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LIKELIHOOD-BASED PREDICTION OF FUTURE WEIBULL RECORD VALUES

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Abstract:

- Point prediction of future record values from a sequence of independent and identically distributed two-parameter Weibull random variables using the maximum likelihood method is considered. Two possible likelihood functions for prediction, the predictive and the observed predictive likelihood functions, are considered and the associated predictors are derived. Mean squared error and Pitman closeness criterion are used for comparing the prediction procedures.

Key-Words:

- *Point prediction; maximum observed likelihood predictor; maximum likelihood predictor; upper record values; Pitman closeness; Weibull distribution*

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1. INTRODUCTION

In many practical applications such as materials testing, meteorology and hydrology, only record data is available for statistical analysis. Then, for a sequence of successive, increasing record values, an appropriate model are upper record values, first studied in [5]. Suppose that $X_1, X_2, \dots$ is an infinite sequence of independent and identically distributed (i.i.d.) continuous random variables with cumulative distribution function (cdf) F. An observation is called an *(upper) record value* provided it is greater than all previously observed values. More specifically, defining the record times as

$$L(1) = 1, \quad L(n+1) = \min\{j > L(n) \mid X_j > X_{L(n)}\}, \quad n \in \mathbb{N},$$

the sequence $(R_n)_{n\in\mathbb{N}} = (X_{L(n)})_{n\in\mathbb{N}}$ is referred to as the sequence of (upper) record values based on $(X_n)_{n\in\mathbb{N}}$; see [2], [15]. The structure of record values also appears in the context of minimal repair of a system, and, under mild conditions, the epoch times of a non-homogeneous Poisson process and upper record values are equal in distribution; see [11]. If not the record values themselves, but the successively k-th largest values $(R_n^{(k)})_{n\in\mathbb{N}}$, $k \in \mathbb{N}$, in an i.i.d. sequence of random variables are of interest, the appropriate description is provided by the model of *k-th record values* introduced in [9].
We consider the problem of providing a prediction value for the occurrence of a future Weibull record value R_s based on the first r, $r < s$, (observed) Weibull record values $R_\star = (R_1, \dots, R_r)$. In addition to the modeling of repairable systems mentioned above, the Weibull record values model has been used in the literature to model reliability growth (see [7]) and software reliability (see [13]). The point prediction problem for Weibull record values has recently been studied in [16], where, in particular, the maximum likelihood predictor of R_s based on $R_\star$ was derived. In fact, predictive analysis of Weibull record values dates as back as [14] and [10], where exact prediction intervals for R_s were constructed. For Bayesian predictive analysis of Weibull record values, the reader is referred to, e.g., [3], [18], [19]. For statistical inference based on record values from Weibull distributions and application, we also refer to [23] and [20].
The *maximum likelihood prediction* procedure is frequently examined in the literature and commonly applied in the context of an ordered data model such as the model of upper record values; see [12]. The maximum likelihood prediction procedure derives a predictor of a r.v. Y based on a possibly p-dimensional random vector X with joint pdf $f_\theta^{X,Y}$, $\theta \in \Theta$, by maximizing the predictive likelihood function L_{rv} of Y and θ given that $X = x$, which takes the form

$$L_{rv}(y, \theta | x) = f_\theta^{X,Y}(x, y),$$

with respect to θ and y. The functions π_{MLP} and $\hat{\theta}_{ML}$ are called, respectively, maximum likelihood predictor (MLP) of Y and predictive maximum likelihood estimator (PMLE) of θ, if, for any $x \in \mathbb{R}^p$,

$$L_{rv}(\pi_{MLP}(x), \hat{\theta}_{ML}(x)) = \max_{(y,\theta)\in\mathbb{R}\times\Theta} L_{rv}(y, \theta | x).$$

Recently, a new likelihood-based general-purpose prediction procedure, the so-called *maximum observed likelihood prediction* method has been introduced and studied in [22]; see also [21]. By means of this procedure, a predictor of Y based on X is obtained by maximizing the observed predictive likelihood function L_{obs} defined by

$$L_{obs}(y, \theta|x) = f_\theta^{X|Y}(x|y)$$

with respect to θ and y. Then, any functions π_{MOLP} and $\hat{\theta}_{MOL}$ are referred to, respectively, as maximum observed likelihood predictor (MOLP) of Y and predictive maximum observed likelihood estimator (PMOLE) of θ, provided that, for any $x \in \mathbb{R}^p$,

$$L_{obs}(\pi_{MOLP}(x), \hat{\theta}_{MOL}(x)) = \max_{(y,\theta)\in\mathbb{R}\times\Theta} L_{obs}(y, \theta|x).$$

If, in a general parametric family $\{F_\theta \mid \theta \in \Theta\}$ of continuous cdfs, the s-th record R_s is predicted based on $R_\star = (R_1, \ldots, R_r)$, then the maximum observed likelihood predictor is given by

$$\pi_{MOLP}^{(s)} = F_{\hat{\theta}(R_\star)}^{-1}\left(1 - (1 - F_{\hat{\theta}(R_\star)}(R_r))^{\frac{s-1}{r}}\right), \tag{1.1}$$

where the function $\hat{\theta}$ is such that

$$\Psi(\hat{\theta}(r_\star), r_\star) = \max_{\substack{\theta\in\Theta:\\(\theta,r_\star)\in\mathcal{Z}_r}} \Psi(\theta, r_\star); \tag{1.2}$$

see [22, Theorem 3.3], [21, Theorem 5.3]. In equation (1.2), the function Ψ is given by

$$\Psi(\theta, r_\star) = \prod_{i=1}^{r} \frac{f_\theta(r_i)/(1 - F_\theta(r_i))}{\ln(1 - F_\theta(r_r))}, \quad (\theta, r_\star) \in \mathcal{Z}_r, \tag{1.3}$$

with $\mathcal{Z}_r = \{(\theta, r_1, \ldots, r_r) \in \Theta \times \mathbb{R}_<^r \mid (r_1, \ldots, r_r) \in (\alpha(F_\theta), \omega(F_\theta))_<^r\}$, where, for an interval $I \subseteq \mathbb{R}$ and $n \in \mathbb{N}$, $I_<^n = \{(x_1, \ldots, x_n) \in I^n \mid x_1 < x_2 < \ldots < x_n\}$, and $\omega(F)$ and $\alpha(F)$ denote the left and right endpoints of the support of a cdf F.
In order to facilitate building some intuition for the difference between the predictive likelihood and the observed predictive likelihood function-based prediction procedures, let us slightly rewrite the associated likelihood functions. First, observe that the predictive likelihood function can be constructed by taking the product of the conditional density function of Y given X and the density function of X, that is

$$L_{rv}(y, \theta|x) = f_\theta^{Y|X}(y|x) f_\theta^X(x).$$

Thus, in maximizing the predictive likelihood function the information on the variability in Y as described by the conditional density function $f_\theta^{Y|X}$ is reduced to the mode of the conditional density of Y given X yielding a prediction value, which, given the observed data, is the most probable value of Y under a model that best fits the observed data as well as the prediction value. In principal, any functional of the conditional density of Y given X could be used to derive a prediction value of Y but the choice of the mode has the appealing advantage of allowing to formally extend the maximum likelihood method from the parametric to the predictive inference setup. Next, we have that

$$L_{obs}(y, \theta|x) = f_\theta^X(x) \frac{f_\theta^{Y|X}(y|x)}{f_\theta^Y(y)},$$

which shows that the maximum observed likelihood prediction procedure excludes the variability in Y from consideration and effectively turns the prediction problem into an estimation problem for the model $f_\theta^{X|Y}(x|y)$, $\theta \in \Theta$, $y > x_p$. In this model the variability of the observed data depends on the value of the quantity of interest, that is Y, which allows to draw inference on Y purely within the classical maximum likelihood framework that is to perform optimization with respect to quantities that are model parameters. Alternatively, the above representation can be interpreted to suggest that the prediction procedure yields a prediction value, which is associated with the highest relative increase in the conditional density of Y given X compared to the unconditional density of Y.
In contrast to the MOLP, a general expression such as in (1.1) does not seem to exist for the MLP. Moreover, from expression (1.1), we find when predicting the very next record value ($s = r + 1$) that the MOLP becomes trivial in the sense that the last observed record value serves as predictor for the next one.
We examine the MLP and the MOLP of future Weibull record values, derive representations and compare their performance via the mean squared error and the Pitman closeness criterion. A predictor π_1 of R_s is said to be Pitman closer to Y than a predictor π_2 if

$$P(|\pi_1 - R_s| < |\pi_2 - R_s|) > \frac{1}{2}, \tag{1.4}$$

and, if (1.4) holds, π_1 is said to be preferable to π_2 in Pitman closeness sense.

