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Orderings and ageing of reliability systems with dependent components under Archimedian copulas

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Abstract:

- In this paper, we have considered systems with dependent components having a joint distribution modeled by an Archimedean copula and with component lifetimes following accelerated failure time and modified proportional hazards distributions. We have then established characterization results specifically for series, fail-safe, 2-out-of-n and parallel systems through comparisons with average systems in terms of mean residual life, hazard rate and reversed hazard rate orders. We have also discussed various stochastic orderings and ageing results for the residual lives of parallel and series systems. The results established here are quite general, and several examples have been used to illustrate all the results and their reliability implications.

Key-Words:

- *Stochastic orders; Ageing faster in hazard rate; Ageing faster in reversed hazard rate; Series systems; Parallel systems; Fail-safe systems.*

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1. INTRODUCTION

Many coherent reliability systems, such as series, parallel, fail-safe and r-out-of-n systems, have all become useful and essential reliability structures in practice. For example, in the architecture of network circuits, series circuit configurations are often used to manage voltage drops to add to equal voltage, and for all the components in the circuit to share the same equal current and the resistance to sum to equal total resistance. Similarly, parallel circuit configurations are made use of so that all the components in the circuit can share the same equal voltage, and with branch current adding to equal total current and resistance diminishing to equal total resistance.

A fail-safe system is one that is designed so as to remain safe in the event of a failure; it is not designed to prevent failure, but it is intended to mitigate failure when it does occur. An elevator is a good example of a fail-safe system as it is designed with special brakes that are held back by the tension of the cable, so that if the cable does snap, the loss of tension would force the special brakes to be applied, thus averting an accident. Another recent practical application of fail-safe system (2-out-of-3 system, to be specific) is in the autonomous parking system in a car which consists of three computers and a sensor to determine an appropriate parking manoeuvre in a given situation. While the three computers take the specific information from the sensor into account and plan the steering and acceleration to successfully park, they would compare their results and only if at least two of them are in agreement, the car would park with that manoeuvre agreed by the majority of computers.

It is, therefore, quite important to understand the reliability and ageing characteristics of such coherent reliability systems commonly used in practice. Stochastic orders are useful tools for the purpose of comparative reliability evaluation and relative ageing of systems; one may refer to the book length accounts by Müller and Stoyan [26] and Shaked and Shanthikumar [33] for various stochastic orders, ageing notions and their applications to a wide range of problems arising from different fields. The earliest and pioneering work in this regard was carried out nearly five decades ago by Pledger and Proschan [28] and Proschan and Sethuraman [29]. There have been numerous subsequent developments in this direction, too many to list here, as a matter of fact. But, interested readers may refer to the following articles for some key results: Deshpande and Kochar [9], Saunders [32], Boland et al. [7], Kochar and Korwar [17], Dykestra et al. [11], Khaledi and Kochar [15], Kochar and Xu [18], Zhao and Balakrishnan [34], Zhao et al. [29], Balakrishnan et al. [1], and Barmalzan et al. [5]. Detailed reviews of all the developments in this regard have also been presented by Kochar [16] and Balakrishnan and Zhao [3].

Even though there is a huge body of literature on various types of comparisons of different reliability systems, as witnessed in the reviews of Kochar [16] and Balakrishnan and Zhao [4], most of the references cited therein and also all the papers mentioned above only deal with the case of independent and non-identical components. Very few papers have dealt with the case when the components in a system are dependent; see, for example, Rezapour and Alamatsaz [31], Li and Fang [21], Ding and Zhang [10], Cai et al. [8], Fang et al. [12], and Barmalzan et al. [6].

Many systems in practice will include a number of components that are homogeneous, like battery packs, circuits, airbags, etc.; but, the assumption that their lifetimes are independent may not be realistic and yet is one that is usually made in order to make the corresponding models and subsequent derivations simpler. As the components in a system will be functioning simultaneously, the functioning of one is likely to impact the functioning of others. Moreover, these components may all be manufactured by the same producer, and so may share the same manufacturing environment. It is, therefore, quite reasonable to expect some dependence between them!

In this work, we consider reliability systems with dependent components, with the joint distribution being modeled by a general Archimedean copula, and the lifetime of components following accelerated failure time and modified proportional hazards distributions. We then establish several characterization results for series, fail-safe, 2-out-of-n and parallel systems through comparisons with average systems in terms of hazard rate, reversed hazard rate and mean residual life orders.

There are several different ways to model dependence [see Kotz et al. [19]], and one convenient way is through the use of copulas [Nelsen [27]]. Here, in this work, we use an Archimedean copula to represent the joint distribution of the lifetimes of n components in the system, as it is a well-known family of copulas with many prominent copulas, such as independence, Ali-Mikhail-Haq, Gumbel-Hougaard, Clayton, and Frank copulas, all as special cases. It is for this reason that we assume the Archimedean copula to model the joint distribution of lifetimes of components.

The rest of this paper proceeds as follows. In Section 2, we briefly introduce some basic stochastic orders, ageing notions and copulas that are most pertinent for the discussions to follow in the subsequent sections; in addition, we provide a description of the accelerated failure time and modified proportional hazards families of distributions that are used to model the marginal distributions of lifetimes of components. In Section 3, we establish various stochastic orderings and ageing results for the residual lives of parallel systems. In Section 4, we similarly establish stochastic orderings and ageing results for the residual lives of series systems. In Section 5, we develop some characterization results for some coherent systems when the components follow an accelerated failure time model based on a comparison with an average system. Similarly, in Section 6, we present some characterization results for some coherent systems when the components follow a modified proportional hazards distribution based on a comparison with an average system. Finally, in Section 7, we present some concluding remarks and also some problems that will be of interest for further research.

2. DEFINITIONS AND KEY NOTIONS

We describe in this section some basic concepts about stochastic orders, copulas and two general families of lifetime distributions that are essential for subsequent developments. We assume through out that all random variables under consideration are lifetime variables and so are nonnegative, and we use "increasing" to mean "nondecreasing" and "decreasing" to mean "nonincreasing". We assume all the expectations involved to exist, and for ease of notation, we use $a \overset{sgn}{=} b$ to denote that both sides of an equality have the same sign.

2.1. Stochastic orders

Let X and Y be random variables with density functions f_X and f_Y, distribution functions F_X and F_Y, survival functions $\bar{F}_X = 1 - F_X$ and $\bar{F}_Y = 1 - F_Y$, hazard rate functions $h_X = f_X/\bar{F}_X$ and $h_Y = f_Y/\bar{F}_Y$, and reversed hazard rate functions $\tilde{h}_X = f_X/F_X$ and $\tilde{h}_Y = f_Y/F_Y$, respectively.

Definition 2.1. Then, X is said to be larger than Y in

(i) usual stochastic order (denoted by $X \geq_{\text{st}} Y$) if $\bar{F}_X(t) \geq \bar{F}_Y(t)$, for all $t \in \mathbb{R}$, or equivalently, $\mathbb{E}[\phi(X)] \geq \mathbb{E}[\phi(Y)]$ for all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$;

(ii) hazard rate order (denoted by $X \geq_{\text{hr}} Y$) if and only if $h_Y(t) \geq h_X(t)$, for all $t \in \mathbb{R}$, or equivalently, $\bar{F}_X(t)/\bar{F}_Y(t)$ is increasing in $t \in \mathbb{R}$;

(iii) reversed hazard rate order (denoted by $X \geq_{\text{rh}} Y$) if and only if $\tilde{h}_X(t) \geq \tilde{h}_Y(t)$, for all $t \in \mathbb{R}$, or equivalently, $F_X(t)/F_Y(t)$ is increasing in $t \in \mathbb{R}$;

(iv) mean residual life order (denoted by $X \geq_{\text{mrl}} Y$) if $E(X_t) \geq E(Y_t)$, for all $t \in \mathbb{R}$, where $E(X_t) = E(X - t|X > t)$ and $E(Y_t) = E(Y - t|Y > t)$ are the mean residual lives of X and Y, respectively.

Then, the following implications are well-known between these orders:

$$X \geq_{\text{hr[rh]}} Y \Longrightarrow X \geq_{\text{st}} Y;$$

see, for example, Müller and Stoyan [26] and Shaked and Shanthikumar [33] for extensive discussions on various stochastic orderings, their inter-relationships, and their properties and applications.

