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ESTIMATION IN WEIBULL DISTRIBUTION UNDER PROGRESSIVELY TYPE-I HYBRID CENSORED DATA

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Abstract:

- In this article, we consider the estimation of unknown parameters of Weibull distribution when the lifetime data are observed in the presence of progressively type–I hybrid censoring scheme. The Newton–Raphson algorithm, Expectation–Maximization (EM) algorithm and Stochastic EM algorithm are utilized to derive the maximum likelihood estimates for the unknown parameters. Moreover, Bayesian estimators using Tierney–Kadane Method and Markov Chain Monte Carlo method are obtained under three different loss functions, namely, squared error loss, linear–exponential and generalized entropy loss functions. Also, the shrinkage pre–test estimators are derived. An extensive Monte Carlo simulation experiment is conducted under different schemes so that the performances of the listed estimators are compared using mean squared error, confidence interval length and coverage probabilities. Asymptotic normality and MCMC samples are used to obtain the confidence intervals and highest posterior density intervals respectively. Further, a real data example is presented to illustrate the methods. Finally, some conclusive remarks are presented.

Key-Words:

- *Bayesian estimation; EM algorithm; SEM algorithm; Tierney-Kadane's approximation; progressively type-I hybrid censoring; Weibull distribution.*

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1. INTRODUCTION

Censored data occurs commonly in reliability and survival analysis. There are mainly two censoring schemes which are Type–I censoring where the life-testing experiment stops at a predetermined time, say T and Type–II censoring, where the life–testing experiment stops when predetermined number of failures, say m, are observed. Epstein [19] proposed the hybrid censoring scheme which is the mixture of Type–I and Type–II censoring schemes. The hybrid censoring scheme has become quite popular in the reliability and life–testing experiments so far. For example, see the papers of of Chen and Bhattacharya [13], Childs et al. [15], Kundu and Joarder [25], Balakrishnan and Kundu [10]. It is worth mentioning that the book of f Balakrishnan and Cramer [8] discussed the topics of progressive censoring and progressive hybrid censoring in detail as separate chapters. In these schemes, it is allowed to remove the units only at the terminal points of the experiments. However, Kundu and Joarder [25] introduced another scheme which is called the Type–I progressively hybrid censoring scheme (Type–I PHCS) such that it allows removals of units during the test time. For more information on progressive censoring, we refer to to Balakrishnan and Aggarwala [7], Balakrishnan [6] and Balakrishnan and Cramer [8]. Type-I PHCS can be viewed as a mixture of Type-I progressive censoring and hybrid censoring as follows: Assume that there are n identical units in a lifetime experiment with the progressive censoring scheme $(R_1, R_2, ..., R_m)$, $1 \leq m \leq n$ and the lifetime experiment ends at a predetermined time $T \in (0, \infty)$ and n, m, R_i's are all fixed non–negative integers. At the time of first failure, say $X_{1:m:n}$, R_1 units randomly removed from the remaining $n-1$ units. Similarly, when the second failure occurs at the time $X_{2:m:n}$, R_2 units are removed from the remaining $n - R_1 - 2$ units. This process continues up to the end of experiment which occurs at the time $\min(X_{m:m:n}, T)$. Therefore, if the mth failure occurs before time T, the experiment ends at the time $X_{m:m:n}$ and all the remaining units $R_m = n - \sum_{i=1}^{m-1} R_i - m$ are removed. However, if the experiment ends at time T with only J failures, $0 \leq J < m$, then all the remaining units $R_J^* = n - \sum_{i=1}^{J} R_i - J$ are removed and the test ends at time T. Therefore, under Type–I PHCS we have the following two cases:

- Case I: $\{X_{1:m:n}, X_{2:m:n}, ..., X_{m:m:n}\}$ if $X_{m:m:n} \leq T$.
- Case II: $\{X_{1:m:n}, X_{2:m:n}, ..., X_{J:m:n}\}$ if $X_{J:m:n} < T < X_{J+1:m:n}$.

Due to the fact that the lifetime distributions of many experimental units can be modeled by a two–parameter Weibull distribution which is one of the most commonly used model in reliability and lifetime data analysis, we consider the Weibull distribution in this paper. The probability distribution function (PDF) and cumulative distribution function (CDF) of two parameter Weibull distribution are given as follows

$$f(x; \alpha, \beta) = \alpha\beta x^{\alpha-1} \exp\{-\beta x^{\alpha}\} \tag{1.1}$$

$$F(x;\alpha,\beta) = 1 - \exp\{-\beta x^{\alpha}\} \tag{1.2}$$

where $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter.

