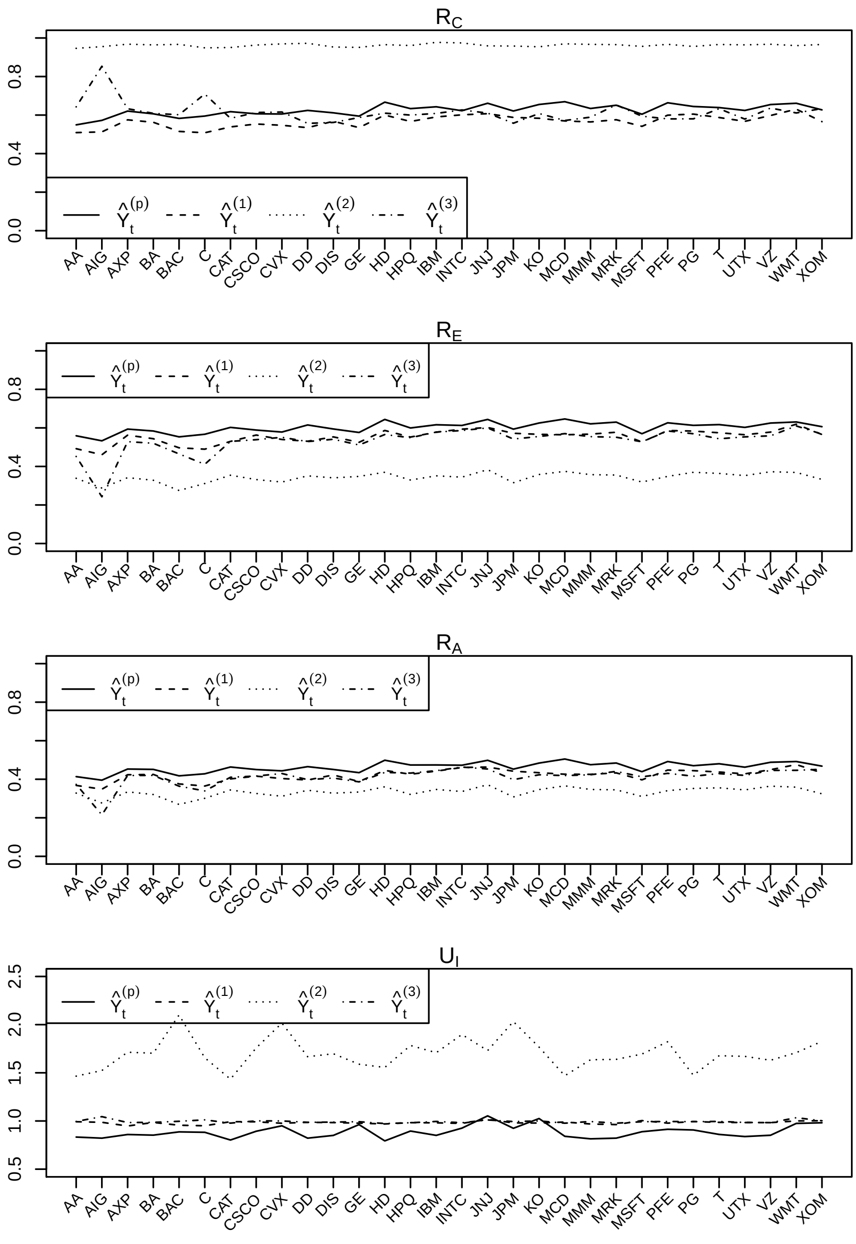


Figure 2: The average values of R_C, R_E, R_A and U_I of $\hat{Y}_t^{(p)}$ (solid line), $\hat{Y}_t^{(1)}$ (dashed line), $\hat{Y}_t^{(2)}$ (dotted line) and $\hat{Y}_t^{(3)}$ (dash-dotted line) for 30 different time series from June 27, 2008 to June 24, 2009.