2.2. Ageing notions

Ageing, in reliability analysis, describes the variation in the performance of a unit over time. Several different measures and measure-based stochastic orders have been discussed in the literature pertaining to ageing characteristics of life distributions. Two most commonly used notions are through hazard and reversed hazard rates.

Definition 2.2. A random variable X is said to be ageing faster than Y in

(i) hazard rate (denoted by $X \geq_{\mathrm{c}} Y$) if $h_Y(t)/h_X(t)$ is increasing in $t \in \mathbb{R}$ (Kalashnikov and Rachev, [14]);

(ii) reversed hazard rate (denoted by $X \geq_{\mathrm{b}} Y$) if $\tilde{h}_X(t)/\tilde{h}_Y(t)$ is increasing in $t \in \mathbb{R}$ (Rezaei et al., [30]).

For more details on the relative ageing by increasing hazard ratio and reversed hazard ratio functions, one may refer to Lai and Xie [20], Misra and Francis [25] and Hazra and Misra [14].

2.3. Archimedean copulas

As mentioned earlier in Section 1, a plethora of stochastic orders and stochastic comparisons of random variables have been discussed in the literature; but, most of them involve only comparisons of marginal distributions of the underlying variables, without taking into account possible dependence between variables, with some exceptions, of course! Here, we consider characterizations of some reliability systems assuming the components to be dependent under an Archimedean copula.

Archimedean copulas are widely used for modeling dependence between variables due to their mathematical tractability as well as their ability to model a wide range of dependence structures. For a decreasing continuous function $\phi : [0,\infty) \longrightarrow [0,1]$ with $\phi(0) = 1$, $\phi(+\infty) = 0$ and $\psi = \phi^{-1}$ being the pseudo-inverse,

$$C_\phi(u_1, \cdots, u_n) = \phi(\psi(u_1) + \cdots + \psi(u_n)), \qquad u_i \in [0,1], \tag{2.1}$$

is said to be an Archimedean copula with generator ϕ if $(-1)^k \phi^{[k]}(x) \geq 0$ for $k = 0, \cdots, n-2$ and $(-1)^{n-2}\phi^{[n-2]}(x)$ is decreasing and convex, with $\phi^{[k]}(x)$ denoting the k-the derivative of the generator $\phi(x)$ with respect to x.

2.4. Accelerated failure time and modified proportional hazards distributions

Let $X_1, \cdots, X_n$ be random variables with X_i having $h_i(t)$, for $i = 1, \cdots, n$, as marginal hazard functions. Then, they are said to have an accelerated failure time family of distributions if, for all $t \geq 0$, $h_i(t) = h(\lambda_i t)$, for $i = 1, \cdots, n$, where $h(\cdot)$ is some baseline hazard function and $\lambda_i > 0$ are scale parameters (also called acceleration constants). Upon noting now that the cumulative hazard rate functions of X_i are given by $H_i(t) = \frac{1}{\lambda_i} H(\lambda_i t)$, and then using the relationship between cumulative hazard function and survival function of a distribution, we arrive at the form of cumulative distribution function for this family as

$$S_i(t) = e^{-H_i(t)} = e^{-\frac{1}{\lambda_i} H(\lambda_i t)} = \{e^{-H(\lambda_i t)}\}^{1/\lambda_i} = \{S(\lambda_i t)\}^{1/\lambda_i}, \tag{2.2}$$

for $t \geq 0$, and $i = 1, \cdots, n$; see, for example, Marshall and Olkin (2007) for details.

In the context of nonparametric rank tests, two families of distributions with

$$G_1(x) = (F(x))^\alpha, \ \alpha > 0, \qquad \bar{G}_2(x) = (S(x))^\beta, \ \beta > 0, \tag{2.3}$$

known as "Lehmann families", have been used extensively as nonparametric alternatives for tests for stochastic orderings. Upon combining the two families in (2.3), we can obtain an unified family of distributions with cumulative distribution function of the form

$$G(x) = 1 - \{1 - (F(x))^\alpha\}^\beta, \qquad \alpha, \beta > 0, \tag{2.4}$$

where $F(\cdot)$ is some baseline distribution function. Now, we may introduce acceleration constants λ_i ($i = 1, \cdots, n$), as in (2.2), to arrive at a general form of accelerated failure time distribution with its cumulative distribution function as

$$F_i(t) = 1 - \{1 - (F(\lambda_i t))^\alpha\}^\beta, \quad t > 0, \alpha, \beta > 0, \tag{2.5}$$

for $i = 1, \cdots, n$. It is evident that the accelerated failure time model in (2.2) is a special case of (2.5) when $\alpha = 1$ and $\beta = 1/\lambda_i$.

Yet another flexible family of useful lifetime distributions, offered by Marshll and Olkin [24], has a survival function of the form

$$S^*(t) = \frac{\alpha S(t)}{1 - \bar{\alpha} S(t)}, \qquad t > 0, \, 0 < \alpha < 1, \bar{\alpha} = 1 - \alpha, \tag{2.6}$$

where S is some baseline survival function and α is referred to as a tilt parameter. Here again, by introducing acceleration constants λ_i ($i = 1, \cdots, n$), as in (2.2), we arrive at a family of modified proportional hazards family of distributions with its survival function as

$$S_i(t) = \frac{\alpha S(\lambda_i t)}{1 - \bar{\alpha} S(\lambda_i t)}, \qquad t > 0, \lambda_i > 0, 0 < \alpha < 1, \bar{\alpha} = 1 - \alpha, \tag{2.7}$$

for $i = 1, \cdots, n$. The name "modified proportional hazards model" stems from the fact that the hazard functions of S and S^* in (2.6) satisfy the relationship

$$h_{S^*}(t) = h_S(t)\ \frac{1}{1 - \bar{\alpha}S(t)}, \tag{2.8}$$

which is indeed a modification of the proportional hazards assumption, with the multiplicative term varying over t, rather than being a constant.

3. RESULTS FOR RESIDUAL LIVES OF PARALLEL SYSTEMS

Let $X_{n:n}$ denote the lifetime of a parallel system consisting of n dependent components whose joint distribution is given by an Archimedean copula. Then, the survival function, density function, hazard rate function and reversed hazard rate function of the residual life variable $X_{n:n}(t)$ at x, given that the parallel system has survived till time t, are given by

$$F_{X_{n:n}(t)}(x) = \frac{\phi\left(n\psi\left[F(x+t)\right]\right) - \phi\left(n\psi\left[F(t)\right]\right)}{1 - \phi\left(n\psi\left[F(t)\right]\right)}, \qquad x, t \geq 0, \tag{3.1}$$

$$f_{X_{n:n}(t)}(x) = \frac{nf(x+t)\,\psi'\left[F(x+t)\right]\phi'\left(n\psi\left[F(x+t)\right]\right)}{1 - \phi\left(n\psi\left[F(t)\right]\right)}, \qquad x, t \geq 0, \tag{3.2}$$

$$h_{X_{n:n}(t)}(x) = \frac{nf(x+t)\,\psi'\left[F(x+t)\right]\phi'\left(n\psi\left[F(x+t)\right]\right)}{1 - \phi\left(n\psi\left[F(x+t)\right]\right)}, \qquad x, t \geq 0, \tag{3.3}$$

$$\tilde{h}_{X_{n:n}(t)}(x) = \frac{nf(x+t)\,\psi'\left[F(x+t)\right]\phi'\left(n\psi\left[F(x+t)\right]\right)}{\phi\left(n\psi\left[F(x+t)\right]\right) - \phi\left(n\psi\left[F(t)\right]\right)} \qquad x, t \geq 0, \tag{3.4}$$

where ϕ is the generator and $\psi = \phi^{-1}$. One question that we may ask here is, between two parallel systems with n and m components, which one is more reliable. Of course, this can be formulated using any particular stochastic order, as seen in the following theorems.

