# GAMMA-ADMISSIBILITY IN A NON-REGULAR FAMILY WITH SQUARED-LOG ERROR LOSS

Authors: Shirin Moradi Zahraie
– Department of Statistics, Yazd University,
Yazd, Iran (sh_moradi_zahraie@stu.yazd.ac.ir)

Hojatollah Zakerzadeh
– Department of Statistics, Yazd University,
Yazd, Iran (hzaker@yazd.ac.ir)

Abstract:

- Review the admissibility of estimators under a vague prior information leads to the concept of gamma-admissibility. In this paper, the problem of estimation in a non-regular family of distributions under a squared log error loss function is considered. We find sufficient condition for a generalized Bayes estimator of a parametric function to be gamma-admissible. Some examples are given.

Key-Words:

- *Gamma-admissibility; generalized Bayes estimator; non-regular distribution; squared-log error loss function.*

AMS Subject Classification:

- 62C15, 62F15.

## 1. INTRODUCTION

Admissibility of estimator is an important problem in statistical decision theory; Consequently, this problem has been considered by many authors under various type of loss functions both in an exponential and in a non-regular family of distributions. For example under squared error loss function (Karlin (1958), Ghosh & Meeden (1977), Ralescu & Ralescu (1981), Sinha & Gupta (1984), Hoffmann (1985), Pulskamp & Ralescu (1991), Kim (1994) and Kim & Meeden (1994)), under entropy loss function (Sanjari Farsipour (2003,2007)) and under LINEX loss function (Tanaka (2010,2011,2012)) and squared-log error loss function (Zakerzadeh & Moradi Zahraie (2015)).

A Bayesian approach to a statistical problem requires defining a prior distribution over the parameter space. Many Bayesians believe that just one prior can be elicited. In practice, it is more frequently the case that the prior knowledge is vague and any elicited prior distribution is only an approximation to the true one. So, we elect to restrict attention to a given flexible family of priors and we choose one member from that family, which seems to best match our personal beliefs.

A gamma-admissible ($\Gamma$-admissible) approach is used which allows to take into account vague prior information on the distribution of the unknown parameter $\theta$. The uncertainty about a prior is assumed by introducing a class $\Gamma$ of priors. If prior information is scarce, the class $\Gamma$ under consideration is large and a decision is close to a admissible decision. In the extreme case when no information is available the $\Gamma$-admissible setup is equivalent to the usual admissible setup. If, on the other hand, the statistician has an exactly prior information and the class $\Gamma$ contains a single prior, then the $\Gamma$-admissible decision is an usual Bayes decision. So it is a middle ground between the subjective Bayes setup and full admissible.

Eichenauer-Herrmann (1992) gained a sufficient condition for an estimator of the form $(aX + b)/(cX + d)$ to be $\Gamma$-admissible under the squared error loss in a one-parameter exponential family.

The most popular convex and symmetric loss function is the squared error loss function which is widely used in decision theory due to its simple mathematical properties. However in some cases, it does not represent the true loss structure. This loss function is symmetric in nature i.e. it gives equal weightage to both over and under estimation. In real life, we encounter many situations where over-estimation may be more serious than under-estimation or vice versa. As an example, in construction an underestimate of the peak water level is usually much more serious than an overestimation.

The squared-log error loss function was introduced by Brown (1968). For an estimator $\delta$ of estimand $h(\theta)$, it is given by

$$L(\nabla) =: L(h(\theta), \delta) = \left[\ln\left(\frac{\delta}{h(\theta)}\right)\right]^2, \tag{1.1}$$

where both $h(\theta)$ and $\delta$ are positive and $\nabla := \delta/h(\theta)$.

We need the following definitions to express properties of the loss (1.1).

**Definition 1.1.** A real function $g(x)$ is *quasi-convex*, if for any given real number $r$, the set of all $x$ such that $g(x) \leq r$ is convex. Any convex function is quasi-convex, but the converse is not necessarily true.