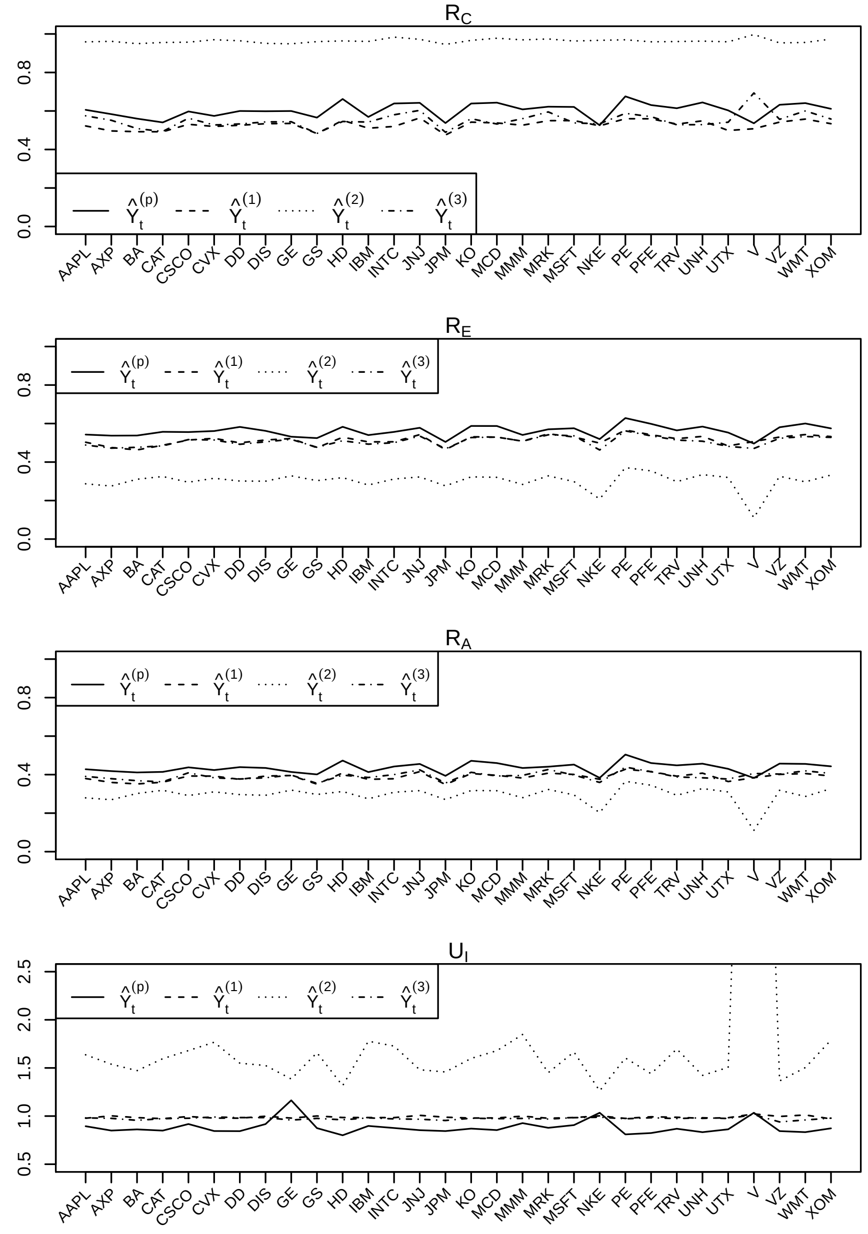
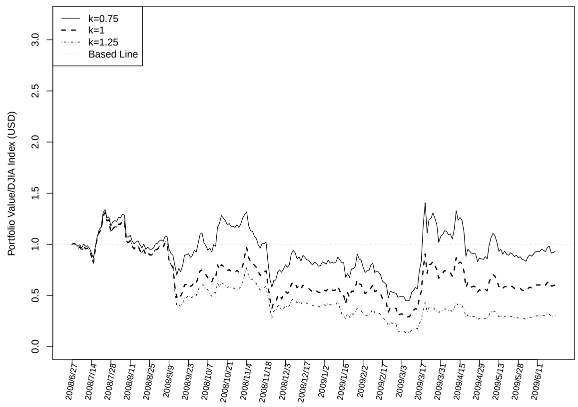
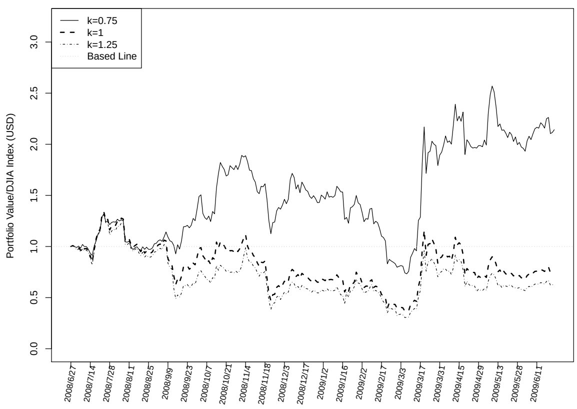


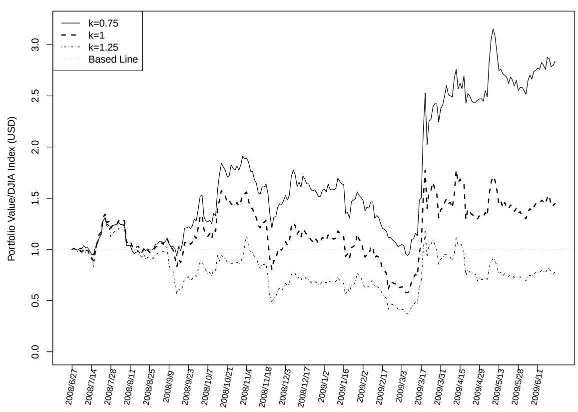
Figure 3: The average values of R_C, R_E, R_A and U_I of $\hat{Y}_t^{(p)}$ (solid line), $\hat{Y}_t^{(1)}$ (dashed line), $\hat{Y}_t^{(2)}$ (dotted line) and $\hat{Y}_t^{(3)}$ (dash-dotted line) for 30 different time series from June 29, 2015 to June 23, 2016.