Theorem 3.1. *If $u \ln'\left[1 - \phi(u)\right]$ is decreasing in $u \in \mathbb{R}^+$, then for $m \geq n$, we have $X_{m:m}(t) \geq_{hr} X_{n:n}(t)$.*

Proof: With the hazard rate function of $X_{n:n}(t)$ as given in (3.3), for obtaining the desired result, it is sufficient to show that $h_{X_{n:n}(t)}(x) - h_{X_{m:m}(t)}(x) \leq 0$, for any $x \in \mathbb{R}^+$. We have

$$\begin{aligned}
I(x) &= h_{X_{n:n}(t)}(x) - h_{X_{m:m}(t)}(x) \\
&= \frac{f(x+t)\psi'\left(F(x+t)\right)}{\psi\left(F(x+t)\right)}\left\{\frac{n\psi\left(F(x+t)\right)\phi'\left(n\psi\left(F(x+t)\right)\right)}{1-\phi\left(n\psi\left(F(x+t)\right)\right)}\right. \\
&- \left.\frac{m\psi\left(F(x+t)\right)\phi'\left(m\psi\left(F(x+t)\right)\right)}{1-\phi\left(m\psi\left(F(x+t)\right)\right)}\right\} \\
&\overset{sgn}{=} u\ln'\left[1-\phi(u)\right]\Big|_{u=n\psi(F(x+t))} - u\ln'\left[1-\phi(u)\right]\Big|_{u=m\psi(F(x+t))}.
\end{aligned} \tag{3.5}$$

Now, by using the decreasing property of $u \ln' [1-\phi(u)]$ with respect to $u \in \mathbb{R}^+$, for $m \geq n$, we readily observe from (3.5) that $h_{X_{n:n}(t)}(x) \geq h_{X_{m:m}(t)}(x)$, for $x \in \mathbb{R}^+$. Thus, the theorem gets established. □

Remark 3.1. Theorem 3.1 shows that, for some Archimedean copulas, parallel systems with more redundancy is more reliable in the sense of hazard rate order; that is, a parallel system with less (dependent) components will possess a higher hazard rate than a parallel system with less components.

Example 3.1. It should be mentioned that the condition "$u \ln' [1-\phi(u)]$ is decreasing" in Theorem 3.1 is quite general and holds for many Archimedean copulas. We now demonstrate this with the following examples:

1. If $\phi_1(u) = e^{-u^\theta}$, for $\theta \in \mathbb{R}^+$ (Gumbel copula, Nelsen [27]), we have

$$u \ln' [1-\phi_1(u)] = -\frac{t\phi_1'(u)}{1-\phi_1(u)} = \frac{\theta u^\theta e^{-u^\theta}}{1-e^{-u^\theta}},$$

which is decreasing in $u \in \mathbb{R}^+$;

2. If $\phi_2(u) = 1-(1-e^{-u})^\theta$, for $\theta \in [0,1)$ (Li and Li [22]), we have

$$u \ln' [1-\phi_2(u)] = -\frac{u\phi_2'(u)}{1-\phi_2(u)} = \frac{\theta u e^{-u}}{1-e^{-u}},$$

which is decreasing in $u \in \mathbb{R}^+$;

3. If $\phi_3(u) = \frac{1}{\sqrt{u}+1}$ (Li and Li [22]), we have

$$u \ln' [1-\phi_3(u)] = -\frac{u\phi_3'(u)}{1-\phi_3(u)} = \frac{1}{4(\sqrt{u}+1)},$$

which is decreasing in $u \in \mathbb{R}^+$;

4. If $\phi_4(u) = \frac{1}{2}e^u \left(e^u - \frac{1}{2}\right)^{-1}$ (Ali-Mikhail-Haq copula, Nelsen [27]), we have

$$u \ln' [1-\phi_4(u)] = -\frac{u\phi_4'(u)}{1-\phi_4(u)} = \frac{ue^u}{2(e^u-\frac{1}{2})(e^u-1)},$$

which is decreasing in $u \in \mathbb{R}^+$.

Example 3.2. Consider the standard exponential distribution as baseline distribution function. Assume that $\phi(u) = \frac{1}{\sqrt{u}+1}$, $t = 5$, $n = 5$ and $m = 10$. Figure 1 presents plots of the hazard rate functions of $h_{X_{5:5}}(1/x-1)$ and $h_{X_{10:10}}(1/x-1)$, from which it can be observed that the value of $h_{X_{10:10}(5)}(1/x-1)$ is always smaller than that of $h_{X_{5:5}(5)}(1/x-1)$ on the interval $(0,1)$. Thus, the results of Theorem 3.1 is validated in this case.

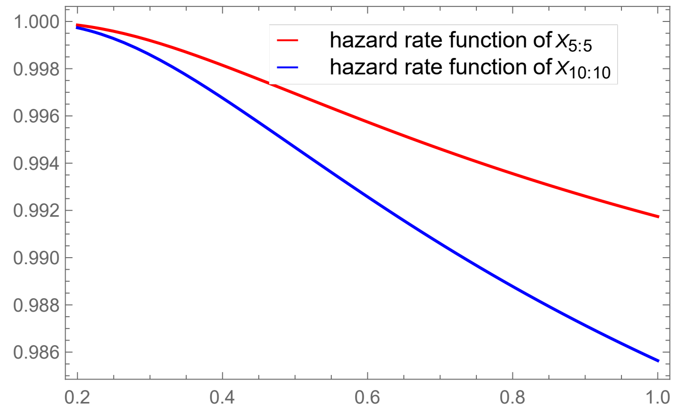

Figure 1: Plots of hazard rate functions of $h_{X_{5:5}}(1/x-1)$ and $h_{X_{10:10}}(1/x-1)$.

Theorem 3.2. *If $u\ln'\left[\phi\left(m\psi\left(F(t)\right)\right)-\phi\left(u\right)\right]$ is increasing with respect to $u\in\mathbb{R}_+$, then for $m\geq n$, we have $X_{n:n}(t)\geq_{rh}X_{m:m}(t)$.*

Proof: With reversed hazard rate function of $X_{n:n}(t)$ as given in (3.4), for establishing the desired result, we need to show that $\tilde{h}_{X_{n:n}(t)}(x)\leq\tilde{h}_{X_{m:m}(t)}(x)$, for any $x\in\mathbb{R}^+$. Because $\phi'(x)\leq 0$, we have

$$
\begin{aligned}
I(x) &= \tilde{h}_{X_{n:n}(t)}(x)-\tilde{h}_{X_{m:m}(t)}(x)\\
&= \frac{f(x+t)\psi'\left(F(x+t)\right)}{\psi\left(F(x+t)\right)}\left\{\frac{n\psi\left(F(x+t)\right)\phi'\left(n\psi\left(F(x+t)\right)\right)}{\phi\left(n\psi\left[F(x+t)\right]\right)-\phi\left(n\psi\left[F(t)\right]\right)}\right.\\
&- \left.\frac{m\psi\left(F(x+t)\right)\phi'\left(m\psi\left(F(x+t)\right)\right)}{\phi\left(m\psi\left[F(x+t)\right]\right)-\phi\left(m\psi\left[F(t)\right]\right)}\right\}\\
&\geq \frac{f(x+t)\psi'\left(F(x+t)\right)}{\psi\left(F(x+t)\right)}\left\{\frac{n\psi\left(F(x+t)\right)\phi'\left(n\psi\left(F(x+t)\right)\right)}{\phi\left(n\psi\left[F(x+t)\right]\right)-\phi\left(m\psi\left[F(t)\right]\right)}\right.\\
&- \left.\frac{m\psi\left(F(x+t)\right)\phi'\left(m\psi\left(F(x+t)\right)\right)}{\phi\left(m\psi\left[F(x+t)\right]\right)-\phi\left(m\psi\left[F(t)\right]\right)}\right\}\\
&\overset{sgn}{=} u\ln'\left[\phi\left(u\right)-\phi\left(m\psi\left(F(t)\right)\right)\right]\big|_{u=m\psi(F(x+t))}\\
&- u\ln'\left[\phi\left(u\right)-\phi\left(m\psi\left(F(t)\right)\right)\right]\big|_{u=n\psi(F(x+t))}\,. \qquad (3.6)
\end{aligned}
$$

Using the increasing property of $u\ln'\left[\phi\left(m\psi\left(F(t)\right)\right)-\phi\left(u\right)\right]$ with respect to $u\in\mathbb{R}^+$, for $m\geq n$, we readily observe from (3.6) that $I(x)\geq 0$, for $x\in\mathbb{R}^+$. Thus, the theorem gets established. □

Remark 3.2. Theorem 3.2 shows that, for some Archimedean copulas, a parallel system with more (dependent) components will possess a higher reversed hazard rate than a parallel system with less components.