**Definition 1.2.** A loss function $L(h(\theta), \delta)$ is (for any $\varepsilon > 0$):

- *downside damaging* if $L(\delta - \varepsilon, \delta) \geq L(\delta + \varepsilon, \delta)$,
- *upside damaging* if $L(\delta - \varepsilon, \delta) \leq L(\delta + \varepsilon, \delta)$,
- *symmetric* if the loss function is both downside and upside damaging.

**Remark 1.1.** With downside damaging loss function, under-estimation is penalized more heavily, per unit distance, than over-estimation and with upside damaging loss function it is reversed.

**Remark 1.2.** If a loss function be downside damaging or upside damaging, then it is called *asymmetric*. By using asymmetric loss functions one is able to deal with cases where it is more damaging to miss the target on one side than the other.

**Definition 1.3.** The $L(h(\theta), \delta)$ is a *precautionary loss function* if and only if

1. $L(h(\theta), \delta)$ is downside damaging, and
2. for each fixed $\theta$, $L(h(\theta), \delta) \to \infty$ as $\delta \to 0$.

**Definition 1.4.** The $L(h(\theta), \delta)$ is a *balanced loss function*, if $L(h(\theta), \delta) \to \infty$ as $\delta \to 0$ or $\delta \to \infty$. A balanced loss function takes both error of estimation and goodness of fit into account but the unbalanced loss function only considers error of estimation.

From Figure 1, we see that the loss (1.1) has the below properties:

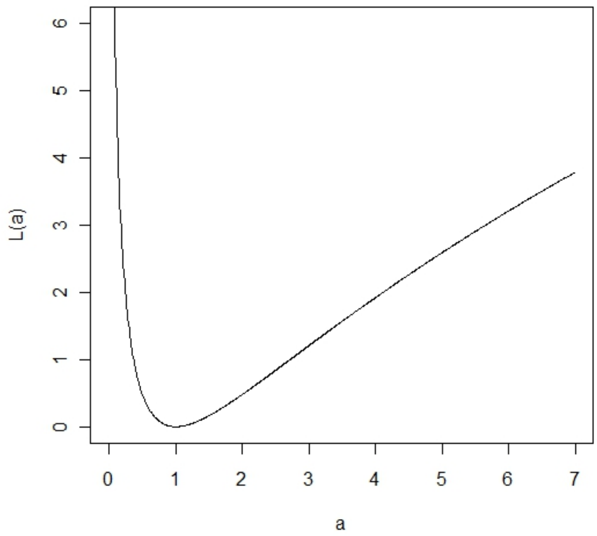


**Figure 1**: Plot of the squared-log error loss function.

(i) It is asymmetric.

(ii) It is quasi-convex.

(iii) It is a balanced loss function.

(iv) It is a precautionary loss function.

(v) When $0 < \nabla < 1$, it rises rapidly to infinity at zero; it has a unique minimum at $\nabla = 1$ and when $\nabla > 1$ it increases sublinearly.

For estimation under (1.1), see Sanjari Farsipour and Zakerzadeh (2005,2006), Mahmoudi and Zakerzadeh (2011), Kiapour and Nematollahi (2011), Nematollahi and Jafari Tabrizi (2012) and Zakerzadeh and Moradi Zahraie (2015).

In this paper we consider the $\Gamma$-admissibility of generalized Bayes estimators in a non-regular family of distributions under the loss (1.1) where class $\Gamma$ consists of all distributions which are compatible with the vague prior information. To this end, in Section 2, we state some preliminary definitions and results. In Section 3, we will obtain main theorem. Finally, in Section 4, we give an application of the $\Gamma$-admissibility in proving the $\Gamma$-minimaxity of estimators. Some examples are given.

## 2. Preliminaries

### 2.1. Definition of Γ-admissibility

In the present paper it is assumed that vague prior density on the distribution of the unknown parameter $\theta$ is available. Let $\Pi$ denote the set of all priors, i.e. Borel probability measures on the parameter interval $\Theta$ and $\Gamma$ be a non-empty subset of $\Pi$. Suppose that the available vague prior information can be described by the set $\Gamma$, in the sense that $\Gamma$ contains all prior which are compatible with the vague prior information.

Eichenauer-Herrmann (1992) defined the Γ-admissibility of an estimator as follows.