2. LIKELIHOOD-BASED PREDICTORS FOR WEIBULL RECORD VALUES

Let $(R_n)_{n\in\mathbb{N}}$ be the sequence of Weibull record values. The density, cumulative distribution and quantile functions of the two-parameter Weibull distribution $\mathcal{W}eibull(\sigma, p)$ with scale parameter $\sigma \in \mathbb{R}_+$ and shape parameter $p \in \mathbb{R}_+$ are given by

$$\begin{aligned} f_\theta(x) &= \frac{p}{\sigma}\left(\frac{x}{\sigma}\right)^{p-1} \exp\left\{-\left(\frac{x}{\sigma}\right)^p\right\}, \quad x \in \mathbb{R}_+, \\ F_\theta(x) &= 1 - \exp\left\{-\left(\frac{x}{\sigma}\right)^p\right\}, \quad x \in \mathbb{R}_+, \\ F_\theta^{-1}(x) &= \sigma(-\ln(1-x))^{\frac{1}{p}}, \quad x \in [0, 1), \end{aligned} \tag{2.1}$$

where $\theta = (\sigma, p) \in \mathbb{R}_+^2$ is the vector of the distributional parameters. For $r, s \in \mathbb{N}$, $r < s-1$, we derive the MOLP as well as the MLP of the future record R_s based on $R_\star = (R_1, \ldots, R_r)$. The density functions of the distribution of $R_\star$ as well as of the conditional distribution of R_s given $R_r = r_r$, $r_r \in (-\infty, \omega(F_\theta))$, can be stated in terms of f_θ and F_θ as follows (see [2]):

(2.2)
$$f_\theta^{R_\star}(r_1, \ldots, r_r) = \left(\prod_{i=1}^{r-1} \frac{f_\theta(r_i)}{1 - F_\theta(r_i)} \right) f_\theta(r_r) \mathbb{1}_{[\alpha(F_\theta), \omega(F_\theta))^r}(r_1, \ldots, r_r), \qquad (r_1, \ldots, r_r) \in \mathbb{R}_<^r,$$

(2.3)
$$f_\theta^{R_s|R_r}(r_s|r_r) = \frac{1}{(s-r-1)!} \frac{f_\theta(r_s)}{1 - F_\theta(r_r)} \left(-\ln\left(\frac{1 - F_\theta(r_s)}{1 - F_\theta(r_r)} \right) \right)^{s-r-1} \mathbb{1}_{(r_r, \omega(F_\theta))}(r_s), \qquad r_s \in \mathbb{R}.$$

The MOLP in the Weibull case can be explicitly stated.

Proposition 2.1. For $s \geq 3$, let $R_1, \ldots, R_s$ be the first s Weibull record values. For $r \in \mathbb{N}$, $2 \leq r < s-1$, the unique MOLP of R_s and the PMOLE of p based on $R_\star$ are given by

$$\pi_{MOLP}^{(s)} = \left(\frac{s-1}{r} \right)^{1/\hat{p}_{MOL}} R_r \quad \text{and} \quad \hat{p}_{MOL} = -\frac{r}{\ln\left(\prod_{i=1}^r \frac{R_i}{R_r} \right)}.$$

Proof: With f_θ and F_θ as above, the function $\Psi(\cdot|r_\star)$, $r_\star = (r_1, \ldots, r_r) \in (0, \infty)_<^r$, in (1.2) reads

$$\Psi(\theta|r_\star) = p^r \left(\prod_{i=1}^r \frac{r_i}{r_r} \right)^{p-1} \frac{1}{r_r^r}, \quad \sigma \in \mathbb{R}_+, \ p \in \mathbb{R}_+. \tag{2.4}$$

The function Ψ does not depend on the scale parameter σ, thus, we only need to find a maximizing function with respect to p. Let

$$\hat{\theta}(r_\star) = \left(\hat{\sigma}(r_\star), -r/\ln\left(\prod_{i=1}^r r_i/r_r \right) \right),$$

where $\hat{\beta}$ is an arbitrary measurable function on $\mathbb{R}_<^r$ with values in $\mathbb{R}_+$. Then, $\hat{\theta}$ satisfies (1.2) with $\Psi(\cdot|r_\star)$ given by (2.4). Together with

$$F_\theta^{-1}\left(1 - (1 - F_\theta(R_r))^{\frac{s-1}{r}} \right) = \left(\frac{s-1}{r} \right)^{\frac{1}{p}} R_r,$$

we find the stated form of the MOLP. □

Remark 2.2. (i) The PMOLE and the MLE of p coincide. For the MLEs of σ and p we refer to [14].

(ii) The MOLP can also be written as $\pi^{(s)}_{MOLP} = (s-1)^{1/\hat{p}}\hat{\sigma}$, where $\hat{p}$ and $\hat{\sigma}$ are the MLEs of p and σ, respectively.

The maximum likelihood predictor of a future Weibull record value was derived in [16] (see also [21, Section 5.3.5]). The respective result is contained in the following theorem.

Proposition 2.3. For $s \geq 3$, let $R_1, \ldots, R_s$ be the first s Weibull record values. For $r \in \mathbb{N}$, $2 \leq r < s-1$, the unique MLP of R_s based on $R_\star$ is given by

$$\pi^{(s)}_{MLP} = s^{-1/\hat{p}_{ML}}\hat{\sigma}_{ML}.$$

Here, $\hat{\sigma}_{ML}$ and $\hat{p}_{ML}$ are the PMLEs of σ and p.
The PMLE of σ takes the form

$$\hat{\sigma}_{ML} = \left(\frac{s+1/\hat{p}_{ML}-1}{s(r+1/\hat{p}_{ML})}\right)^{1/\hat{p}_{ML}} R_r,$$

while the PMLE of p is obtained as the unique positive solution of

$$p^2 \ln\left(\prod_{i=1}^{r-1}\frac{R_i}{R_r}\right) + (r+1)p = \ln\left(\frac{r+1/p}{s+1/p-1}\right)$$

with respect to $p \in \mathbb{R}_+$. For $s = r+1$, the MLP takes the form $\pi^{(s)}_{MLP} = R_r$.

The following remark collects, in particular, some results concerning the existence of the MOLP and the MLP in the case of the three-parameter Weibull distribution, where, in (2.1), x is replaced by $x-\mu$ for some location parameter μ.

Remark 2.4. (i) It is straightforward to see that the MLP can also be expressed in the form

$$\pi^{(s)}_{MLP} = \left(\frac{s+1/\hat{p}_{ML}-1}{r+1/\hat{p}_{ML}}\right)^{1/\hat{p}_{ML}} R_r.$$

(ii) In case the underlying distribution depends on an unknown location parameter $\mu \in \mathbb{R}$, neither the MLP nor the MOLP exists. Indeed, consider first the derivation of the MOLP. Then, for $r_\star = (r_1, \ldots, r_r) \in \mathbb{R}^r_<$, we want to determine the global maximum of the function

$$\Psi(\mu, p|r_\star) = p^r \left(\prod_{i=1}^{r-1}\frac{r_i-\mu}{r_r-\mu}\right)^{p-1} \frac{1}{(r_r-\mu)^r}, \quad (\mu, \sigma) \in (-\infty, r_1) \times \mathbb{R}_+.$$

We have

$$\Psi\left(\mu, -r/\ln\left(\prod_{i=1}^{r-1}\frac{r_i-\mu}{r_r-\mu}\right)\middle| r_\star\right) \sim h\left(-\ln\left(\prod_{i=1}^{r-1}\frac{r_i-\mu}{r_r-\mu}\right)\right)\frac{r^r e^{-r}}{(r_r-r_1)^r},$$

as $\mu \underset{\mu<r_1}{\to} r_1$,, where $h(x) = e^x/x^r$, $x \in \mathbb{R}_+$. Since $\lim\limits_{x\to\infty} h(x) = \infty$,

$$\lim_{\substack{\mu\to r_1\\ \mu<r_1}} \Psi\left(\mu, -r/\ln\left(\prod_{i=1}^{r-1}\frac{r_i-\mu}{r_r-\mu}\right)\middle| r_\star\right) = \infty.$$

Hence, function Ψ does not possess a finite global maximum.