Theorem 3.3. *If $u \ln'\left[-\frac{\phi'(u)}{1-\phi(u)}\right]$ is decreasing in $u \in \mathbb{R}^+$, then for $m \geq n$, we have $X_{n:n}(t) \geq_c X_{m:m}(t)$.*

Proof: With the hazard rate functions of $X_{n:n}(t)$ and $X_{m:m}(t)$ as given in (3.3), we have

$$\begin{aligned} I(x) &= \frac{h_{X_{n:n}(t)}(x)}{h_{X_{m:m}(t)}(x)} \\ &= \frac{n}{m} \times \frac{\phi'\left(n\psi\left\{F(x+t)\right]\right)}{1-\phi\left(n\psi\left[F(x+t)\right]\right)} \times \left\{\frac{\phi'\left(m\psi\left[F(x+t)\right]\right)}{1-\phi\left(m\psi\left[F(x+t)\right]\right)}\right\}^{-1}. \end{aligned}$$

Because $\phi(x)$ is decreasing, we obtain, for $m \geq n$,

$$\begin{aligned} I'(x) &\overset{sgn}{=} \left\{\frac{\phi'\left(n\psi(F(x+t))\right)}{1-\phi\left(n\psi(F(x+t))\right)}\right\}' \times \frac{\phi'\left(m\psi(F(x+t))\right)}{1-\phi\left(m\psi(F(x+t))\right)} \\ &\quad -\frac{\phi'\left(n\psi(F(x+t))\right)}{1-\phi\left(n\psi(F(x+t))\right)} \times \left\{\frac{\phi'\left(m\psi(F(x+t))\right)}{1-\phi\left(m\psi(F(x+t))\right)}\right\}' \\ &\overset{sgn}{=} -n\psi(F(x+t))\left\{\frac{\phi''(n\psi(F(x+t)))}{\phi'(n\psi(F(x+t)))}+\frac{\phi'(n\psi(F(x+t)))}{1-\phi(n\psi(F(x+t)))}\right\} \\ &\quad +m\psi(F(x+t))\left\{\frac{\phi''(m\psi(F(x+t)))}{\phi'(m\psi(F(x+t)))}+\frac{\phi'(m\psi(F(x+t)))}{1-\phi(m\psi(F(x+t)))}\right\} \\ &= u\ln'\left[-\frac{\phi'(u)}{(1-\phi(u))}\right]\Bigg|_{u=m\psi(F(x+t))} - u\ln'\left[-\frac{\phi'(u)}{(1-\phi(u))}\right]\Bigg|_{u=n\psi(F(x+t))}. \end{aligned}$$

Due to the assumption that $u \ln'\left[-\frac{\phi'(u)}{1-\phi(u)}\right]$ is decreasing in $u \in \mathbb{R}^+$, we get the required result from the above equation. $\square$

Remark 3.3. Theorem 3.3 shows that, for some Archimedean copulas, a parallel system with less redundancy (with dependence between components) ages faster in hazard rate than a parallel system with more redundancy. Some illustrations of the result in Theorem 3.3 can be seen in Part (i) of Example 3.4 of Ding and Zhang [10].

Theorem 3.4. *If $u \ln'\left[-\frac{\phi'(u)}{\phi(u)-\phi(m\psi[F(t)])}\right]$ is decreasing in $u \in \mathbb{R}^+$, then for $m \geq n$, we have $X_{m:m}(t) \geq_b X_{n:n}(t)$.*

Proof: With the reversed hazard rate functions of $X_{m:m}(t)$ and $X_{n:n}(t)$ as given in (3.4), we have

$$\begin{aligned} I(x) &= \frac{\tilde{h}_{X_{n:n}(t)}(x)}{\tilde{h}_{X_{m:m}(t)}(x)} \\ &= \frac{n}{m} \times \frac{\phi'\left(n\psi\left[F(x+t)\right]\right)}{\phi\left(n\psi\left[F(x+t)\right]\right)-\phi\left(n\psi\left[F(t)\right]\right)} \\ &\quad \times \left\{\frac{\phi'\left(m\psi\left\{F(x+t)\right]\right)}{\phi\left(m\psi\left[F(x+t)\right]\right)-\phi\left(m\psi\left[F(t)\right]\right)}\right\}^{-1}. \end{aligned}$$

Because $\phi(x)$ is decreasing, we obtain, for $m \geq n$,

$$\begin{aligned}
I'(x) &\overset{sgn}{=} \left\{ \frac{\phi'\left(n\psi(F(x+t))\right)}{\phi\left(n\psi\left[F(x+t)\right]\right) - \phi\left(n\psi\left[F(t)\right]\right)} \right\}' \times \frac{\phi'\left(m\psi(F(x+t))\right)}{\phi\left(m\psi\left[F(x+t)\right]\right) - \phi\left(m\psi\left[F(t)\right]\right)} \\
&\quad - \frac{\phi'\left(n\psi(F(x+t))\right)}{\phi\left(n\psi\left[F(x+t)\right]\right) - \phi\left(n\psi\left[F(t)\right]\right)} \times \left\{ \frac{\phi'\left(m\psi(F(x+t))\right)}{\phi\left(m\psi\left[F(x+t)\right]\right) - \phi\left(m\psi\left[F(t)\right]\right)} \right\}' \\
&\overset{sgn}{=} -n\psi(F(x+t)) \left\{ \frac{\phi''(n\psi(F(x+t)))}{\phi'(n\psi(F(x+t)))} - \frac{\phi'(n\psi(F(x+t)))}{\phi\left(n\psi\left[F(x+t)\right]\right) - \phi\left(n\psi\left[F(t)\right]\right)} \right\} \\
&\quad + m\psi(F(x+t)) \left\{ \frac{\phi''(m\psi(F(x+t)))}{\phi'(m\psi(F(x+t)))} - \frac{\phi'(m\psi(F(x+t)))}{\phi\left(m\psi\left[F(x+t)\right]\right) - \phi\left(m\psi\left[F(t)\right]\right)} \right\} \\
&\leq -n\psi(F(x+t)) \left\{ \frac{\phi''(n\psi(F(x+t)))}{\phi'(n\psi(F(x+t)))} - \frac{\phi'(n\psi(F(x+t)))}{\phi\left(n\psi\left[F(x+t)\right]\right) - \phi\left(m\psi\left[F(t)\right]\right)} \right\} \\
&\quad + m\psi(F(x+t)) \left\{ \frac{\phi''(m\psi(F(x+t)))}{\phi'(m\psi(F(x+t)))} - \frac{\phi'(m\psi(F(x+t)))}{\phi\left(m\psi\left[F(x+t)\right]\right) - \phi\left(m\psi\left[F(t)\right]\right)} \right\} \\
&= u \ln' \left[- \frac{\phi'(u)}{\phi(u) - \phi\left(m\psi\left[F(t)\right]\right)} \right] \Bigg|_{u=m\psi(F(x+t))} \\
&\quad - u \ln' \left[- \frac{\phi'(u)}{\phi(u) - \phi\left(m\psi\left[F(t)\right]\right)} \right] \Bigg|_{u=n\psi(F(x+t))} .
\end{aligned}$$

Due to assumption that $u \ln' \left[- \frac{\phi'(u)}{\phi(u) - \phi(m\psi[F(t)])} \right]$ is decreasing in $u \in \mathbb{R}^+$, from the above equation, we find $I(x)$ to be decreasing, as required. □

Remark 3.4. Theorem 3.4 shows that, for some Archimedean copulas, under the decreasing property of the function $u \ln' \left[- \frac{\phi'(u)}{\phi(u) - \phi(m\psi[F(t)])} \right]$ with respect to $u \in \mathbb{R}^+$, a parallel system with more redundancy ages faster in terms of the reversed hazard rate than a parallel system with less redundancy.