(a) $\alpha = 0.05$

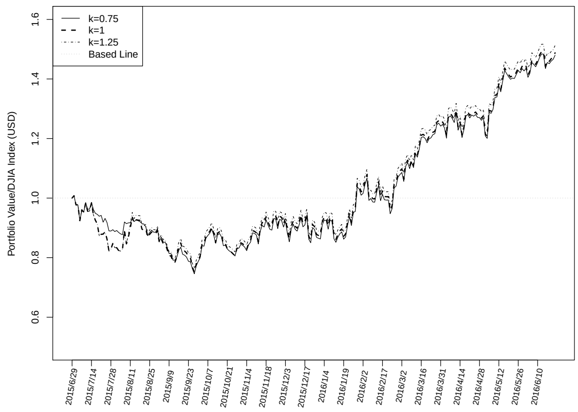


(b) $\alpha = 0.20$

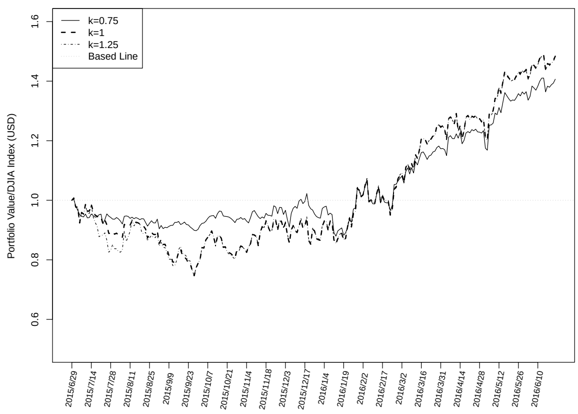


(c) $\alpha = 0.35$

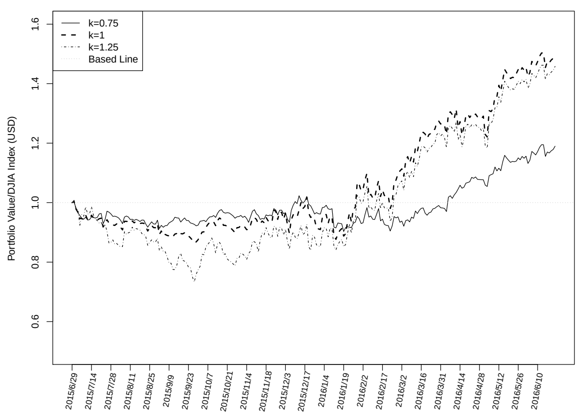
Figure 4: The ratios of the capitals of different trading strategies to the Dow Jones Industrial Average Index in 2008-2009.



(a) $\alpha = 0.05$

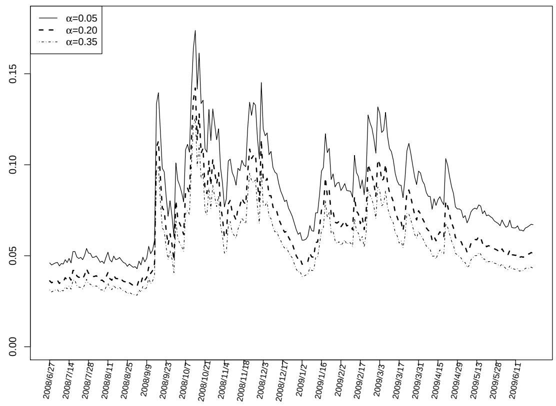


(b) $\alpha = 0.20$

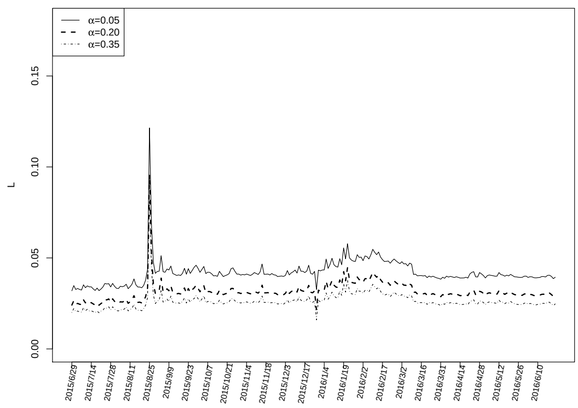


(c) $\alpha = 0.35$

Figure 5: The ratios of the capitals of different trading strategies to the Dow Jones Industrial Average Index in 2015-2016.



(a) L in 2008-2009



(b) L in 2015-2016

Figure 6: The values of L in 2008-2009 and 2015-2016.

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