Next, consider the derivation of the MLP. There, for $(r_1, \ldots, r_r) \in \mathbb{R}^r_<$, we want, in particular, to maximize the function

$$G(\mu, p) = p^{r+1}\left(\prod_{i=1}^{r-1}\frac{r_i-\mu}{r_r-\mu}\right)^{p-1}\frac{1}{(r_r-\mu)^{r+1}}\frac{(r+1/p)^{r+1/p}}{(s+1/p-1)^{s+1/p-1}},$$
$$(\mu,\sigma) \in (-\infty, r_1)\times\mathbb{R}_+.$$

Since

$$\frac{(r+1/p)^{r+1/p}}{(s+1/p-1)^{s+1/p-1}} = \exp\left\{-(s-r-1)\ln(s+1/p-1) - \frac{(s-r-1)}{rp+1} + o(1)\right\},$$

as $p \to 0$, we have that

$$G\left(\mu, -r/\ln\left(\prod_{i=1}^{r-1}\frac{r_i-\mu}{r_r-\mu}\right)\right) \sim g\left(-\ln\left(\prod_{i=1}^{r-1}\frac{r_i-\mu}{r_r-\mu}\right)\right)\frac{r^{r+1}e^{-(s-1)}}{(r_r-r_1)^{r+1}},$$

as $\mu \underset{\mu<r_1}{\to} r_1$, where $g(x) = e^{x\left(1-\frac{(s-r-1)\ln(x(\frac{1}{r+1}+\frac{s-1}{x}))}{x}\right)}/x^{r+1}$, $x \in \mathbb{R}_+$. Since $\lim\limits_{x\to\infty} g(x) = \infty$, we conclude that

$$\lim_{\substack{\mu\to r_1\\ \mu<r_1}} G\left(\mu, -r/\ln\left(\prod_{i=1}^{r-1}\frac{r_i-\mu}{r_r-\mu}\right)\right) = \infty.$$

Hence, function G does not possess a finite global maximum.

3. EVALUATION IN TERMS OF THE BIAS AND THE MSE

In the following, $\mathcal{G}amma(a,b)$, $a, b \in \mathbb{R}_+$, denotes the gamma distribution with parameters a, b with density function $f(x) = b^a x^{a-1}\exp\{-bx\}/\Gamma(a)$, $x > 0$, where $\Gamma(a)$ is the gamma function evaluated at a.

Lemma 3.1. For $s \geq 3$, let $R_1, \ldots, R_s$ be the first s Weibull record values. For $r \in \mathbb{N}$, $2 \leq r < s-1$, the bias of the MOLP of R_s based on $R_\star$ is finite if and only if $\frac{1}{r}\ln\left(\frac{s-1}{r}\right) < p$, in which case it is given by

$$\mathrm{E}(R_s - \pi^{(s)}_{MOLP}) = \sigma\frac{\Gamma\left(s+\frac{1}{p}\right)}{\Gamma(s)}\left(1 - \frac{\prod_{i=r}^{s-1}\left(1+\frac{1}{pi}\right)^{-1}}{\left(1-\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}}\right).$$

If $\frac{1}{r}\ln\left(\frac{s-1}{r}\right) \geq p$, then $\mathrm{E}(R_s - \pi^{(s)}_{MOLP}) = -\infty$.

Proof: To prove the statement, we derive the expression for the expectation of $\pi^{(s)}_{MOLP}$ and use that the integral is linear if one of the integrand functions is integrable (cf. [17, p. 135]). By [14, p. 42], R_r and $\hat{p}_{MOL}$ are independent and $pr/\hat{p}_{MOL} \sim \mathcal{G}amma(r-1,1)$. Using results in [2, section 2.7.1], we conclude that

$$\begin{aligned}
&\mathrm{E}(\pi^{(s)}_{MOLP})\\
&\quad= \mathrm{E}\left(\left(\frac{s-1}{r}\right)^{1/\hat{p}_{MOL}} R_r\right) = \mathrm{E}(R_r)\mathrm{E}\left(\left(\frac{s-1}{r}\right)^{1/\hat{p}_{MOL}}\right)\\
&\quad= \mathrm{E}(R_r)\mathrm{E}\left(\exp\left\{\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\frac{pr}{\hat{p}_{MOL}}\right\}\right) = \mathrm{E}(R_r)\frac{1}{\left(1-\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}}\\
&\quad= \sigma\frac{\Gamma\left(r+\frac{1}{p}\right)}{\Gamma(r)}\frac{1}{\left(1-\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}} = \sigma\frac{\Gamma\left(s+\frac{1}{p}\right)}{\Gamma(s)}\frac{\prod_{i=r}^{s-1}(1+\frac{1}{pi})^{-1}}{\left(1-\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}}\\
&\quad= \mathrm{E}(R_s)\frac{\prod_{i=r}^{s-1}(1+\frac{1}{pi})^{-1}}{\left(1-\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}},
\end{aligned}$$

where in the fourth equality we used the expression for the moment generating function of the $\mathcal{G}amma(r-1,1)$ distribution to evaluate $E\left(\exp\left\{\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\frac{pr}{\hat{p}}\right\}\right)$, which is finite if and only if $\frac{1}{pr}\ln\left(\frac{s-1}{r}\right) < 1$, as well as the fact that $\Gamma(x+1) = \Gamma(x)x$, $x \in \mathbb{R}_+$. Now, linearity of the integral yields the desired conclusion. □

Lemma 3.2. For $s \geq 3$, let $R_1, \ldots, R_s$ be the first s Weibull record values. For $r \in \mathbb{N}$, $2 \leq r < s-1$, the MSE of the MOLP of R_s based on $R_\star$ is finite if and only if $\frac{2}{r}\ln(\frac{s-1}{r}) < p$, in which case it is given by

$$\mathrm{MSE}(\pi^{(s)}_{MOLP}) = \sigma^2\frac{\Gamma(s+\frac{2}{p})}{\Gamma(s)}\left(1 - 2\frac{\prod\limits_{i=r}^{s-1}(1+\frac{1}{pi})^{-1}}{\left(1-\frac{1}{pr}\ln(\frac{s-1}{r})\right)^{r-1}} + \frac{\prod\limits_{i=r}^{s-1}(1+\frac{2}{pi})^{-1}}{\left(1-\frac{2}{pr}\ln(\frac{s-1}{r})\right)^{r-1}}\right).$$

Proof: To prove the statement, we use that

$$(R_s - \pi_{MOLP}^{(s)})^2 = R_s^2 - 2R_sR_r\left(\frac{s-1}{r}\right)^{1/\hat{p}_{MOL}} + (\pi_{MOLP}^{(s)})^2$$

as well as the fact that the integral is linear if the integrand can be written as a sum of an integrable and a quasi-integrable function (cf. [17, p. 135]). By [2, Theorem 3.3.1], we have

$$\begin{aligned}\mathrm{E}(R_s^2) &= \sigma^2\frac{\Gamma(s+\frac{2}{p})}{\Gamma(s)},\\ \mathrm{E}(R_rR_s) &= \sigma^2\frac{\Gamma(s+\frac{2}{p})}{\Gamma(s)}\prod_{i=r}^{s-1}\left(1+\frac{1}{pi}\right)^{-1},\end{aligned}$$

and a similar argument as in the proof of Lemma 3.1 yields

$$\begin{aligned}\mathrm{E}((\pi_{MOLP}^{(s)})^2) &= \begin{cases}\sigma^2\frac{\Gamma(r+\frac{2}{p})}{\Gamma(r)}\frac{1}{\left(1-\frac{2}{pr}\ln(\frac{s-1}{r})\right)^{r-1}} & \frac{2}{r}\ln\left(\frac{s-1}{r}\right)<p\\ \infty & \frac{2}{r}\ln\left(\frac{s-1}{r}\right)\geq p\end{cases}\\ &= \begin{cases}\sigma^2\frac{\Gamma(s+\frac{2}{p})}{\Gamma(s)}\frac{\prod_{i=r}^{s-1}(1+\frac{2}{pi})^{-1}}{\left(1-\frac{2}{pr}\ln(\frac{s-1}{r})\right)^{r-1}} & \frac{2}{r}\ln\left(\frac{s-1}{r}\right)<p\\ \infty & \frac{2}{r}\ln\left(\frac{s-1}{r}\right)\geq p\end{cases}.\end{aligned}$$

Combining these results, we conclude that

$$\begin{aligned}&\mathrm{MSE}(\pi_{MOLP}^{(s)})\\ &\quad= \mathrm{E}((R_s-\pi_{MOLP}^{(s)})^2)\\ &\quad= \begin{cases}\sigma^2\frac{\Gamma(s+\frac{2}{p})}{\Gamma(s)}\left(1-2\frac{\prod_{i=r}^{s-1}(1+\frac{1}{pi})^{-1}}{\left(1-\frac{1}{pr}\ln(\frac{s-1}{r})\right)^{r-1}}+\frac{\prod_{i=r}^{s-1}(1+\frac{2}{pi})^{-1}}{\left(1-\frac{2}{pr}\ln(\frac{s-1}{r})\right)^{r-1}}\right) & \frac{2}{r}\ln\left(\frac{s-1}{r}\right)<p\\ \infty & 1<\frac{pr}{\ln(\frac{s-1}{r})}\leq 2\end{cases}.\end{aligned}$$

Finally, it remains to show that $\mathrm{MSE}(\pi_{MOLP}^{(s)})=\infty$ for $p\leq\frac{1}{r}\ln(\frac{s-1}{r})$. Lemma 3.1 implies that $\mathrm{E}(|R_s-\pi_{MOLP}^{(s)}|)=\infty$ for $p\leq\frac{1}{r}\ln(\frac{s-1}{r})$. By the well-known embedding theorem for Lebesgue spaces (cf. [17, Example 8.4.9 (2)]), we find $\mathrm{E}((R_s-\pi_{MOLP}^{(s)})^2)=\infty$ for $p\leq\frac{1}{r}\ln(\frac{s-1}{r})$. □

Table 1 contain the biases and MSEs of the MLP (estimated from 10^7 Monte Carlo replications) and the MOLP for various values of r, s and p, and with $\sigma=1$. Results in boldface represent all best results in terms of the MSE among the prediction methods, provided the best result is achieved by the MLP. The simulation results indicate that the MOLP exhibits superior performance based on the MSE in most cases. There are a few exceptions though, which suggest that the MLP has a lower MSE in cases when p, r are small ($p=0.5$, $r=3$) and $s=r+2$.