Ng et al. [34] used the estimation method, along with Fisher information matrix, in the context of optimal progressive censoring schemes for the Weibull distribution. Banerjee and Kundu [11] considered the statistical inference on Weibull parameters when the data are Type–II hybrid censored, maximum likelihood estimation (MLE), approximate MLE and Bayes estimation techniques were studied by the authors. Balakrishnan and Kateri [9] proposed an alternative approach based on a graphical method, which also shows the existence and uniqueness of the MLEs. Lin et al. [28] studied the MLEs and the approximate MLEs (AMLEs) of the parameters of Weibull distribution under adaptive type–II progressive hybrid censoring. Huang and Wu [21] discussed the maximum likelihood estimation and Bayesian estimation of Weibull parameters under progressively type–II censoring scheme. Lin et al. [30] investigated the maximum likelihood estimation and Bayesian estimation for a two–parameter Weibull distribution based on adaptive type–I progressively hybrid censored data which was introduced by Lin and Huang [29]. Jia et al. [20] studied the exact inference on Weibull parameters under multiple type–I censoring. Mokhtari et al. [32] discussed the approximate and Bayesian inferential procedures for the progressively type–II hybrid censored data from the Weibull distribution. However, this type of censoring is identical to what we called as type–I progressive hybrid censored data. This paper will be different from [32] in three directions. Firstly, we introduce a new approach for inference about the Weibull distribution based on expectation–maximization (EM) and stochastic expectation–maximization (SEM) methods. We will show that both EM and SEM will result to have better estimates in the sense of having smaller biases and mean square errors. Secondly, we will derive the shrinkage estimators based on the ML estimates resulting to have higher deficiencies. Finally, in the Bayesian approach, different loss functions such as squared error loss (SEL), linear–exponential (LINEX), and general entropy loss (GEL) will be applied with both informative and non–informative priors.

The rest of the paper is organized as follows: In Section 2, MLE of the parameters are introduced by using Newton–Raphson (NR) algorithm, EM algorithm and SEM algorithm, also the Fisher information matrix is obtained. In Section 3, Bayes estimation for the parameters of Weibull distribution under the assumption of independent priors using different loss functions such as SEL, LINEX and GEL loss functions. Moreover, Tierney and Kadane [44] (T-K) approximations under these loss functions are also computed and Markov-Chain Monte Carlo (MCMC) method is also presented to estimate the parameters. In Section 4, a shrinkage pre–test estimation method is discussed. Extensive Monte Carlo simulations are conducted and results are discussed in Section 5. A real data example is presented in Section 6 to illustrate the findings of the study. Finally, some conclusive remarks are given in Section 7.

2. Maximum Likelihood Estimation

Let $\mathbf{X} = (X_{1:m:n}, \ldots, X_{r:m:n})$ represents the Type-I progressively hybrid censored sample of size r from a sample of size n drawn from a population with probability distribution given in Equation (1.1). Throughout this paper, we will denote $X_{i:m:n}$ by $X_{(i)}, i = 1, 2, \ldots, r$. Then the likelihood function of (α, β) given the observed data $\mathbf{x}$ can be written as

$$L(\alpha, \beta \mid \mathbf{x}) \propto \prod_{i=1}^{r} f(x_{(i)}; \alpha, \beta)\left[1 - F(x_{(i)}; \alpha, \beta)\right]^{R_i}\left[1 - F(\mathcal{C}; \alpha, \beta)\right]^{R_T}, \tag{2.1}$$

where $r = m, \mathcal{C} = x_{(m)}, R_T = 0$ in Case I, and $r = d, \mathcal{C} = T, R_T = n - d - \sum_{i=1}^{d} R_i$ in Case II. Based on the observed data, the log-likelihood function can be expressed as

$$\begin{aligned} l(\alpha, \beta \mid \mathbf{x}) &= \ln L(\alpha, \beta \mid \mathbf{x}) \\ &= r\ln(\alpha\beta) + (\alpha - 1)\sum_{i=1}^{r} \ln\left(x_{(i)}\right) - \beta \sum_{i=1}^{r}\left\{x_{(i)}^{\alpha}(1 + R_i)\right\} - \beta\mathcal{C}^{\alpha} R_T. \end{aligned} \tag{2.2}$$