**Definition 2.1.** An estimator $\delta^*$ is called Γ-admissible, if

$$r(\pi, \delta) \leq r(\pi, \delta^*), \qquad \pi \in \Gamma,$$

for some estimator $\delta$ implies that

$$r(\pi, \delta) = r(\pi, \delta^*), \qquad \pi \in \Gamma,$$

where $r(\pi, \delta)$ is the Bayes risk of $\delta$.

**Remark 2.1.** From Definition 2.1, it is obvious that

- A Π-admissible estimator is admissible.

- A $\{\pi\}$-admissible estimator is simply a Bayes strategy with respect to the prior $\pi$.

- In general neither Γ-admissibility implies admissibility nor admissibility implies Γ-admissibility.

Hence, the available results on admissibility cannot be applied in order to prove the Γ-admissibility of an estimator. Consequently, it is necessary to study the problem of Γ-admissibility of estimators.

### 2.2. A non-regular family of distributions

Let $X$ be a random variable whose probability density function with respect to some $\sigma$-finite measure $\mu$ is given by

$$f_X(x;\theta) = \begin{cases} q(\theta)r(x) & \underline{\theta} < x < \theta, \\ 0 & \text{otherwise}, \end{cases}$$

where $\theta \in \Theta =: (\underline{\theta}, \bar{\theta})$ and $\Theta$ is a nondegenerate interval (possibly infinite) on the real line. Also $r(x)$ is a positive $\mu$-measurable function of $x$ and

$$q^{-1}(\theta) = \int_{\underline{\theta}}^{\theta} r(x) d\mu(x) < \infty$$

for $\theta \in \Theta$. This family is known as *a non-regular family of distributions.*

Suppose $\pi(\theta)$ be a prior (possibly improper) by its Lebesgue density $p_\pi(\theta)$ over $\Theta$ which is positive and continuous. Let $h(\theta)$ be a continuous function to be estimated from $\Theta$ to $\mathbb{R}$ and the loss to be (1.1). The generalized Bayes estimator of $h(\theta)$ with respect to $\pi(\theta)$ is given by $\delta_\pi(X)$, where

$$\delta_\pi(x) = \exp\left\{ \frac{\int_x^{\bar{\theta}} \{\ln(h(\theta))\} q(\theta) p_\pi(\theta) d\theta}{\int_x^{\bar{\theta}} q(\theta) p_\pi(\theta) d\theta} \right\} \tag{2.1}$$

for $\underline{\theta} < x < \bar{\theta}$, provided that the integrals in (2.1) exist and are finite.

## 3. Main results

In this section, the main results will obtain.

For some real number $\lambda_0$ let $a, b : [\lambda_0, \infty) \mapsto \Theta$ be continuously differentiable functions with $a(\lambda_0) < b(\lambda_0)$, where $a$ and $b$ are supposed to be strictly decreasing and strictly increasing, respectively. For $\lambda \geq \lambda_0$ a prior $\pi_\lambda$ is defined by its Lebesgue density $p_{\pi_\lambda}$ of the form

$$p_{\pi_\lambda}(\theta) := \left( \int_{a(\lambda)}^{b(\lambda)} p_\pi(t) dt \right)^{-1} I_{[a(\lambda), b(\lambda)]}(\theta) p_\pi(\theta).$$

Throughout this paper, we restrict estimators to the class

$$\Delta := \{\delta | \text{(A1) and (A2) are satisfied}\},$$

where

(A1) $E_\theta[\{\ln(\delta(X))\}^2] < \infty$ for all $\theta \in \Theta$,

(A2) $\int_{a(\lambda)}^{b(\lambda)} E_\theta[\{\ln(\frac{\delta(X)}{h(\theta)})\}^2] p_\pi(\theta) d\theta < \infty$ for $a(\lambda) < b(\lambda)$ and $\lambda \geq \lambda_0$ .

**Remark 3.1.** In the statistical game $(\Gamma, \Delta, r)$, a $\Gamma$-admissible estimator is an admissible strategy of the second player.

The next lemma is essential to obtain our results.