It should be noted that the MSE of the MOLP can become large (or even infinite) for small values of p in $(0,1)$, small values of r and a higher gap between r and s (see Table 1). This is due to the fact that, in theses cases, $(2/r)\ln((s-1)/r)$ is close to p from below (or exceeds p), which yields large (or infinite) MSEs by means of Lemma 3.2. However, the situation of a small r combined with a large gap between r and s is not meaningful in practice. Moreover, one can observe that the difference in performance becomes smaller as the sample size increases.

4. COMPARISON IN TERMS OF PITMAN'S MEASURE OF CLOSENESS

Since the MLP $\pi^{(s)}_{MLP}$ of R_s based on $R_\star$ is not given in closed form, we are not able to derive an analytic expression for the Pitman efficiency of the MOLP $\pi^{(s)}_{MOLP}$ relative to $\pi^{(s)}_{MLP}$. We therefore aim at establishing a lower bound on the Pitman efficiency. The following lemmas are required to establish the desired result.

Lemma 4.1. For $s \geq 3$, let $R_1,\ldots,R_s$ be the first s Weibull record values. For $r \in \mathbb{N}$, $2 \leq r < s-1$, the MOLP of R_s is always greater than the MLP of R_s.

Proof: Indeed, we know that, for $(r_1,\ldots,r_r) \in (0,\infty)^r_<$, the PMLE of p satisfies the equation

$$p\left(p\ln\left(\prod_{i=1}^{r-1}\frac{r_i}{r_r}\right)+r+1\right)=\ln\left(\frac{r+1/p}{s+1/p-1}\right).$$

Note that $\ln\left(\frac{r+1/p}{s+1/p-1}\right)<0$, $p\in\mathbb{R}_+$. Consequently, since $\ln(\prod_{i=1}^{r-1} r_i/r_r)<0$, the solution of the above equation is always greater than $-(r+1)/\ln(\prod_{i=1}^{r-1} r_i/r_r)$ $(=\frac{r+1}{r}\hat{p}_{MOL})$. Now, for $\alpha,\ \beta\in\mathbb{R}$, $0<\alpha<\beta$, consider the functions

$$f_{\alpha,\beta}(t)=\frac{\beta+t}{\alpha+t},\quad g_{\alpha,\beta}(t)=f_{\alpha,\beta}(t)^t,\quad t\in(-\alpha,\infty).$$

Differentiating $f_{\alpha,\beta}$ and $g_{\alpha,\beta}$ yields

$$f'_{\alpha,\beta}(t)=-\frac{\beta-\alpha}{(\alpha+t)^2},\quad g'_{\alpha,\beta}(t)=f_{\alpha,\beta}(t)\left(\ln(f_{\alpha,\beta}(t))+t\frac{f'_{\alpha,\beta}(t)}{f_{\alpha,\beta}(t)}\right),\quad t\in(-\alpha,\infty).$$

Obviously, $f'_{\alpha,\beta}(t)<0$, $t\in(-\alpha,\infty)$. Hence, $f_{\alpha,\beta}$ is a strictly decreasing function.

$r\backslash s$	Predictor	$r+2$								$r+3$							
		$p=0.5$		$p=1.5$		$p=2$		$p=2.5$		$p=0.5$		$p=1.5$		$p=2$		$p=2.5$	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
3	MLP	15.299	579.159	0.679	0.808	0.408	0.292	0.287	0.145	24.711	**1,309.337**	0.914	1.363	0.534	0.471	0.369	0.228
	MOLP	11.629	569.880	0.571	0.730	0.348	0.264	0.246	0.132	14.406	2,590.372	0.689	1.257	0.414	0.427	0.290	0.205
5	MLP	19.869	984.181	0.524	0.548	0.292	0.171	0.196	0.077	29.635	**2,008.898**	0.670	0.881	0.365	0.266	0.241	0.118
	MOLP	15.387	885.968	0.446	0.496	0.251	0.155	0.170	0.071	18.515	2,042.598	0.508	0.787	0.284	0.237	0.190	0.105
10	MLP	30.423	2,409.340	0.372	0.323	0.186	0.082	0.117	0.033	40.279	4,281.257	0.439	0.486	0.217	0.120	0.135	0.047
	MOLP	25.197	2,169.620	0.332	0.300	0.168	0.076	0.106	0.031	28.332	3,842.880	0.357	0.441	0.180	0.110	0.113	0.043
15	MLP	40.638	4,442.341	0.309	0.239	0.145	0.053	0.088	0.020	50.312	7,340.106	0.350	0.348	0.163	0.076	0.098	0.028
	MOLP	35.132	4,065.112	0.283	0.226	0.134	0.051	0.082	0.019	38.237	6,601.259	0.298	0.322	0.141	0.071	0.086	0.026
20	MLP	50.727	7,064.397	0.273	0.194	0.123	0.039	0.072	0.014	60.330	11,192.520	0.302	0.277	0.134	0.056	0.079	0.019
	MOLP	45.099	6,562.983	0.254	0.185	0.115	0.038	0.068	0.013	48.183	10,183.050	0.265	0.260	0.119	0.052	0.071	0.018

$r\backslash s$	Predictor	$r+4$								$r+5$							
		$p=0.5$		$p=1.5$		$p=2$		$p=2.5$		$p=0.5$		$p=1.5$		$p=2$		$p=2.5$	
		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
3	MLP	36.179	**2,526.103**	1.145	2.028	0.655	0.676	0.447	0.320	49.757	–	1.372	2.801	0.771	0.903	0.520	0.419
	MOLP	14.526	60,224.680	0.799	1.990	0.475	0.632	0.331	0.293	8.623	–	0.901	2.981	0.532	0.880	0.369	0.394
5	MLP	41.340	**3,612.705**	0.817	1.276	0.437	0.373	0.285	0.162	55.020	**5,967.848**	0.964	1.733	0.507	0.492	0.328	0.211
	MOLP	20.992	4,742.856	0.568	1.161	0.315	0.335	0.210	0.145	22.353	11,142.580	0.626	1.623	0.344	0.449	0.229	0.190
10	MLP	51.544	6,913.748	0.508	0.672	0.247	0.163	0.153	0.063	64.231	10,459.760	0.579	0.883	0.278	0.209	0.170	0.080
	MOLP	31.323	6,464.899	0.382	0.608	0.191	0.148	0.120	0.057	34.083	10,444.290	0.407	0.802	0.203	0.190	0.127	0.073
15	MLP	61.189	11,164.670	0.392	0.469	0.181	0.101	0.108	0.037	73.097	16,080.470	0.435	0.604	0.198	0.128	0.118	0.046
	MOLP	41.279	10,142.230	0.313	0.432	0.147	0.094	0.089	0.034	44.221	14,965.170	0.328	0.555	0.154	0.118	0.093	0.042
20	MLP	70.786	16,400.880	0.331	0.368	0.146	0.073	0.085	0.025	82.231	22,872.340	0.360	0.467	0.158	0.092	0.091	0.031
	MOLP	51.232	14,927.090	0.275	0.343	0.124	0.068	0.073	0.024	54.225	21,027.620	0.285	0.434	0.128	0.085	0.075	0.029

Table 1: Values of the biases and MSEs of the MLP (estimated from 10^7 Monte Carlo replications) and the MOLP of R_s based on $\mathcal{Weibull}(1,p)$ record values $R_1,\ldots,R_r$ for selected r and $s\in\{r+2,r+3,r+4,r+5\}$ for $p\in\{0.5,1.5,2,2.5\}$. Boldface: MSEs of MLP provided it is minimal among the MSEs of both predictors.