4. RESULTS FOR RESIDUAL LIVES OF SERIES SYSTEMS

Let $X_{1:n}$ denote the lifetime of a series system consisting of n dependent components whose joint distribution is given by an Archimedean copula. Then, the distribution function, density function, hazard rate function and reversed hazard rate function of residual life variable $X_{1:n}(t)$ at x, given that the series system has survived till time t, are given by

$$\text{(4.1)} \quad \bar{F}_{X_{1:n}(t)}(x) = \frac{\phi\left(n\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(n\psi\left(\bar{F}(t)\right)\right)}, \qquad x, t > 0,$$

$$\text{(4.2)} \quad f_{X_{1:n}(t)}(x) = \frac{nf(x+t)\,\psi'\left(\bar{F}(x+t)\right)\phi'\left(n\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(n\psi\left(\bar{F}(t)\right)\right)}, \qquad x, t > 0,$$

$$(4.3)\; h_{X_{1:n}(t)}(x) = \frac{nf(x+t)\,\psi'\left(\bar{F}(x+t)\right)\phi'\left(n\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(n\psi\left(\bar{F}(x+t)\right)\right)}, \qquad x,t>0,$$

$$(4.4)\; \tilde{h}_{X_{1:n}(t)}(x) = \frac{nf(x+t)\,\psi'\left(\bar{F}(x+t)\right)\phi'\left(n\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(n\psi\left(\bar{F}(t)\right)\right)-\phi\left(n\psi\left(\bar{F}(x+t)\right)\right)}, \qquad x,t>0,$$

respectively, where ϕ is the generator and $\psi=\phi^{-1}$. Now, we examine between two series systems with n and m components, which one is more reliable.

Theorem 4.1. *If $u\ln'\phi(u)$ is decreasing in $u\in\mathbb{R}^+$, then for $m\geq n$, we have $X_{1:n}(t)\geq_{hr}X_{1:m}(t)$.*

Proof: With the hazard rate functions of $X_{1:n}(t)$ and $X_{1:m}(t)$ as given in (4.3), we have

$$\begin{aligned} I(x) &= h_{X_{1:n}(t)}(x)-h_{X_{1:m}(t)}(x)\\ &= \frac{f(x+t)\psi'\left(\bar{F}(x+t)\right)}{\psi\left(\bar{F}(x+t)\right)}\\ &\times \left\{\frac{n\psi\left(\bar{F}(x+t)\right)\phi'\left(n\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(n\psi\left(\bar{F}(x+t)\right)\right)}-\frac{m\psi\left(\bar{F}(x+t)\right)\phi'\left(m\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(m\psi\left(\bar{F}(x+t)\right)\right)}\right\}\\ &\overset{sgn}{=} u\ln'\phi(u)\mid_{u=m\psi\left(\bar{F}(x+t)\right)}-u\ln'\phi(u)\mid_{u=n\psi\left(\bar{F}(x+t)\right)}. \end{aligned}$$

By using the decreasing property of $u\ln'\phi(u)$, for $m\geq n$, we readily observe that $I(x)\leq 0$. Thus, the theorem gets established. □

Remark 4.1. Theorem 4.1 shows that, for some Archimedean copulas, a series system with less (dependent) components is more reliable in the sense of hazard rate order; that is, a series system with less (dependent) components will possess a lower hazard function than a series system with more components.

Example 4.1. The condition "$u\ln'\phi(u)$ is decreasing" in Theorem 4.1 is quite general and can be verified for many well-known Archimedean copulas. For example, we consider the following:

1. If $\phi_1(u)=e^{-u^\theta}$, for $\theta\in\mathbb{R}^+$ (Gumbel copula, Nelsen [27]), we have

$$u\ln'\left[\phi_1(u)\right]=-\theta u^\theta,$$

which is decreasing in $u\in\mathbb{R}^+$;

2. If $\phi_2(u)=(\theta u+1)^{-\frac{1}{\theta}}$ (Clayton copula, Nelsen [27]), we have

$$u\ln'\left[\phi_2(u)\right]=-\frac{u}{\theta u+1},$$

which is decreasing in $u\in\mathbb{R}^+$.

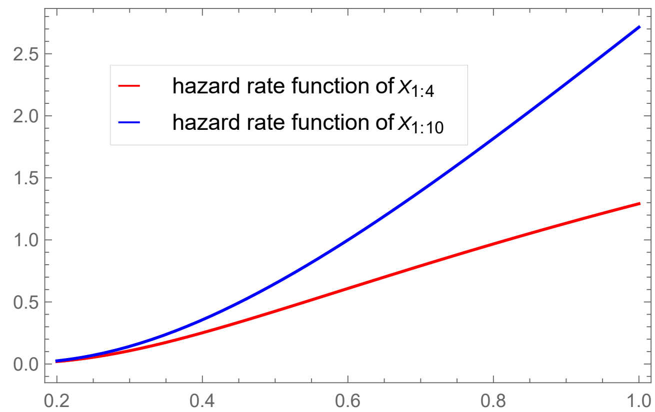

Figure 2: Plots of hazard rate functions of $h_{X_{1:4}}(1/x-1)$ and $h_{X_{1:10}}(1/x-1)$.

Example 4.2. Consider the standard exponential distribution as baseline distribution function. Assume that $\phi(u) = (\theta u + 1)^{-\frac{1}{\theta}}$, $\theta = 2$, $t = 2$, $n = 4$ and $m = 10$. Figure 2 presents plots of the hazard rate functions of $h_{X_{1:4}}(1/x-1)$ and $h_{X_{1:10}}(1/x-1)$, from which it can be observed that the value of $h_{X_{1:14}}(1/x-1)$ is always smaller than that of $h_{X_{1:10}}(1/x-1)$ on the interval $(0,1)$. Thus, the result of Theorem 4.1 is validated in this case.

Theorem 4.2. *If $u \ln' \left[\phi\left(n\psi\left(\bar{F}(t)\right)\right) - \phi(u)\right]$ is decreasing in $u \in \mathbb{R}^+$, then for $m \geq n$, we have $X_{1:n}(t) \geq_{rh} X_{1:m}(t)$.*

Proof: With the reversed hazard rate functions of $X_{1:n}(t)$ and $X_{1:m}(t)$ as given in (4.4), for $m \geq n$, we have

$$\begin{aligned}
I(x) &= \tilde{h}_{X_{1:n}(t)}(x) - \tilde{h}_{X_{1:m}(t)}(x) \\
&= \frac{f(x+t)\psi'\left(\bar{F}(x+t)\right)}{\psi\left(\bar{F}(x+t)\right)} \left\{ \frac{n\psi\left(\bar{F}(x+t)\right)\phi'\left(n\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(n\psi\left(\bar{F}(t)\right)\right) - \phi\left(n\psi\left(\bar{F}(x+t)\right)\right)} \right. \\
&- \left. \frac{n\psi\left(\bar{F}(x+t)\right)\phi'\left(m\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(m\psi\left(\bar{F}(t)\right)\right) - \phi\left(m\psi\left(\bar{F}(x+t)\right)\right)} \right\} \\
&\geq \frac{f(x+t)\psi'\left(\bar{F}(x+t)\right)}{\psi\left(\bar{F}(x+t)\right)} \left\{ \frac{n\psi\left(\bar{F}(x+t)\right)\phi'\left(n\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(n\psi\left(\bar{F}(t)\right)\right) - \phi\left(n\psi\left(\bar{F}(x+t)\right)\right)} \right. \\
&- \left. \frac{n\psi\left(\bar{F}(x+t)\right)\phi'\left(m\psi\left(\bar{F}(x+t)\right)\right)}{\phi\left(n\psi\left(\bar{F}(t)\right)\right) - \phi\left(m\psi\left(\bar{F}(x+t)\right)\right)} \right\} \\
&\stackrel{sgn}{=} u \ln' \left[\phi\left(n\psi\left(\bar{F}(t)\right)\right) - \phi(u)\right] |_{u=n\psi\left(\bar{F}(x+t)\right)} \\
&- u \ln' \left[\phi\left(n\psi\left(\bar{F}(t)\right)\right) - \phi(u)\right] |_{u=m\psi\left(\bar{F}(x+t)\right)} . \qquad (4.5)
\end{aligned}$$

Using the decreasing property of $u \ln' \left[\phi\left(n\psi\left(\bar{F}(t)\right)\right) - \phi(u)\right]$ in $u \in \mathbb{R}^+$, for $m \geq n$, we readily observe from (4.5) that $I(x) \geq 0$. Thus, the theorem gets established. $\square$

Remark 4.2. Theorem 4.2 shows that, for some Archimedean copulas, a series system with less (dependent) components will possess lower reversed hazard rate than a series system with more components.