**Lemma 3.1.** *Let $S(\theta)$ be a continuous and non-negative function over $\Theta = (\underline{\theta}, \bar{\theta})$ and $G(\lambda) := \int_{a(\lambda)}^{b(\lambda)} S(\theta)d\theta$. Suppose that there exists a positive function $R(\theta)$ such that*

$$G(\lambda) \leq 4(\min\{R(b(\lambda))b'(\lambda), -R(a(\lambda))a'(\lambda)\})^{-1/2}(G'(\lambda))^{1/2}$$

*for $\lambda \geq \lambda_0$. If*

$$\int_{\lambda_0}^{\infty} \min\{R(b(\lambda))b'(\lambda), -R(a(\lambda))a'(\lambda)\}d\lambda = \infty,$$

*then $S(\theta) = 0$ for a.a. $\theta \in \Theta$.*

**Proof:** See Eichenauer-Herrmann (1992). □

**Theorem 3.1.** *Suppose that $\delta_\pi \in \Delta$ and put*

$$K(x,\theta) := \int_x^{\theta} \{\ln(\frac{\delta_\pi(x)}{h(t)})\}q(t)p_\pi(t)dt,$$

*and*

$$\gamma(\theta) := \frac{1}{p_\pi(\theta)q(\theta)} \int_{\underline{\theta}}^{\theta} r(x)K^2(x,\theta)d\mu(x).$$

*If $\pi_\lambda \in \Gamma$ for all $\lambda \geq \lambda_0$ and*

$$\int_{\lambda_0}^{\infty} \min\{\gamma^{-1}(b(\lambda))b'(\lambda), -\gamma^{-1}(a(\lambda))a'(\lambda)\}d\lambda = \infty, \tag{3.1}$$

*then $\delta_\pi(X)$ is $\Gamma$-admissible under the loss (1.1).*

**Proof:** Let $\delta \in \Delta$ be an estimator such that $r(\pi, \delta) \leq r(\pi, \delta_\pi)$ for every prior $\pi \in \Gamma$. Since $\pi_\lambda \in \Gamma$ for $\lambda \geq \lambda_0$, we must have

$$\begin{aligned} 0 \; &\leq \; \left(\int_{a(\lambda)}^{b(\lambda)} p_\pi(t)dt\right)\{r(\pi_\lambda, \delta_\pi) - r(\pi_\lambda, \delta)\} \\ &= \int_{a(\lambda)}^{b(\lambda)} E_\theta[L(\delta_\pi, h(\theta)) - L(\delta, h(\theta))]p_\pi(\theta)d\theta \end{aligned}$$

for all $\theta \in \Theta$. From Condition (A1), we see that it is equivalent to

$$\begin{aligned} 0 \; &\leq \; \int_{a(\lambda)}^{b(\lambda)} E_\theta\left[\left\{\ln\left(\frac{\delta(X)}{\delta_\pi(X)}\right)\right\}^2\right]p_\pi(\theta)d\theta \\ &\leq \; 2\int_{a(\lambda)}^{b(\lambda)} E_\theta\left[\left\{\ln\left(\frac{\delta_\pi(X)}{h(\theta)}\right)\right\}\left\{\ln\left(\frac{\delta_\pi(X)}{\delta(X)}\right)\right\}\right]p_\pi(\theta)d\theta, \end{aligned}$$

for all $\theta \in \Theta$.

An application of the Fubini's theorem gives

$$
\begin{aligned}
0 \;\le\; & \int_{a(\lambda)}^{b(\lambda)} \int_{\underline{\theta}}^{\theta} \left\{ \ln\left(\frac{\delta(x)}{\delta_\pi(x)}\right)\right\}^2 r(x)q(\theta)p_\pi(\theta)d\mu(x)d\theta \\
\le\; & 2\int_{\underline{\theta}}^{b(\lambda)} \left[\int_x^{b(\lambda)} \left\{\ln\left(\frac{\delta_\pi(x)}{h(\theta)}\right)\right\} p_\pi(\theta)q(\theta)d\theta\right] \left\{\ln\left(\frac{\delta_\pi(x)}{\delta(x)}\right)\right\} r(x)d\mu(x) \\
-\; & 2\int_{\underline{\theta}}^{a(\lambda)} \left[\int_x^{a(\lambda)} \left\{\ln\left(\frac{\delta_\pi(x)}{h(\theta)}\right)\right\} p_\pi(\theta)q(\theta)d\theta\right] \left\{\ln\left(\frac{\delta_\pi(x)}{\delta(x)}\right)\right\} r(x)d\mu(x),
\end{aligned}
$$

which is guaranteed by Condition (A2).