Furthermore, for $t \in (-\alpha, \infty)$,

$$\begin{aligned}\ln(f_{\alpha,\beta}(t)) + t\frac{f'_{\alpha,\beta}(t)}{f_{\alpha,\beta}(t)} &= \ln\left(1 + \frac{\beta-\alpha}{\alpha+t}\right) + t\frac{\alpha-\beta}{(\alpha+t)(\beta+t)} \\ &> \frac{\beta-\alpha}{\beta+t} + t\frac{\alpha-\beta}{(\alpha+t)(\beta+t)} \\ &= \frac{\alpha(\beta-\alpha)}{(\alpha+t)(\beta+t)} > 0,\end{aligned}$$

where we used the inequality $x/(x+1) < \ln(1+x)$, for $x > -1$, $x \neq 0$. Thus, $g_{\alpha,\beta}$ is a strictly increasing function. Using the preceding results, we obtain

$$\begin{aligned}\pi^{(s)}_{MLP} &= g_{r,s-1}(1/\hat{p}_{ML})R_r \\ &< g_{r,s-1}(1/\hat{p}_{MOL})R_r \\ &= f_{r,s-1}(1/\hat{p}_{MOL})^{1/\hat{p}_{MOL}}R_r \\ &< f_{r,s-1}(0)^{1/\hat{p}_{MOL}}R_r = \pi^{(s)}_{MOLP}.\end{aligned}$$

□

Lemma 4.2. For $s \geq 3$, let $R_1, \ldots, R_s$ be the first s Weibull record values. For $r \in \mathbb{N}$, $2 \leq r < s-1$, the probability of R_s exceeding its MOLP based on $R_\star$ is given by

$$P(\pi^{(s)}_{MOLP} < R_s) = \sum_{j=r}^{s-1}\sum_{i=0}^{s-1-j}(-1)^i\binom{s-1}{j}\binom{s-1-j}{i}\frac{1}{\left(1 + \frac{i+j}{r}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}}.$$

In particular, this probability is independent of the distributional parameters σ and p.

Proof: Observe that

$$\pi^{(s)}_{MOLP} < R_s \quad \Longleftrightarrow \quad \frac{s-1}{r} < \left(\frac{R_s}{R_r}\right)^{\hat{p}_{MOL}}.$$

Let G denote the cumulative distribution function of $(R_s/R_r)^{1/\hat{p}_{MOL}}$. By the results in [14, section 4], G admits the representation

$$G(t) = \int_0^\infty H\left(\frac{r}{s-r}\left(t^{\frac{z}{2r}} - 1\right)\Big|2(s-r), 2r\right) f_{\chi^2}(z|2(r-1))dz, \quad t \in (1,\infty).$$

Here, for $n, m \in \mathbb{N}$, $H(\cdot|n,m)$ denotes the cumulative distribution function of the F distribution with parameters n and m, and $f_{\chi^2}(\cdot|n)$ denotes the density function of the χ^2 distribution with parameter n. First, note that

$$H\left(\frac{r}{s-r}\left(t^{\frac{z}{2r}} - 1\right)\Big|2(s-r), 2r\right) = I_{1-t^{-\frac{z}{2r}}}(s-r, r) = 1 - I_{t^{-\frac{z}{2r}}}(r, s-r).$$

Consequently,

$$P(\pi^{(s)}_{MOLP} < R_s) = 1 - G\left(\frac{s-1}{r}\right) = \int\limits_0^\infty I_{\left(\frac{r}{s-1}\right)^{\frac{z}{2r}}}(r, s-r) f_{\chi^2}(z|2(r-1))dz.$$

Furthermore, since the parameters of the regularized incomplete beta function are integers, we have, by the relation of the regularized incomplete beta function to the binomial expansion (see [8, (6.6.4)]),

$$\begin{aligned} I_x(r, s-r) &= \sum_{j=r}^{s-1} \binom{s-1}{j} x^j (1-x)^{s-1-j} \\ &= \sum_{j=r}^{s-1} \sum_{i=0}^{s-1-j} (-1)^i \binom{s-1}{j} \binom{s-1-j}{i} x^{i+j}, \quad x \in (0,1). \end{aligned}$$

From the preceding results we infer that

$$\begin{aligned} &P(\pi^{(s)}_{MOLP} < R_s) \\ &\quad = \int\limits_0^\infty I_{\left(\frac{r}{s-1}\right)^{\frac{z}{2r}}}(r, s-r) f_{\chi^2}(z|2(r-1))dz \\ &\quad = \sum_{j=r}^{s-1} \sum_{i=0}^{s-1-j} (-1)^i \binom{s-1}{j} \binom{s-1-j}{i} \int_0^\infty \left(\frac{r}{s-1}\right)^{\frac{i+j}{2r} z} f_{\chi^2}(z|2(r-1))dz \\ &\quad = \sum_{j=r}^{s-1} \sum_{i=0}^{s-1-j} (-1)^i \binom{s-1}{j} \binom{s-1-j}{i} \int_0^\infty e^{\frac{i+j}{2r} \ln\left(\frac{r}{s-1}\right) z} f_{\chi^2}(z|2(r-1))dz \\ &\quad = \sum_{j=r}^{s-1} \sum_{i=0}^{s-1-j} (-1)^i \binom{s-1}{j} \binom{s-1-j}{i} \frac{1}{\left(1 + \frac{i+j}{r} \ln\left(\frac{s-1}{r}\right)\right)^{r-1}}, \end{aligned}$$

where in the last equality we used the expression for the moment generating function of the $\chi^2(2(r-1))$ distribution to evaluate the integrals $\int_0^\infty e^{\frac{i+j}{2r} \ln\left(\frac{r}{s-1}\right) z} f_{\chi^2}(z|2(r-1))dz$. This concludes the proof. □

Remark 4.3. The proof of Lemma 4.2 yields a finite sum representation of the cumulative distribution function G of $(R_s/R_r)^{1/\hat{p}_{MOL}}$:

$$G(t) = 1 - \sum_{j=r}^{s-1} \sum_{i=0}^{s-1-j} (-1)^i \binom{s-1}{j} \binom{s-1-j}{i} \frac{1}{\left(1 + \frac{i+j}{r} \ln(t)\right)^{r-1}}, \quad t \in (1, \infty).$$

By exploiting the presence of alternating binomial sums in the above representation, a more compact representation of G can be obtained. More precisely, we have that

$$G(t) = 1 - \sum_{j=r}^{s-1} (-1)^{s-1-j} \binom{s-1}{j} \Delta^{s-1-j} f_{r,j,t}, \quad t \in (1, \infty),$$

where

$$f_{r,j,t}(i) = \frac{1}{\left(1 + \frac{i+j}{r}\ln(t)\right)^{r-1}}, \quad 0 \le i \le s-1-j,$$

and, for $j = r, \ldots, s-1$, the $(s-1-j)$-th forward difference is computed for $i = 0$. Using this finite sum representation allows to avoid applying numeric integration for evaluation of G (cf. [14, section 6]). Since alternating sums can be numerically problematic, for an efficient and accurate implementation of G, it is advisable to use high precision arithmetic. See `sumBinomMpfr()` in `R` package `Rmpfr` and its documentation.

Proposition 4.4. For $s \ge 3$, let $R_1, \ldots, R_s$ be the first s Weibull record values. For $r \in \mathbb{N}$, $2 \le r < s-1$, let $\pi^{(s)}_{MOLP}$ and $\pi^{(s)}_{MLP}$ be the MOLP and the MLP of R_s based on $R_\star$, respectively. Then

$$P(|\pi^{(s)}_{MOLP} - R_s| < |\pi^{(s)}_{MLP} - R_s|) > \sum_{j=r}^{s-1} \sum_{i=0}^{s-1-j} (-1)^i \binom{s-1}{j} \binom{s-1-j}{i} \frac{1}{\left(1 + \frac{i+j}{r}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}}.$$

Proof: Due to Lemma 4.1,

$$P(|\pi^{(s)}_{MOLP} - R_s| < |\pi^{(s)}_{MLP} - R_s|) > P(R_s > \pi^{(s)}_{MOLP}).$$

Hence, Lemma 4.2 yields the desired result. □

Figure 1 contains the contour plots of the lower bound on the Pitman efficiency $\text{PE}(\text{MOLP}, \text{MLP}) = P\left(|R_s - \pi^{(s)}_{MOLP}| < |R_s - \pi^{(s)}_{MLP}|\right)$ of the MOLP of R_s relative to the MLP of R_s based on $R_\star$ for r, s such that $2 \le r \le 20$ and $r+1 < s \le r+10$. Table 2 contains values of the lower bound on as well as estimated Pitman efficiencies for selected r and s, and, in the case of estimated Pitman efficiencies, for shape parameter values $p = 0.5, 1.5, 2, 2.5$. Observe that while the lower bound on the Pitman efficiencies does not depend on the distributional parameters, the Pitman efficiencies do depend on the shape parameter p. Each estimated Pitman efficiency was computed based on 10^6 simulated samples of Weibull record values. From the contour plot of the lower bound on the Pitman efficiency, the MOLP seems to be superior to the MLP in terms of Pitman closeness for r, s such that $2 \le r \le 20$ and $r+1 < s \le r+10$. The estimated Pitman efficiencies presented in Table 2 as well as additional simulation results suggest that for fixed r and s the Pitman efficiency is a decreasing function of p. Furthermore, the simulation results indicate that the lower bound from Proposition 4.4 is the tighter, the bigger r and the smaller $s-r$ are.
The superior performance of the MOLP in terms of the Pitman efficiency compared to the MLP even for small values of r and $p < 1$, for which the MOLP