Theorem 4.3. *If $u\ln'\left[-\frac{\phi'(u)}{\phi(u)}\right]$ is decreasing (increasing) in $u \in \mathbb{R}^+$, then for $m \geq n$, we have $X_{1:m}(t) \geq_c (\leq_c) X_{1:n}(t)$.*

Proof: With the hazard rate functions of $X_{1:m}(t)$ and $X_{1:n}(t)$ as given in (4.3), we have

$$\begin{aligned} I(x) &= \frac{h_{X_{1:n}(t)}(x)}{h_{X_{1:m}(t)}(x)} \\ &= \frac{n}{m} \times \frac{\phi'\left(n\psi\left[\bar{F}(x+t)\right]\right)}{\phi\left(n\psi\left[\bar{F}(x+t)\right]\right)} \times \left\{\frac{\phi'\left(m\psi\left[\bar{F}(x+t)\right]\right)}{\phi\left(m\psi\left[\bar{F}(x+t)\right]\right)}\right\}^{-1}. \end{aligned}$$

By differentiating this function, we find

$$\begin{aligned} I'(x) &\overset{sgn}{=} \left\{\frac{\phi'\left(n\psi(\bar{F}(x+t))\right)}{\phi\left(n\psi(\bar{F}(x+t))\right)}\right\}' \times \frac{\phi'\left(m\psi(\bar{F}(x+t))\right)}{\phi\left(m\psi(\bar{F}(x+t))\right)} \\ &\quad - \frac{\phi'\left(n\psi(\bar{F}(x+t))\right)}{\phi\left(n\psi(\bar{F}(x+t))\right)} \times \left\{\frac{\phi'\left(m\psi(\bar{F}(x+t))\right)}{\phi\left(m\psi(\bar{F}(x+t))\right)}\right\}' \\ &\overset{sgn}{=} n\psi(\bar{F}(x+t))\left\{\frac{\phi''(n\psi(\bar{F}(x+t)))}{\phi'(n\psi(\bar{F}(x+t)))} - \frac{\phi'(n\psi(\bar{F}(x+t)))}{\phi(n\psi(\bar{F}(x+t)))}\right\} \\ - &\quad m\psi(\bar{F}(x+t))\left\{\frac{\phi''(m\psi(\bar{F}(x+t)))}{\phi'(m\psi(\bar{F}(x+t)))} - \frac{\phi'(m\psi(\bar{F}(x+t)))}{\phi(m\psi(\bar{F}(x+t)))}\right\} \\ &= u\ln'\left[-\frac{\phi'(u)}{\phi(u)}\right]\Bigg|_{u=n\psi(\bar{F}(x+t))} - u\ln'\left[-\frac{\phi'(u)}{\phi(u)}\right]\Bigg|_{u=m\psi(\bar{F}(x+t))} \\ &\geq (\leq)\ 0, \end{aligned}$$

according to whether $u\ln'\left[-\frac{\phi'(u)}{\phi(u)}\right]$ is decreasing (or increasing) in $u \in \mathbb{R}^+$, for $m \geq n$. Thus, the theorem gets established. □

Remark 4.3. Theorem 4.3 shows that, for some Archimedean copulas, under the decreasing (increasing) property of the function $u\ln'\left[-\frac{\phi'(u)}{\phi(u)}\right]$, a series system with less (dependent) components ages faster (ages slower) in terms of hazard rate than a series system with more components. Some illustrations of the result in Theorem 4.3 can be seen in Part (ii) of Example 3.4 of Ding and Zhang [10].

Theorem 4.4. *If* $u\ln'\left[-\frac{\phi'(u)}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi(u)}\right]$ *is decreasing in* $u \in \mathbb{R}^+$, *then for* $m \geq n$, *we have* $X_{1:n}(t) \geq_b X_{1:m}(t)$.

Proof: With the reversed hazard rate functions of $X_{1:m}(t)$ and $X_{1:n}(t)$ as given in (4.4), we have

$$\begin{aligned}
I(x) &= \frac{\tilde{h}_{X_{1:n}(t)}(x)}{\tilde{h}_{X_{1:m}(t)}(x)} \\
&= \frac{n}{m} \times \frac{\phi'\left(n\psi\left[\bar{F}(x+t)\right]\right)}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi\left(n\psi\left[\bar{F}(x+t)\right]\right)} \\
&\times \left\{\frac{\phi'\left(m\psi\left[\bar{F}(x+t)\right]\right)}{\phi\left(m\psi\left[\bar{F}(t)\right]\right)-\phi\left(m\psi\left[\bar{F}(x+t)\right]\right)}\right\}^{-1}.
\end{aligned}$$

As $\phi(x)$ is decreasing, for $m \geq n$, we obtain

$$\begin{aligned}
I'(x) &\overset{sgn}{=} \left\{\frac{\phi'\left(n\psi(\bar{F}(x+t))\right)}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi\left(n\psi(\bar{F}(x+t))\right)}\right\}' \times \frac{\phi'\left(m\psi(\bar{F}(x+t))\right)}{\phi\left(m\psi\left[\bar{F}(t)\right]\right)-\phi\left(m\psi(\bar{F}(x+t))\right)} \\
&\quad -\frac{\phi'\left(n\psi(\bar{F}(x+t))\right)}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi\left(n\psi(\bar{F}(x+t))\right)} \times \left\{\frac{\phi'\left(m\psi(\bar{F}(x+t))\right)}{\phi\left(m\psi\left[\bar{F}(t)\right]\right)-\phi\left(m\psi(\bar{F}(x+t))\right)}\right\}' \\
&\overset{sgn}{=} n\psi(\bar{F}(x+t))\left\{\frac{\phi''(n\psi(\bar{F}(x+t)))}{\phi'(n\psi(\bar{F}(x+t)))}+\frac{\phi'(n\psi(\bar{F}(x+t)))}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi(n\psi(\bar{F}(x+t)))}\right\} \\
&\quad -m\psi(\bar{F}(x+t))\left\{\frac{\phi''(m\psi(\bar{F}(x+t)))}{\phi'(m\psi(\bar{F}(x+t)))}+\frac{\phi'(m\psi(\bar{F}(x+t)))}{\phi\left(m\psi\left[\bar{F}(t)\right]\right)-\phi(m\psi(\bar{F}(x+t)))}\right\} \\
&\geq n\psi(\bar{F}(x+t))\left\{\frac{\phi''(n\psi(\bar{F}(x+t)))}{\phi'(n\psi(\bar{F}(x+t)))}+\frac{\phi'(n\psi(\bar{F}(x+t)))}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi(n\psi(\bar{F}(x+t)))}\right\} \\
&\quad -m\psi(\bar{F}(x+t))\left\{\frac{\phi''(m\psi(\bar{F}(x+t)))}{\phi'(m\psi(\bar{F}(x+t)))}+\frac{\phi'(m\psi(\bar{F}(x+t)))}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi(m\psi(\bar{F}(x+t)))}\right\} \\
&= u\ln'\left[-\frac{\phi'(u)}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi(u)}\right]\Bigg|_{u=n\psi(\bar{F}(x+t))} \\
&- u\ln'\left[-\frac{\phi'(u)}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi(u)}\right]\Bigg|_{u=m\psi(\bar{F}(x+t))}.
\end{aligned}$$

Due to the assumption that $u\ln'\left[-\frac{\phi'(u)}{\phi\left(n\psi\left[\bar{F}(t)\right]\right)-\phi(u)}\right]$ is decreasing in $u \in \mathbb{R}^+$, we have $I'(x) > 0$. Thus, the theorem gets established. □

Remark 4.4. Theorem 4.4 shows that, for some Archimedean copulas, a series system with less (dependent) components ages faster in terms of reversed hazard rate than a series system with more components.

Example 4.3. We note that the condition "$u \ln' \left[-\frac{\phi'(u)}{\phi\left(m\psi\left[\bar{F}(t)\right]\right) - \phi(u)} \right]$ is decreasing" in Theorem 4.4 holds in many cases. For example, consider $\phi(u(x,t)) = e^{-u}$ and $0 < a(t) \leq 1$ and also $\phi(u(x,t)) < a(t)$ for all $t \in [0, \infty)$. We then have

$$u \ln' \left[-\frac{\phi'(u)}{a - \phi(u)} \right] = u \left\{ \frac{\phi''(u)}{\phi'(u)} - \frac{\phi'(u)}{a - \phi(u)} \right\} = \frac{-au}{a - e^{-u}}$$

to be decreasing in $u \in \mathbb{R}^+$.