Applying the Cauchy-Schwartz inequality, the first term of the right-hand side in the above equation, is less than

$$
2\left\{\int_{\underline{\theta}}^{b(\lambda)} \left\{\ln\left(\frac{\delta(x)}{\delta_\pi(x)}\right)\right\}^2 r(x)d\mu(x)\right\}^{1/2} \left\{\int_{\underline{\theta}}^{b(\lambda)} r(x)K^2(x,b(\lambda))d\mu(x)\right\}^{1/2}.
$$

Hence, if we define

$$
T(\theta) := \int_{\underline{\theta}}^{\theta} \left\{\ln\left(\frac{\delta(x)}{\delta_\pi(x)}\right)\right\}^2 r(x)d\mu(x),
$$

then we have

$$
\begin{aligned}
0 \;\le\; & \int_{a(\lambda)}^{b(\lambda)} T(\theta)q(\theta)p_\pi(\theta)d\theta \\
\le\; & 2\{T(b(\lambda))b'(\lambda)q(b(\lambda))p_\pi(b(\lambda))\}^{1/2}\{\gamma^{-1}(b(\lambda))b'(\lambda)\}^{-1/2} \\
& +2\{-T(a(\lambda))a'(\lambda)q(a(\lambda))p_\pi(a(\lambda))\}^{1/2}\{-\gamma^{-1}(a(\lambda))a'(\lambda)\}^{-1/2} \\
\le\; & 4\left(\min\{\gamma^{-1}(b(\lambda))b'(\lambda), \gamma^{-1}(a(\lambda))a'(\lambda)\}\right)^{-1/2} \\
& \times \left(T(b(\lambda))q(b(\lambda))p_\pi(b(\lambda))b'(\lambda) - T(a(\lambda))q(a(\lambda))p_\pi(a(\lambda))a'(\lambda)\right)^{1/2}
\end{aligned}
$$

for $\lambda \ge \lambda_0$, where the definition of the function $\gamma(\theta)$ has been used. Now a continuous, differentiable and increasing function $H : [\lambda_0, \infty] \to \mathbb{R}$ is defined by

$$
H(\lambda) := \int_{a(\lambda)}^{b(\lambda)} T(\theta)q(\theta)p_\pi(\theta)d\theta.
$$

So the above inequality can be written in the form

$$
H(\lambda) \le 4\left(\min\{\gamma^{-1}(b(\lambda))b'(\lambda), -\gamma^{-1}(a(\lambda))a'(\lambda)\right)^{-1/2} (H'(\lambda))^{1/2}
$$

for $\lambda \ge \lambda_0$. Therefore, from Lemma 3.1 we obtain $T(\theta) = 0$ for $a.a.\theta \in \Theta$, and consequently from (A1), we have $\delta(x) = \delta_\pi(x)$ $a.e.\mu$. This completes the proof. $\square$

**Remark 3.2.** $K(x,\theta)$ can expressed as

$$K(x,\theta)=\frac{\int_x^{\theta}\int_{\theta}^{\bar{\theta}}\{\ln(\frac{h(s)}{h(t)})\}q(s)p_\pi(s)q(t)p_\pi(t)dsdt}{\int_x^{\bar{\theta}}q(u)p_\pi(u)du}$$

by (2.1) and the symmetry of the integrand.