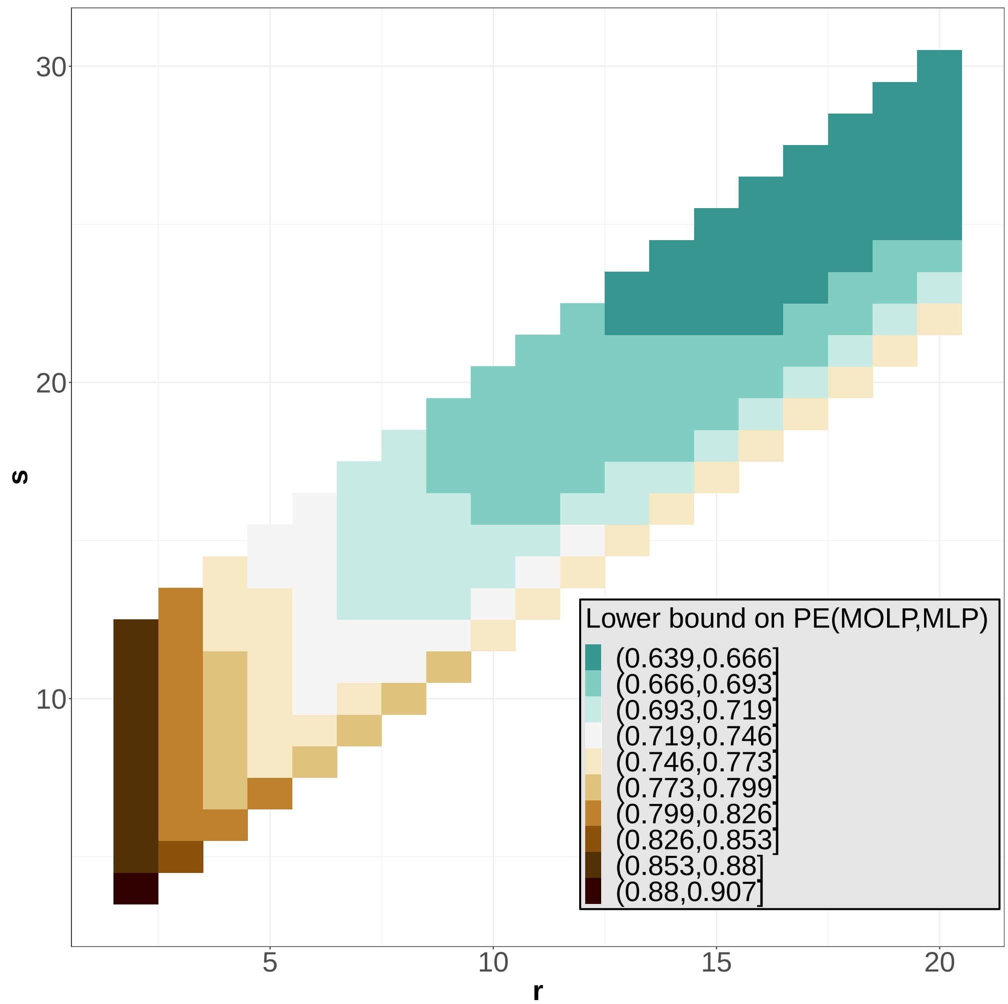

Figure 1: Contour plot of the lower bound on the Pitman efficiency PE(MOLP, MLP) $= P(|R_s - \pi^{(s)}_{MOLP}| < |R_s - \pi^{(s)}_{MLP}|)$ of the MOLP of R_s relative to the MLP of R_s based on Weibull record values $R_1, \ldots, R_r$ for r, s such that $2 \leq r \leq 20$ and $r + 1 < s \leq r + 10$.

performs poorly if evaluated in terms of the MSE (see Table 1) is in line with the intuition underlying the Pitman criterion: namely being only affected by the bias and not accounting for the variability in the predictors. Note also that according to Table 1, despite MOLP's inferior performance in terms of the MSE, it still has a lower bias than the MLP, which supports its superior performance in terms of the Pitman criterion as evidenced by the values in Table 2.

r \ s	p	$r+2$	$r+3$	$r+4$	$r+5$	$r+10$
2		0.906	0.875	0.877	0.874	0.873
	0.5	0.927	0.920	0.917	0.915	0.909
	1.5	0.923	0.918	0.918	0.918	0.920
	2	0.921	0.916	0.917	0.917	0.920
	2.5	0.920	0.915	0.916	0.916	0.920
5		0.805	0.771	0.759	0.754	0.745
	0.5	0.852	0.836	0.832	0.831	0.831
	1.5	0.838	0.818	0.812	0.812	0.818
	2	0.836	0.814	0.809	0.808	0.812
	2.5	0.834	0.813	0.807	0.805	0.809
10		0.771	0.727	0.708	0.698	0.681
	0.5	0.807	0.778	0.770	0.766	0.769
	1.5	0.794	0.760	0.746	0.742	0.739
	2	0.792	0.757	0.743	0.738	0.734
	2.5	0.790	0.755	0.742	0.736	0.731
15		0.760	0.711	0.689	0.677	0.654
	0.5	0.788	0.751	0.738	0.732	0.729
	1.5	0.776	0.735	0.718	0.710	0.701
	2	0.775	0.733	0.716	0.707	0.696
	2.5	0.773	0.731	0.713	0.705	0.693
20		0.754	0.703	0.679	0.665	0.639
	0.5	0.776	0.735	0.719	0.711	0.704
	1.5	0.767	0.722	0.702	0.692	0.677
	2	0.766	0.720	0.700	0.690	0.673
	2.5	0.765	0.719	0.699	0.687	0.671

Table 2: Values of the lower bound (first row in each section) as well as estimated Pitman efficiencies PE(MOLP, MLP) $= P(|R_s - \pi_{MOLP}^{(s)}| < |R_s - \pi_{MLP}^{(s)}|)$ of the MOLP of R_s relative to the MLP of R_s based on Weibull record values $R_1, \ldots, R_r$ for selected r and s, and, in the case of estimated Pitman efficiencies, for $p \in \{0.5, 1.5, 2, 2.5\}$.

5. ASYMPTOTIC RESULTS

In the present section we establish two asymptotic results concerning the behavior of the bias as well as the asymptotic distribution of the prediction error of the MOLP. Hereby, we consider sequences $(r_n)_{n=1}^{\infty}, (s_n)_{n=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$, satisfying

$$r_n < s_n \text{ for all } n \in \mathbb{N} \text{ and } \lim_{n\to\infty} r_n = \infty.$$

However, by an abuse of notation, we write $r, s \to \infty$ when taking limits with respect to n. Also, we will suppress n in the notation.

Proposition 5.1. For $s \geq 3$, let $R_1, \ldots, R_s$ be the first s Weibull record values. The MOLP $\pi^{(s)}_{MOLP}$ of R_s based on $R_\star$ is asymptotically unbiased in the sense that if $\lim_{s,r\to\infty} s/r = \lambda$, for some $\lambda > 1$, then

$$\frac{\mathrm{E}(\pi^{(s)}_{MOLP})}{\mathrm{E}(R_s)} \longrightarrow 1, \quad r, s \to \infty.$$

Proof: Observe that, under the stated assumptions, the condition $\frac{1}{pr}\ln\left(\frac{s-1}{r}\right) < 1$ is satisfied for r large enough, in which case, by the proof of Lemma 3.1,

$$\frac{\mathrm{E}(\pi^{(s)}_{MOLP})}{\mathrm{E}(R_s)} = \frac{\prod_{i=r}^{s-1}(1+\frac{1}{pi})^{-1}}{\left(1-\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}}.$$

For $x \in (-1,1)$, set $\rho(x) = \sum_{k=2}^{\infty}(-1)^{k+1}\frac{x^k}{k}$. Then $\log(1+x) = x + \rho(x)$, $x \in (-1,1)$, and

$$\begin{aligned}\prod_{i=r}^{s-1}\left(1+\frac{1}{pi}\right)^{-1} &= \exp\left\{-\sum_{i=r}^{s-1}\log\left(1+\frac{1}{pi}\right)\right\}\\ &= \exp\left\{-\frac{1}{p}\sum_{i=r}^{s-1}\frac{1}{i}\right\}\exp\left\{-\sum_{i=r}^{s-1}\rho(1/pi)\right\}.\end{aligned}$$