5. SYSTEMS WITH DEPENDENT ACCELERATION FAILURE TIME COMPONENTS

One of the common reliability structures in practice is a r-out-of-n system. This system, consisting of n components, works iff at least r components work. It includes parallel, fail-safe and series systems all as special cases when $r = 1$, $r = n - 1$ and $r = n$, respectively. In this section, we develop some characterization results for these systems when the components are dependent with an Archimedean copula and the component lifetimes follow an accelerated failure time distribution in (2.5) based on a comparison with the "average system". The results established here complete and extend some results of Cai et al. [8].

Using the copula representation for the joint distribution of $X_1, \cdots, X_n$ in (2.1), we have in this case

$$\bar{F}_{1:n}(x) \;=\; \phi\left(\sum_{k=1}^{n} \psi((1 - F^{\alpha}(\lambda_k x))^{\beta}) \right), \quad x > 0, \tag{5.1}$$

$$\begin{aligned} \bar{F}_{2:n}(x) \;=\; & \sum_{l=1}^{n} \phi\left(\sum_{k=1, k \neq l}^{n} \psi((1 - F^{\alpha}(\lambda_k x))^{\beta}) \right) \\ & -\;(n-1)\phi\left(\sum_{k=1}^{n} \psi((1 - F^{\alpha}(\lambda_k x))^{\beta}) \right), \quad x > 0, \end{aligned} \tag{5.2}$$

$$\bar{G}(x) \;=\; \frac{1}{n} \sum_{l=1}^{n} \phi\left(\sum_{k=1, k \neq l}^{n} \psi((1 - F^{\alpha}(\lambda_k x))^{\beta}) \right), \quad x > 0. \tag{5.3}$$

The expressions in (5.1) and (5.2) correspond to the survival functions of the series system (i.e., $r = n$) and of the fail-safe system ($r = n-1$), respectively. The expression in (5.3) corresponds to the survival function of an "average series system", whose lifetime is denoted by Y. This average series system can be explained by a randomization process as follows: From a series system comprising

n components, one randomly selected component may be removed to obtain a series system with $(n-1)$ remaining components; out of the n such $(n-1)$-component series systems, we then randomly select one of them, and that is what the average series system is here. The expression of the survival function given in (5.3) then becomes clear.

Theorem 5.1. *We have:*

(i) $X_{1:n} \le_{mrl} X_{2:n}$ *iff* $X_{1:n} \le_{mrl} Y$;

(ii) $X_{1:n} \le_{hr} X_{2:n}$ *iff* $X_{1:n} \le_{hr} Y$;

(iii) $X_{1:n} \le_{rh} X_{2:n}$ *iff* $X_{1:n} \le_{rh} Y$.

Proof: (i) By definition, $X_{1:n} \le_{mrl} X_{2:n}$ iff $\forall t > 0$, we have

$$\frac{\int_0^\infty \bar{F}_{2:n}(x+t)\,dx}{\bar{F}_{2:n}(t)} \ge \frac{\int_0^\infty \bar{F}_{1:n}(x+t)\,dx}{\bar{F}_{1:n}(t)}. \tag{5.4}$$

Upon using (5.1) and (5.2) in (5.4) and Theorem 2.A.6 of Shaked and Shanthikumar [33] and some simplifications, $\forall t > 0$,

$$\begin{aligned}
&\phi\left(\sum_{i=1}^{n} \psi((1-F^\alpha(\lambda_k t))^\beta)\right) \\
&\times \int_0^\infty \left[\sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi((1-F^\alpha(\lambda_k x+\lambda_k t))^\beta)\right)\right] dx \\
&\ge \sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi((1-F^\alpha(\lambda_k x+\lambda_k t))^\beta)\right) \\
&\times \int_0^\infty \left[\phi\left(\sum_{k=1}^{n} \psi((1-F^\alpha(\lambda_k x+\lambda_k t))^\beta)\right)\right] dx.
\end{aligned} \tag{5.5}$$

Similarly, from (5.1) and (5.3), we see that $Y \ge_{mrl} X_{1:n}$ iff $\forall t > 0$,

$$\begin{aligned}
&\frac{\int_0^\infty \frac{1}{n}\sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi((1-F^\alpha(\lambda_k x+\lambda_k t))^\beta)\right) dx}{\frac{1}{n}\sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi((1-F^\alpha(\lambda_k t))^\beta)\right)} \\
&\ge \frac{\int_0^\infty \phi\left(\sum_{i=1}^{n} \psi((1-F^\alpha(\lambda_k x+\lambda_k t))^\beta)\right) dx}{\phi\left(\sum_{k=1}^{n} \psi((1-F^\alpha(\lambda_k t))^\beta)\right)}.
\end{aligned} \tag{5.6}$$

The equivalence of the inequalities in (5.5) and (5.6) yields Part (i) immediately.
(ii) By definition, $X_{1:n} \le_{hr} X_{2:n}$ iff $\forall x, t > 0$, we have

$$\frac{\bar{F}_{2:n}(x+t)}{\bar{F}_{2:n}(t)} \ge \frac{\bar{F}_{1:n}(x+t)}{\bar{F}_{1:n}(t)}. \tag{5.7}$$

Upon using (5.1) and (5.2) in (5.7) and simplification, $\forall x, t > 0$,

$$\begin{aligned} &\sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi((1-F^{\alpha}(\lambda_k x+\lambda_k t))^{\beta})\right) \\ &\times \left[\phi\left(\sum_{k=1}^{n} \psi((1-F^{\alpha}(\lambda_k t))^{\beta})\right)\right] \\ &\geq \sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi((1-F^{\alpha}(\lambda_k t))^{\beta})\right) \\ &\times \left[\phi\left(\sum_{k=1}^{n} \psi((1-F^{\alpha}(\lambda_k x+\lambda_k t))^{\beta})\right)\right]. \end{aligned} \tag{5.8}$$

Similarly, from (5.1) and (5.3), we see that $Y \geq_{hr} X_{1:n}$ iff $\forall x, t > 0$,

$$\begin{aligned} &\frac{\frac{1}{n}\sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi((1-F^{\alpha}(\lambda_k x+\lambda_k t))^{\beta})\right)}{\frac{1}{n}\sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi((1-F^{\alpha}(\lambda_k t))^{\beta})\right)} \\ &\geq \frac{\phi\left(\sum_{k=1}^{n} \psi((1-F^{\alpha}(\lambda_k x+\lambda_k t))^{\beta})\right)}{\phi\left(\sum_{k=1}^{n} \psi((1-F^{\alpha}(\lambda_k t))^{\beta})\right)}. \end{aligned} \tag{5.9}$$

The equivalence of the inequalities in (5.8) and (5.9) yields Part (ii) immediately.
(iii) This can be proved in a manner similar to Part (ii). □

Next, from the copula representation for the joint distribution of $X_1, \cdots, X_n$ in (2.1), we have, in this case, for $x > 0$,

$$F_{n:n}(x) = \phi\left(\sum_{k=1}^{n} \psi(1-(1-F^{\alpha}(\lambda_k x))^{\beta})\right), \tag{5.10}$$

$$\begin{aligned} F_{n-1:n}(x) &= \sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi(1-(1-F^{\alpha}(\lambda_k x))^{\beta})\right) \\ &- (n-1)\phi\left(\sum_{k=1}^{n} \psi(1-(1-F^{\alpha}(\lambda_k x))^{\beta})\right), \end{aligned} \tag{5.11}$$

and let Z have its distribution function as

$$H(x) = \frac{1}{n}\sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi(1-(1-F^{\alpha}(\lambda_k x))^{\beta})\right), \qquad x > 0. \tag{5.12}$$

The expression in (5.10) corresponds to the survival function of a parallel system (i.e., $r = 1$), while the expression in (5.11) corresponds to the survival function of a 2-out-of-n system. The expression in (5.12) corresponds to the survival function of an "average parallel system", whose lifetime is denoted here by Z. This average parallel system can once again be explained by a randomization process as follows: From a parallel system consisting of n components, one randomly selected component may be removed to obtain a parallel system with $(n-1)$ remaining components; out of the n such $(n-1)$-component parallel systems, we randomly select one of them, and that is what the average parallel system is here. The expression of the survival function given in (5.12) then becomes clear.