Taking the derivatives of Equation (2.2) with respect to α and β and equating them to zero, one can obtain the following likelihood equations for α and β respectively

$$\begin{aligned} \frac{\partial l(\alpha, \beta \mid \mathbf{x})}{\partial \alpha} &= \frac{r}{\alpha} + \sum_{i=1}^{r} \ln\left(x_{(i)}\right) - \beta \sum_{i=1}^{r}\left\{(1 + R_i)\, x_{(i)}^{\alpha} \ln\left(x_{(i)}\right)\right\} \\ &\quad -\beta\mathcal{C}^{\alpha}\ln(\mathcal{C}) R_T = 0 \end{aligned} \tag{2.3}$$

$$\frac{\partial l(\alpha, \beta \mid \mathbf{x})}{\partial \beta} = \frac{r}{\beta} - \sum_{i=1}^{r}\left\{x_{(i)}^{\alpha}(1 + R_i)\right\} - \mathcal{C}^{\alpha} R_T = 0. \tag{2.4}$$

Solving Equation (2.4) yields the MLE of β which is given by

$$\widehat{\beta} = \frac{r}{\mathcal{C}^{\widehat{\alpha}} R_T + \sum_{i=1}^{r}\left\{x_{(i)}^{\widehat{\alpha}}(1 + R_i)\right\}}. \tag{2.5}$$

Now, substituting Equation (2.5) into (2.3), the MLE of α can be obtained by solving the following nonlinear equation:

$$\frac{r}{\widehat{\alpha}} + \frac{r\left[\sum_{i=1}^{r}\left\{(1 + R_i)\, x_{(i)}^{\widehat{\alpha}} \ln(x_{(i)})\right\} + R_T\mathcal{C}^{\widehat{\alpha}}\ln(\mathcal{C})\right]}{R_T\mathcal{C}^{\widehat{\alpha}} + \sum_{i=1}^{r}\left\{x_{(i)}^{\widehat{\alpha}}(1 + R_i)\right\}} = 0.$$

The second partial derivatives of the log-likelihood equation are obtained as follows:

$$\frac{\partial^2 l(\alpha,\beta \mid \mathbf{x})}{\partial \alpha^2} = -\frac{r}{\alpha^2} - \beta \sum_{i=1}^{r} \left\{ (1+R_i)\, x_{(i)}^{\alpha} \ln \left(x_{(i)} \right)^2 \right\} - \beta \mathcal{C}^{\alpha} \ln(C)^2 R_T, \tag{2.6}$$

$$\frac{\partial^2 l(\alpha,\beta \mid \mathbf{x})}{\partial \alpha \partial \beta} = -\sum_{i=1}^{r} \left\{ (1+R_i)\, x_{(i)}^{\alpha} \ln \left(x_{(i)} \right) \right\} - \mathcal{C}^{\alpha} \ln(C) R_T \tag{2.7}$$

$$\frac{\partial^2 l(\alpha,\beta \mid \mathbf{x})}{\partial \beta^2} = \frac{-r}{\beta^2}. \tag{2.8}$$

Now, using Equations (2.6)–(2.8), the Fisher's information matrix $\mathbf{I}(\alpha,\beta)$ can be formed by

$$\mathbf{I}(\alpha,\beta) = E \begin{bmatrix} -\frac{\partial^2 l(\alpha,\beta|\mathbf{x})}{\partial \alpha^2} & -\frac{\partial^2 l(\alpha,\beta|\mathbf{x})}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 l(\alpha,\beta|\mathbf{x})}{\partial \alpha \partial \beta} & -\frac{\partial^2 l(\alpha,\beta|\mathbf{x})}{\partial \beta^2} \end{bmatrix}. \tag{2.9}$$