**Example 3.1.** Suppose that $X$ be a random variable according to an exponential distribution whose probability density function is given by

$$f_X(x,\theta)=\begin{cases} e^{x-\theta} & x<\theta,\\ 0 & x>\theta,\end{cases}$$

where $\theta(\in\mathbb{R})$ is unknown. The Generalized Bayes estimator of $h(\theta)=e^{\theta}$ with respect to the Lebesgue prior is given by $\delta_\pi(X)=\exp\{X+1\}$ which is of the form $ah(X)(a>0)$. A direct calculation gives $K(x,\theta)=e^{-\theta}(\theta-x)$ and $\gamma(\theta)=2$. Let class $\Gamma_0$ consists of all priors with mean 0, i.e., $\Gamma_0:=\{\pi\in\Pi|\int_\Theta\theta p_\pi(\theta)d\theta=0\}$. Define functions $a$ and $b$ by $a(\lambda)=-\lambda$ and $b(\lambda)=\lambda$ for $\lambda\geq\lambda_0>0$, i.e., the prior $\pi_\lambda$ is the uniform distribution on the interval $[-\lambda,\lambda]$. Hence, $\pi_\lambda\in\Gamma_0$ for all $\lambda\geq\lambda_0$. Since (3.1) is satisfied, Theorem 3.1 implies that $\delta_\pi(X)$ is $\Gamma_0$-admissible under the loss (1.1).

**Remark 3.3.** It is difficult to express $\gamma(\theta)$ explicitly and it can have a complicated form, so to apply Theorem 3.1, we have to seek the suitable upper bound of $\gamma(\theta)$. For the case when $h(\theta)$ is bounded, we can get the next corollary.

**Corollary 3.1.** *Suppose that $h(\theta)$ is bounded and $\delta_\pi\in\Delta$. Put*

$$\tilde{K}(x,\theta):=\frac{\int_\theta^{\bar{\theta}}q(s)p_\pi(s)ds\int_x^{\theta}q(t)p_\pi(t)dt}{\int_x^{\bar{\theta}}q(u)p_\pi(u)du},$$

*and*

$$\tilde{\gamma}(\theta):=\frac{1}{p_\pi(\theta)q(\theta)}\int_{\underline{\theta}}^{\theta}r(x)\tilde{K}^2(x,\theta)d\mu(x).$$

*If $\pi_\lambda\in\Gamma$ for all $\lambda\geq\lambda_0$ and*

$$\int_{\lambda_0}^{\infty}\min\{\tilde{\gamma}^{-1}(b(\lambda))b'(\lambda),-\tilde{\gamma}^{-1}(a(\lambda))a'(\lambda)\}d\lambda=\infty,$$

*then $\delta_\pi(X)$ is $\Gamma$-admissible under the loss (1.1).*

**Proof:** It can be easily shown that there exists a constant $C$ such that $|K(x,\theta)|\leq C\tilde{K}(x,\theta)$ for all $(x,\theta)\in\{(x,\theta)|\underline{\theta}<x<\theta<\bar{\theta}\}$. This completes the proof by Theorem 3.1. $\square$

**Example 3.2.** Suppose that $X_1, ..., X_n$ are independent and identically distributed random variables according to a uniform distribution over the interval $(0, \theta)$ where $\theta(\in \mathbb{R}^+)$ is unknown. Then the probability density function of the sufficient statistic $X = X_{(n)}$ is given by

$$f_X(x, \theta) = \begin{cases} \frac{n}{\theta^n} x^{n-1} & 0 < x < \theta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $h(\theta)$ be bounded and $\pi(\theta) = 1/\theta$. We can easily obtain

$$\tilde{K}(x, \theta) = (1/(n\theta^n)) \left\{1 - (x/\theta)^n\right\},$$

and

$$\tilde{\gamma}(\theta) = \theta/(3n^2).$$

We assume that $\Gamma_m := \{\pi \in \Pi | \int_\Theta \theta p_\pi(\theta) d\theta = m\}$, i.e., $\Gamma_m$ consists of all priors with mean $m$. Define functions $a$ and $b$ by $a(\lambda) = m \ln(\lambda)/(\lambda - 1)$ and $b(\lambda) = \lambda a(\lambda)$ for $\lambda \geq \lambda_0 > 1$. Since