Since $\rho(x) = O(x^2)$ as $x \to 0$, we have $\rho(1/pi) = O(1/i^2)$ as $i \to \infty$. Consequently, $\lim_{r,s\to\infty}\sum_{i=r}^{s-1}\rho(1/pi) = 0$. Moreover, $\lim_{r,s\to\infty}\sum_{i=r}^{s-1}1/i = \ln(\lambda)$. Hence, we obtain that

$$\lim_{r,s\to\infty}\prod_{i=r}^{s-1}\left(1+\frac{1}{pi}\right)^{-1} = \left(\frac{1}{\lambda}\right)^{\frac{1}{p}}.$$

The claim now follows from the fact that

$$\lim_{r,s\to\infty}\left(1-\frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1} = \exp\left\{-\frac{1}{p}\ln(\lambda)\right\} = \left(\frac{1}{\lambda}\right)^{\frac{1}{p}}.$$

□

Remark 5.2. The shifted Stirling's approximation for the (real) Gamma function reads (see [1, formula (5.11.7)])

$$\Gamma(x+a) = \sqrt{2\pi} e^{-x} x^{x+a-\frac{1}{2}} e^{o(1)}, \quad \text{as } x \to \infty.$$

Hence,

$$\frac{\Gamma(s+1/p)}{\Gamma(s)} = s^{1/p} e^{o(1)}, \quad \text{as } s \to \infty.$$

To prove that $\pi_{MOLP}^{(s)}$ is unbiased in the limit, i.e., $\lim_{r,s\to\infty} \mathrm{E}(R_s - \pi_{MOLP}^{(s)}) = 0$, where r and s are supposed to satisfy $\lim_{r,s\to\infty} \frac{s}{r} = \lambda > 1$, one has to prove that

$$s^{1/p}\left(1 - \frac{\prod_{i=r+1}^{s}(1+\frac{1}{pi})^{-1}}{\left(1 - \frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1}}\right) \to 0, \quad r, s \to \infty,$$

which is equivalent to showing that

$$s^{1/p}\left(\left(1 - \frac{1}{pr}\ln\left(\frac{s-1}{r}\right)\right)^{r-1} - \prod_{i=r+1}^{s}\left(1+\frac{1}{pi}\right)^{-1}\right) \to 0, \quad r, s \to \infty.$$

Numerical computation indicates that this is true.

We continue with a result concerning the asymptotic distribution of the prediction error of the MOLP.

Proposition 5.3. For $s \geq 3$, let $R_1, \ldots, R_s$ be the first s Weibull record values. The prediction error of the MOLP $\pi_{MOLP}^{(s)}$ of R_s based on $R_\star$ has an asymptotic normal distribution. More specifically, we have that

$$\alpha_s(\sigma,p)\left(R_s - \left(\frac{s-1}{r}\right)^{\frac{1}{\hat{p}_{MOL}}} R_r\right) \longrightarrow \mathcal{N}\left(0, \lambda + \lambda\ln^2(\lambda) - 1\right), \quad r, s \longrightarrow \infty,$$

where it is assumed that there exists a $\lambda \in (1,\infty)$ such that $\lim_{r,s\to\infty}(\lambda - s/r)\sqrt{r} = 0$, and the sequence of normalizing constants is given by $\alpha_s(\sigma,p) = \frac{p}{\sigma} s^{\frac{1}{2}-\frac{1}{p}}$.

Proof: First, recall that, by result (7) in [14], $\hat{p}_{MOL}$ is independent of R_r and R_s and $pr/\hat{p}_{MOL} \sim \mathcal{G}amma(r-1,1)$. Let $(Y_n)_{n=1}^\infty$ and $(Z_n)_{n=1}^\infty$ be two independent sequences of i.i.d. random variables, $Y_1,\ Z_1 \sim \mathcal{E}xp(1)$. By [2, equation (2.3.3)], for any $r, s \in \mathbb{N}$, $r < s$, the identity $(R_r, R_s) \overset{d}{=}$

$\sigma((\sum_{i=1}^r Y_i)^{1/p}, (\sum_{i=1}^s Y_i)^{1/p})$ holds true. Combining these results, we conclude that

$$\frac{1}{\sigma}\left(R_s - \left(\frac{s-1}{p}\right)^{\frac{1}{\hat{p}_{MOL}}} R_r\right)$$

$$\stackrel{d}{=} \left(\sum_{i=1}^r Y_i + \sum_{i=r+1}^s Y_i\right)^{1/p} - \left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\sum_{i=1}^{r-1} Z_i} \left(\sum_{i=1}^r Y_i\right)^{1/p}$$

$$= \left(s + \left\{\frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{\frac{r}{s}} + \frac{\sum_{i=r+1}^s Y_i - (s-r)}{\sqrt{s-r}}\sqrt{\frac{s-r}{s}}\right\}\sqrt{s}\right)^{1/p} - s^{1/p}$$

$$+ s^{1/p} - \lambda^{1/p}\left(r + \frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{r}\right)^{1/p}$$

$$+ \left\{\lambda^{1/p} - \left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\left((r-1) + \frac{\sum_{i=1}^{r-1} Z_i - (r-1)}{\sqrt{r-1}}\sqrt{r-1}\right)}\right\}$$

$$\times \left(r + \frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{r}\right)^{1/p}$$

(5.1)

$$=: A + B + C.$$

We first treat the terms denoted by A. By [6, Theorem 5.6.1],

$$\frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}, \ \frac{\sum_{i=r+1}^s Y_i - (s-r)}{\sqrt{s-r}} \quad \xrightarrow{d} \quad \mathcal{N}(0,1), \quad r, s \to \infty. \tag{5.2}$$

Hence, applying Theorem 2.8 in [4] as well as Slutsky's lemma, we conclude that

(5.3)

$$\frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{\frac{r}{s}} + \frac{\sum_{i=r+1}^s Y_i - (s-r)}{\sqrt{s-r}}\sqrt{\frac{s-r}{s}} \quad \xrightarrow{d} \quad \mathcal{N}(0,1), \quad r, s \to \infty.$$

By expanding the function $f(x) = (1+x)^{1/p}$, $x > -1$, around $x = 0$, $f(x) = 1 + \frac{x}{p} + o(x)$, as $x \to 0$. Consequently,

$$A = \left(s + \left\{\frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{\frac{r}{s}} + \frac{\sum_{i=r+1}^s Y_i - (s-r)}{\sqrt{s-r}}\sqrt{\frac{s-r}{s}}\right\}\sqrt{s}\right)^{1/p} - s^{1/p}$$

$$= s^{1/p}\left[\left(1 + \left\{\frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{\frac{r}{s}} + \frac{\sum_{i=r+1}^s Y_i - (s-r)}{\sqrt{s-r}}\sqrt{\frac{s-r}{s}}\right\}\frac{1}{\sqrt{s}}\right)^{1/p} - 1\right]$$

(5.4)

$$= \frac{s^{\frac{1}{p}-\frac{1}{2}}}{p}\left\{\frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{\frac{r}{s}} + \frac{\sum_{i=r+1}^s Y_i - (s-r)}{\sqrt{s-r}}\sqrt{\frac{s-r}{s}}\right\}(1 + o_P(1)), \quad r, s \to \infty.$$

Similarly, we have

$$B = s^{1/p} - \lambda^{1/p}\left(r + \frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{r}\right)^{1/p}$$
$$= -s^{1/p}\left[\left(\lambda\frac{r}{s} + \frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{\lambda^2\frac{r}{s}\frac{1}{\sqrt{s}}}\right)^{1/p} - 1\right]$$
$$= -s^{1/p}\left[\left(1 + o(1/\sqrt{s}) + \frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{\lambda^2\frac{r}{s}\frac{1}{\sqrt{s}}}\right)^{1/p} - 1\right], \quad r,s\to\infty$$
$$(5.5)\quad = -\frac{s^{\frac{1}{p}-\frac{1}{2}}}{p}\left(\frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{\lambda^2\frac{r}{s}} + o(1)\right)(1 + o_P(1)), \quad r,s\to\infty,$$

where the second-last equality is due to $\lim_{r,s\to\infty}(\lambda - s/r)\sqrt{r} = 0$. Finally, we treat the terms denoted by C. First, we rewrite C as

$$C = \left\{\lambda^{1/p} - \left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\left((r-1)+\frac{\sum_{i=1}^{r-1} Z_i-(r-1)}{\sqrt{r-1}}\sqrt{r-1}\right)}\right\}\left(r + \frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{r}\right)^{1/p}$$
$$= \frac{s^{\frac{1}{p}-\frac{1}{2}}}{p}\left\{\lambda^{1/p} - \left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\left((r-1)+\frac{\sum_{i=1}^{r-1} Z_i-(r-1)}{\sqrt{r-1}}\sqrt{r-1}\right)}\right\}$$
$$\times\frac{\left(r + \frac{\sum_{i=1}^r Y_i - r}{\sqrt{r}}\sqrt{r}\right)^{1/p} - r^{1/p}}{s^{\frac{1}{p}-\frac{1}{2}}/p}$$
$$+\frac{s^{\frac{1}{p}-\frac{1}{2}}}{p}\left\{\lambda^{1/p} - \left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\left((r-1)+\frac{\sum_{i=1}^{r-1} Z_i-(r-1)}{\sqrt{r-1}}\sqrt{r-1}\right)}\right\}\left(\frac{r}{s}\right)^{1/p} p\sqrt{s}$$
$$(5.6)\quad =: D + E.$$