It is well-known that (see [27]) the distribution of MLEs $\left(\widehat{\alpha}, \widehat{\beta}\right)$ is a bivariate normal distribution with

$$N\left((\alpha,\beta),\ \mathbf{I}^{-1}(\alpha,\beta)\right)$$

where $\mathbf{I}^{-1}(\alpha,\beta)$ is the covariance matrix. Moreover, one can approximate the covariance matrix evaluated at $(\widehat{\alpha}, \widehat{\beta})$ by the following observed information matrix

$$\mathbf{I}\left(\widehat{\alpha}, \widehat{\beta}\right) = \begin{bmatrix} -\frac{\partial^2 l(\alpha,\beta|\mathbf{x})}{\partial \alpha^2} & -\frac{\partial^2 l(\alpha,\beta|\mathbf{x})}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 l(\alpha,\beta|\mathbf{x})}{\partial \alpha \partial \beta} & -\frac{\partial^2 l(\alpha,\beta|\mathbf{x})}{\partial \beta^2} \end{bmatrix}_{(\widehat{\alpha}, \widehat{\beta})}. \tag{2.10}$$

2.1. Expectation-Maximization Algorithm

The EM algorithm proposed by Dempster et al. [17] can be used to obtain the MLEs of the parameters α and β. It is known that the EM algorithm converges more reliably than NR. Since Type-I PHCS can be considered as an incomplete data problem (see [33]), it is possible to apply EM algorithm to obtain the MLEs of the parameters. Now, let us denote the incomplete (censored) data by $\mathbf{Z} = (Z_1, Z_2, ..., Z_r)$ where $Z_j = \left(Z_{j1}, Z_{j2}, ..., Z_{jR_j}\right)$, $j = 1, 2, ..., r$ such that Z_j denotes the lifetimes of censored units at the time of $x_{(j)}$. Similarly, let Z_T denotes the lifetimes of censored units at the time of T. Now, combining both the observed and censored data, one can obtain the complete data which is given by $\mathbf{W} = (\mathbf{X}, \mathbf{Z})$. The corresponding likelihood equation of the complete data can be obtained as follows:

$$L_W(\alpha,\beta|\mathbf{x}) = \prod_{i=1}^{r} \left\{ f(x_{(i)};\alpha,\beta) \prod_{j=1}^{R_i} f(z_{ij};\alpha,\beta) \right\} \prod_{j=1}^{R_T} f(z_{Tj};\alpha,\beta) \tag{2.11}$$

Therefore, the log-likelihood equation can be easily obtained by taking the natural logarithm of Equation (2.11) as follows:

$$\begin{aligned}
l_W(\alpha,\beta|\mathbf{x}) &= \ln\left(L_W(\alpha,\beta|\mathbf{x})\right) = \sum_{i=1}^{r}\ln\left(\alpha\beta x_{(i)}^{\alpha-1}\exp\left\{-\beta x_{(i)}^{\alpha}\right\}\right) \\
&\quad + \sum_{i=1}^{r}\sum_{j=1}^{R_i}\ln\left(\alpha\beta z_{ij}^{\alpha-1}\exp\left\{-\beta z_{ij}^{\alpha}\right\}\right) + \sum_{j=1}^{R_T}\ln\left(\alpha\beta z_{Tj}^{\alpha-1}\exp\left\{-\beta z_{Tj}^{\alpha}\right\}\right) \\
&= n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{i=1}^{r}\ln\left(x_{(i)}\right) - \beta\sum_{i=1}^{r}x_{(i)}^{\alpha} + (\alpha-1)\sum_{i=1}^{r}\sum_{j=1}^{R_i}\ln\left(z_{ij}\right) \\
&\quad -\beta\sum_{i=1}^{r}\sum_{j=1}^{R_i}z_{ij}^{\alpha} + (\alpha-1)\sum_{j=1,r\neq m}^{R_T}\ln\left(z_{Tj}\right) - \beta\sum_{j=1,r\neq m}^{R_T}z_{Tj}^{\alpha} \qquad (2.12)
\end{aligned}$$

Note that the last two terms of Equation(2.12), should be considered only for the Case II. Based on the complete sample, the MLEs of the parameters α and β can be obtained by taking the derivatives of (2.12) with respect to α and β respectively and equating them to zero as follows:

$$\begin{aligned}
\frac{\partial l_W(\alpha,\beta|\mathbf{x})}{\partial\alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{r}\ln\left(x_{(i)}\right) - \beta\sum_{i=1}^{r}x_{(i)}^{\alpha}\ln\left(x_{(i)}\right) + \sum_{i=1}^{r}\sum_{j=1}^{R_i}\ln\left(z_{ij}\right) \\
&\quad -\beta\sum_{i=1}^{r}\sum_{j=1}^{R_i}z_{ij}^{\alpha}\ln\left(z_{ij}\right) + \sum_{j=1,r\neq m}^{R_T}\ln\left(z_{Tj}\right) - \beta\sum_{j=1,r\neq m}^{R_T}z_{Tj}^{\alpha}\ln\left(z_{Tj}\right) = 0, \qquad (2.13)
\end{aligned}$$