Theorem 5.2. *In the special case when $n = 2$, we have:*

(i) $X_{n-1:n} \leq_{mrl} X_{n:n}$ iff $Z \leq_{mrl} X_{n:n}$;

(ii) $X_{n-1:n} \leq_{hr} X_{n:n}$ iff $Z \leq_{hr} X_{n:n}$;

(iii) $X_{n-1:n} \leq_{rh} X_{n:n}$ iff $Z \leq_{rh} X_{n:n}$.

Proof: This can be established in a manner analogous to Theorem 5.1, and we therefore do not present it here for the sake of brevity. □

We now present a complete characterization result for the special case when $n = 2$.

Theorem 5.3. *We have:*

(i) $X_{1:2} \leq_{mrl} Y \iff X_{1:2} \leq_{mrl} X_{2:2} \iff Z \leq_{mrl} X_{2:2}$;

(ii) $X_{1:2} \leq_{hr} Y \iff X_{1:2} \leq_{hr} X_{2:2} \iff Z \leq_{hr} X_{2:2}$;

(iii) $X_{1:2} \leq_{rh} Y \iff X_{1:2} \leq_{rh} X_{2:2} \iff Z \leq_{rh} X_{2:2}$.

Proof: In Theorem 3.1, we have characterization between $X_{1:n}$ and $X_{2:n}$ based on characterization between $X_{1:n}$ and Y. For the case when $n = 2$, it is simply a characterization between $X_{1:2}$ and $X_{2:2}$ based on characterization between $X_{1:2}$ and Y. Similarly, in Theorem 3.2, we have characterization between $X_{n-1:n}$ and $X_{n:n}$ based on characterization between Z and $X_{n:n}$, which in the case when $n = 2$, is simply a characterization between $X_{1:2}$ and $X_{2:2}$ based on characterization between Z and $X_{2:2}$. As the left hand sides of both results are the same variables, the characterization results on the right hand sides must be equivalent. Thus, the characterization of $X_{1:2}$ and Y must be equivalent to the characterization of Z and $X_{2:2}$. □

6. SYSTEMS WITH DEPENDENT MODIFIED PROPORTIONAL HAZARDS COMPONENTS

In this section, we assume that the n components in a reliability system are dependent with their component lifetimes following a modified proportional hazards model in (2.7) and their joint distribution being represented by an Archimedean copula in (2.1). We then establish some characterization results for series, fail-safe, 2-out-of-n and parallel systems in this general setup using mean residual life, hazard rate and reversed hazard orders based on a comparison with the "average system". The results established here complete and extend some results of Cai et al. [8].

In this case, from (2.1), we have

$$\bar{F}_{1:n}(x) = \phi\left(\sum_{k=1}^{n}\psi\left(\frac{\alpha\bar{F}(\lambda_k x)}{1-\bar{\alpha}\bar{F}(\lambda_k x)}\right)\right), \qquad x>0 \tag{6.1}$$

$$\begin{aligned}\bar{F}_{2:n}(x) &= \sum_{l=1}^{n}\phi\left(\sum_{l=1,k\neq l}^{n}\psi\left(\frac{\alpha\bar{F}(\lambda_k x)}{1-\bar{\alpha}\bar{F}(\lambda_k x)}\right)\right)\\ &\quad -(n-1)\phi\left(\sum_{k=1}^{n}\psi\left(\frac{\alpha\bar{F}(\lambda_k x)}{1-\bar{\alpha}\bar{F}(\lambda_k x)}\right)\right),\ x>0,\end{aligned} \tag{6.2}$$

$$\bar{G}(x) = \frac{1}{n}\sum_{l=1}^{n}\phi\left(\sum_{k=1,k\neq l}^{n}\psi\left(\frac{\alpha\bar{F}(\lambda_k x)}{1-\bar{\alpha}\bar{F}(\lambda_k x)}\right)\right), \qquad x>0, \tag{6.3}$$

where ϕ is the generator and $\psi=\phi^{-1}$. The expressions in (6.1)-(6.3) correspond to the survival functions of series, fail-safe and average series systems in this case, respectively. We use Y to denote the lifetime of the average series system whose survival function is given in (6.3)

Theorem 6.1. *We have:*

(i) $X_{1:n}\le_{mrl} X_{2:n}$ *iff* $X_{1:n}\le_{mrl} Y$;

(ii) $X_{1:n}\le_{hr} X_{2:n}$ *iff* $X_{1:n}\le_{hr} Y$;

(iii) $X_{1:n}\le_{rh} X_{2:n}$ *iff* $X_{1:n}\le_{rh} Y$.

Proof: This can be established in a manner analogous to Theorem 5.1, and we therefore do not present it here for the sake of brevity. □

Next, from the copula representation for the joint distribution of $X_1, \cdots, X_n$ in (2.1), we find in this case

$$F_{n:n}(x) \;=\; \phi\left(\sum_{k=1}^{n} \psi\left(\frac{1-\bar{F}(\lambda_k x)}{1-\bar{\alpha}\bar{F}(\lambda_k x)}\right)\right), \quad x>0, \tag{6.4}$$

$$\begin{aligned} F_{n-1:n}(x) \;&=\; \sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi\left(\frac{1-\bar{F}(\lambda_k x)}{1-\bar{\alpha}\bar{F}(\lambda_k x)}\right)\right) \\ &-\; (n-1)\phi\left(\sum_{k=1}^{n} \psi\left(\frac{1-\bar{F}(\lambda_k x)}{1-\bar{\alpha}\bar{F}(\lambda_k x)}\right)\right), \; x>0, \end{aligned} \tag{6.5}$$

and let Z be a random variable with its distribution function as

$$H(x) = \frac{1}{n}\sum_{l=1}^{n} \phi\left(\sum_{k=1,k\neq l}^{n} \psi\left(\frac{1-\bar{F}(\lambda_k x)}{1-\bar{\alpha}\bar{F}(\lambda_k x)}\right)\right), \qquad x>0. \tag{6.6}$$

The expressions in (6.4)-(6.6) correspond to the distribution functions of parallel, 2-out-of-n and average parallel systems in this case.

Theorem 6.2. *We have:*

(i) $X_{n-1:n} \leq_{mrl} X_{n:n}$ *iff* $Z \leq_{mrl} X_{n:n}$;

(ii) $X_{n-1:n} \leq_{hr} X_{n:n}$ *iff* $Z \leq_{hr} X_{n:n}$;

(iii) $X_{n-1:n} \leq_{rh} X_{n:n}$ *iff* $Z \leq_{rh} X_{n:n}$.

Proof: This can be proved in a manner analogous to Theorem 6.1, and we therefore do not present the proof here for the sake of brevity. □

Theorem 6.3. *In the special case when* $n=2$*, we have:*

(i) $X_{1:2} \leq_{mrl} Y \iff X_{1:2} \leq_{mrl} X_{2:2} \iff Z \leq_{mrl} X_{2:2}$;

(ii) $X_{1:2} \leq_{hr} Y \iff X_{1:2} \leq_{hr} X_{2:2} \iff Z \leq_{hr} X_{2:2}$;

(iii) $X_{1:2} \leq_{rh} Y \iff X_{1:2} \leq_{rh} X_{2:2} \iff Z \leq_{rh} X_{2:2}$.

Proof: This can be proved in a way similar to Theorem 5.3, and we therefore do not describe it here. □

7. CONCLUDING REMARKS

In this work, we have considered reliability systems with dependent components having accelerated failure time and modified proportional hazards distributions and having a joint distribution represented by a general Archimedean copula. We have focused especially on series, fail-safe, 2-out-of-n and parallel systems, and have then established some characterization results for these systems through comparisons with average systems in terms of mean residual life, hazard rate and reversed hazard rate orders. It will naturally be of interest to extend these results to the case of general $(n-r+1)$-out-of-n systems and sequential $(n-r+1)$-out-of-n systems as discussed by Barmalzan et al. [6] under the general setting considered here; one may see Misra and Francis [25] for some results in this regard under a restricted setting. We are currently working on these problems and hope to report the findings in a future paper.

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