By [6, Theorem 5.6.1] and Slutsky's lemma,

$$(5.7)\quad \lambda^{1/p} - \left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\left((r-1)+\frac{\sum_{i=1}^{r-1} Z_i-(r-1)}{\sqrt{r-1}}\sqrt{r-1}\right)} \xrightarrow{P} 0, \quad r,s\to\infty.$$

Next, by (5.2), we have that

$$\frac{\left(r+\frac{\sum_{i=1}^{r} Y_i - r}{\sqrt{r}}\sqrt{r}\right)^{1/p} - r^{1/p}}{s^{\frac{1}{p}-\frac{1}{2}}/p} = \frac{\left(r+\frac{\sum_{i=1}^{r} Y_i - r}{\sqrt{r}}\sqrt{r}\right)^{1/p} - r^{1/p}}{r^{\frac{1}{p}-\frac{1}{2}}/p}\frac{r^{\frac{1}{p}-\frac{1}{2}}}{s^{\frac{1}{p}-\frac{1}{2}}}$$

$$= \frac{\left(1+\frac{\sum_{i=1}^{r} Y_i - r}{\sqrt{r}}\frac{1}{\sqrt{r}}\right)^{1/p} - 1}{r^{-\frac{1}{2}}/p}\frac{r^{\frac{1}{p}-\frac{1}{2}}}{s^{\frac{1}{p}-\frac{1}{2}}}$$

$$= \left(\frac{\sum_{i=1}^{r} Y_i - r}{\sqrt{r}} + o_P(1)\right)\lambda^{\frac{1}{2}-\frac{1}{p}}, \quad r, s \to \infty$$

$$= O_P(1), \quad r, s \to \infty. \tag{5.8}$$

From (5.7) and (5.8) it readily follows that

$$ps^{\frac{1}{2}-\frac{1}{p}} D = o_P(1), \quad r, s \to \infty. \tag{5.9}$$

Next, consider the terms denoted by E. We have $(r/s)^{1/p} p\sqrt{s} \sim p\lambda^{\frac{1}{2}-\frac{1}{p}}\sqrt{r}$, as $r, s \to \infty$. Furthermore, setting

$$W_r = \frac{1}{p}\left(\left(\frac{\sum_{i=1}^{r-1} Z_i - (r-1)}{\sqrt{r-1}}\sqrt{\frac{r-1}{r}}\right)\frac{1}{\sqrt{r}} - \frac{1}{r}\right)\ln\left(\frac{s-1}{r}\right)$$

and observing that $W_r = o_P(1)$, as $r, s \to \infty$, we conclude that

$$\left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\left((r-1)+\frac{\sum_{i=1}^{r-1} Z_i-(r-1)}{\sqrt{r-1}}\sqrt{r-1}\right)} = \left(\frac{s-1}{r}\right)^{\frac{1}{p}} \exp\{W_r\}$$

$$= \left(\frac{s-1}{r}\right)^{\frac{1}{p}} (1 + W_r + W_r^2 + o_P(W_r^2)),$$

as $r, s \to \infty$. Hence, using these asymptotic relations as well as the fact that $W_r^2 = o_P(1/\sqrt{r})$, as $r \to \infty$, we obtain

$$ps^{\frac{1}{2}-\frac{1}{p}} E$$

$$= \left\{\lambda^{1/p} - \left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\left((r-1)+\frac{\sum_{i=1}^{r-1} Z_i-(r-1)}{\sqrt{r-1}}\sqrt{r-1}\right)}\right\}\left(\frac{r}{s}\right)^{1/p} p\sqrt{s}$$

$$= \left\{\lambda^{1/p} - \left(\left[\frac{s-1}{r}\right]^{1/p}\right)^{\frac{1}{r}\left((r-1)+\frac{\sum_{i=1}^{r-1} Z_i-(r-1)}{\sqrt{r-1}}\sqrt{r-1}\right)}\right\} p\lambda^{\frac{1}{2}-\frac{1}{p}}\sqrt{r}, \quad r, s \to \infty$$

$$= p\lambda^{\frac{1}{2}-\frac{1}{p}}\left\{\left(\lambda^{\frac{1}{p}} - \left(\frac{s-1}{r}\right)^{\frac{1}{p}}\right)\sqrt{r} - \left(\frac{s-1}{r}\right)^{\frac{1}{p}}\sqrt{r}W_r - \left(\frac{s-1}{r}\right)^{\frac{1}{p}}\sqrt{r}W_r^2 + o_P(W_r^2)\sqrt{r}\right\},$$

$$r, s \to \infty$$

(5.10)

$$= p\lambda^{\frac{1}{2}-\frac{1}{p}}\left\{-\left(\frac{s-1}{r}\right)^{\frac{1}{p}}\sqrt{r}W_r + o_P(1)\right\},$$

as $r, s \to \infty$. Finally, combining (5.1), (5.4), (5.5), (5.6), (5.9), (5.10), we arrive at

$$\begin{aligned}
&\frac{1}{\sigma}\left(R_s - \left(\frac{s-1}{p}\right)^{\frac{1}{\hat{p}_{MOL}}} R_r\right) \\
&\quad \stackrel{d}{=} \frac{s^{\frac{1}{p}-\frac{1}{2}}}{p}\left\{\frac{\sum_{i=1}^{r} Y_i - r}{\sqrt{r}}\left(\sqrt{\frac{r}{s}} - \sqrt{\lambda^2\frac{r}{s}}\right) + \frac{\sum_{i=r+1}^{s} Y_i - (s-r)}{\sqrt{s-r}}\sqrt{\frac{s-r}{s}}\right\} \\
&\qquad - \frac{s^{\frac{1}{p}-\frac{1}{2}}}{p}\left\{p\lambda^{\frac{1}{2}-\frac{1}{p}}\left(\frac{s-1}{r}\right)^{\frac{1}{p}}\sqrt{r}W_r + o_P(1)\right\}, \quad r, s \to \infty.
\end{aligned}$$

Since $\sqrt{r}W_r \stackrel{d}{\to} \frac{1}{p}\ln(\lambda)\mathcal{N}(0,1)$, as $r, s \to \infty$, Theorem 2.8 in [4] as well as Slutsky's lemma yield that

$$\frac{p}{\sigma}s^{\frac{1}{2}-\frac{1}{p}}\left(R_s - \left(\frac{s-1}{p}\right)^{\frac{1}{\hat{p}_{MOL}}} R_r\right) \stackrel{d}{\longrightarrow} \left(\sqrt{\frac{1}{\lambda}} - \sqrt{\lambda}\right) X_1 + \sqrt{1-\frac{1}{\lambda}}X_2 - \sqrt{\lambda}\ln(\lambda)X_3,$$

as $r, s \to \infty$, where $X_1,\ X_2,\ X_3 \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. Since

$$\left(\frac{1}{\lambda} - \sqrt{\lambda}\right) X_1 + \sqrt{1-\frac{1}{\lambda}}X_2 - \sqrt{\lambda}\ln(\lambda)X_3 \sim \mathcal{N}(0, \lambda + \lambda\ln^2(\lambda) - 1),$$

the statement is proved. □

Remark 5.4. From the preceding result we obtain an approximate prediction interval for R_s with nominal coverage probability $1-\alpha$, which is given by

$$\pi_{MOLP}^{(s)} \pm z_{1-\alpha/2}\frac{\sqrt{(s/r)(1+\ln^2(s/r)) - 1)}}{\alpha_s(\hat{\sigma}_{MLE}, \hat{p}_{MLE})}$$

where $\hat{\sigma}_{MLE}$ is the MLE of σ, $\hat{\sigma}_{MLE} = R_r/r^{1/\hat{p}_{MLE}}$ and $z_{1-\alpha/2}$ denotes the respective quantile of $\mathcal{N}(0,1)$. For the MLEs of σ and p we refer to [14].

6. CONCLUSION

For predicting future record values based on a sequence of observed upper record values with an underlying Weibull distribution, we derive two likelihood-based predictors, namely the maximum likelihood predictor and the maximum observed likelihood predictor. Expressions for the predictors are derived along with properties in terms of bias and mean squared error. The predictors are compared via Pitman's measure of closeness and their performance is examined in a simulation study.

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