$$\frac{\partial l_W(\alpha,\beta|\mathbf{x})}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{r}x_{(i)}^{\alpha} - \sum_{i=1}^{r}\sum_{j=1}^{R_i}z_{ij}^{\alpha} - \sum_{j=1,r\neq m}^{R_T}z_{Tj}^{\alpha} = 0. \qquad (2.14)$$

Now, the conditional expectation of the log–likehood equation of the complete data given the observations should be computed in the E-step of the algorithm. However, the following conditional expectations are necessary to be computed:

$$\begin{aligned}
E\left(\frac{\partial l_W(\alpha,\beta|\mathbf{x})}{\partial\alpha}\bigg|\, x_{(i)},T\right) &= \frac{n}{\alpha} + \sum_{i=1}^{r}\ln\left(x_{(i)}\right) - \beta\sum_{i=1}^{r}x_{(i)}^{\alpha}\ln\left(x_{(i)}\right) \\
&\quad + \sum_{i=1}^{r}\sum_{j=1}^{R_i}E\left[\ln\left(Z_{ij}\right)\left(1-\beta Z_{ij}^{\alpha}\right)\Big|\, Z_{ij} > x_{(i)}\right] \\
&\quad + \sum_{j=1,r\neq m}^{R_T}E\left[\ln\left(Z_{Tj}\right)\left(1-\beta Z_{Tj}^{\alpha}\right)\Big|Z_{Tj} > T\right], \qquad (2.15)
\end{aligned}$$

$$\begin{aligned}
E\left(\frac{\partial l_W(\beta,\beta\mid\mathbf{x})}{\partial\beta}\bigg|x_{(i)},T\right) &= \frac{n}{\beta} - \sum_{i=1}^{r}x_{(i)}^{\alpha} - \sum_{i=1}^{r}\sum_{j=1}^{R_i}E\left[Z_{ij}^{\alpha}\Big|Z_{ij} > x_{(i)}\right] \\
&\quad - \sum_{j=1,r\neq m}^{R_T}E\left[Z_{Tj}^{\alpha}\Big|Z_{Tj} > T\right]. \qquad (2.16)
\end{aligned}$$

In order to compute the expectations given above, making use of the theorem proved in [33], the conditional probability function of the censored data given the observed data can be obtained as follows:

$$f(z_i|\mathcal{C}^*,\alpha,\beta) = \frac{f(z_i,\alpha,\beta)}{1-F(\mathcal{C}^*,\alpha,\beta)}, Z_i > \mathcal{C}^* \tag{2.17}$$

such that $\mathcal{C}^* = x_{(i)}$ for $i = 1, 2, \ldots, r$ and $\mathcal{C}^* = T$ for $i = T$. Thus, the following expectations can be obtained

$$\begin{aligned}\mathcal{E}_1(\mathcal{C}^*,\alpha,\beta) &= E\left[Z^\alpha \middle| Z > \mathcal{C}^*\right] = \frac{1}{1-F(\mathcal{C}^*,\alpha,\beta)}\int_{\mathcal{C}^*}^\infty t^\alpha f(t)dt \\ &= \frac{e^{-\beta\mathcal{C}^{*\alpha}}}{1-F(\mathcal{C}^*,\alpha,\beta)}\frac{(1+\beta\mathcal{C}^{*\alpha})}{\beta},\end{aligned} \tag{2.18}$$

$$\begin{aligned}\mathcal{E}_2(\mathcal{C}^*,\alpha,\beta) &= E\left(\ln(Z)(1-\beta Z^\alpha)\middle| Z > \mathcal{C}^*\right) \\ &= \frac{1}{1-F(\mathcal{C}^*,\alpha,\beta)}\int_{\mathcal{C}^*}^\infty \ln(t)(1-\beta t^\alpha) f(t)dt.\end{aligned} \tag{2.19}$$