$$\int\limits_\Theta \theta p_{\pi_\lambda}(\theta) d(\theta) = \left( \int_{a(\lambda)}^{b(\lambda)} \frac{1}{t} dt \right)^{-1} (b(\lambda) - a(\lambda)) = m$$

for all $\lambda \geq \lambda_0$, so that $\pi_\lambda \in \Gamma_m$. A short calculation yields $a'(\lambda) = m\frac{\lambda - 1 - \lambda \ln(\lambda)}{\lambda(\lambda-1)^2} < 0$ and $b'(\lambda) = m\frac{\lambda-1-\ln(\lambda)}{(\lambda-1)^2} > 0$ for $\lambda \geq \lambda_0$. Because of $\lambda - 1 - \ln(\lambda) < \lambda \ln(\lambda) - \lambda + 1$ for $\lambda \geq \lambda_0$ and $\lim_{\lambda \to \infty} b(\lambda) = \infty$, one obtains

$$\begin{aligned} \int_{\lambda_0}^{\infty} \min\{\tilde{\gamma}^{-1}(b(\lambda))b'(\lambda), -\tilde{\gamma}^{-1}(a(\lambda))a'(\lambda)\} d\lambda &= (3n^2) \int_{\lambda_0}^{\infty} \min\{\frac{b'(\lambda)}{b(\lambda)}, \frac{a'(\lambda)}{a(\lambda)}\} d\lambda \\ &= (3n^2) \int_{\lambda_0}^{\infty} \frac{b'(\lambda)}{b(\lambda)} d\lambda = \infty. \end{aligned}$$

Hence, according to Corollary 3.1 the Generalized Bayes estimator of $h(\theta)$ with respect to $\pi(\theta) = 1/\theta$ is $\Gamma_m$-admissible under the loss (1.1).

**Remark 3.4.** Typically all the result in this paper go through with some modifications for the density

$$f_X(x, \theta) = \begin{cases} q(\theta)r(x) & \theta < x < \bar{\theta}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\theta \in \Theta$ is unknown.

## 4. An application

In the presence of vague prior information frequently the $\Gamma$-minimax approach is used as underlying principle. In this section, we provide the definition of the $\Gamma$-minimaxity of an estimator and then express the relation between this concept and the $\Gamma$-admissibility. Finally, we give an example.

**Definition 4.1.** A $\Gamma$-minimax estimator is a minimax strategy of the second player in the statistical game $(\Gamma, \Delta, r)$; $\delta^*$ is called a $\Gamma$-minimax estimator, if

$$\sup_{\pi \in \Gamma} r(\pi, \delta^*) = \inf_{\delta \in \Delta} \sup_{\pi \in \Gamma} r(\pi, \delta),$$

where $r(\pi, \delta)$ is the Bayes risk of $\delta$.

**Definition 4.2.** A $\Gamma$-minimax estimator $\delta^*$ is said to be unique, if

$$r(\pi, \delta) = r(\pi, \delta^*), \quad \pi \in \Gamma,$$

for any other $\Gamma$-minimax estimator $\delta$.

**Remark 4.1.**

- From Definition 4.2, it is obvious that a unique $\Gamma$-minimax estimator is $\Gamma$-admissible.

- If a $\Gamma$-admissible estimator $\delta$ is an equalizer on $\Gamma$, i.e., $r(., \delta)$ is constant on $\Gamma$, then $\delta$ is a unique $\Gamma$-minimax estimator.

**Example 4.1.** In Example 3.1, we have $E_\theta[X] = \theta - 1$ and $E_\theta[X^2] = \theta^2 - 2\theta + 2$. Thus, from (1.1), the risk function of $\delta_\pi$ is equal to

$$\begin{aligned} R(e^{X+1}, e^\theta) &= E_\theta[\{\ln(e^{X+1}) - \ln(e^\theta)\}^2] \\ &= E_\theta[\{X + 1 - \theta\}^2] \\ &= Var_\theta[X] \\ &= 1. \end{aligned}$$

So, $\delta_\pi$ is an equalizer on $\Gamma_0$, since its risk function is constant. Hence, $\delta_\pi(X) = e^{X+1}$ is the unique $\Gamma_0$-minimax estimator for $e^\theta$.

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