Since it is hard to obtain a closed form solution to Equation (2.19), the integral is approximated via Monte Carlo integration method in the simulation. After updating the missing data with the expectations above in the E–step, the log–likelihood function is maximized in the M–step at the current state, say $\widehat{\alpha}_k$ and $\widehat{\beta}_k$ being the estimators of α and β and the following updating equations are computed:

$$\begin{aligned}\widehat{\alpha}_{k+1} &= n\left\{-\sum_{i=1}^r \ln\left(x_{(i)}\right) + \widehat{\beta}_{k+1}\sum_{i=1}^r x_{(i)}^{\widehat{\alpha}_k}\ln\left(x_{(i)}\right) - \sum_{i=1}^r R_i\mathcal{E}_2\left(x_{(i)},\widehat{\alpha}_k,\widehat{\beta}_{k+1}\right)\right. \\ &\quad \left. -R_T\mathcal{E}_2\left(T,\widehat{\alpha}_k,\widehat{\beta}_{k+1}\right)\right\}^{-1}\end{aligned} \tag{2.20}$$

$$\widehat{\beta}_{k+1} = n\left\{\sum_{i=1}^r x_{(i)}^{\widehat{\alpha}_k} + \sum_{i=1}^r R_i\mathcal{E}_1\left(x_{(i)},\widehat{\alpha}_k,\widehat{\beta}_k\right) + R_T\mathcal{E}_1\left(T,\widehat{\alpha}_k,\widehat{\beta}_k\right)\right\}^{-1}. \tag{2.21}$$

The EM estimates of (α, β) can be computed by an iterative procedure using Equation (2.21) and the iterations can be terminated when $|\widehat{\alpha}_{k+1} - \alpha_k| + |\widehat{\beta}_{k+1} - \beta_k| < \epsilon$ where $\epsilon > 0$ is a small real number.

2.2. Stochastic Expectation-Maximization Algorithm

The computations in the E–step of EM algorithm is complex. Therefore, Wei and Tanner [46] proposed a Monte Carlo version of EM algorithm. However, the M–step of this algorithm may take so much time. Diebolt and Celeux [16] introduced a stochastic–EM (SEM) algorithm by considering a simulated values from the conditional distribution. Asl et al. [4] used this algorithm successfully. In the SEM algorithm, firstly, one needs to generate R_i number of samples of z_{ij} where $i = 1, 2, ..., r$ and $j = 1, 2, ..., R_i$ using the following conditional CDF

$$F\left(z_{ij}; \alpha, \beta | z_{ij} > x_{(i)}\right) = \frac{F\left(z_{ij}; \alpha, \beta\right) - F\left(x_{(i)}; \alpha, \beta\right)}{1 - F\left(x_{(i)}; \alpha, \beta\right)}, z_{ij} > x_{(i)}. \tag{2.22}$$

Now, using Equations (2.13) and (2.14), the estimators of (α, β) at the $k+1$ step of the algorithm can be obtained as follows:

$$\begin{aligned} \widehat{\alpha}_{k+1} = n \Bigg[& -\sum_{i=1}^{r} \ln\left(x_{(i)}\right) + \widehat{\beta}_{k+1} \sum_{i=1}^{r} x_{(i)}^{\widehat{\alpha}_k} \ln\left(x_{(i)}\right) - \sum_{i=1}^{r}\sum_{j=1}^{R_i} \ln\left(z_{ij}\right)\left(1 - \widehat{\beta}_{k+1} z_{ij}^{\widehat{\alpha}_k}\right) \\ & - \sum_{j=1, r\neq m}^{R_T} \ln\left(z_{Tj}\right)\left(1 - \widehat{\beta}_{k+1} z_{Tj}^{\widehat{\alpha}_k}\right) \Bigg]^{-1} \end{aligned} \tag{2.23}$$

$$\widehat{\beta}_{k+1} = n\left[\sum_{i=1}^{r} x_{(i)}^{\widehat{\alpha}_k} + \sum_{i=1}^{r}\sum_{j=1}^{R_i} z_{ij}^{\widehat{\alpha}_k} + \sum_{j=1, r\neq m}^{R_T} z_{Tj}^{\widehat{\alpha}_k}\right]^{-1}. \tag{2.24}$$

Similarly, the iterations can be terminated when $|\widehat{\alpha}_{k+1} - \alpha_k| + |\widehat{\beta}_{k+1} - \beta_k| < \epsilon$ where $\epsilon > 0$ is a small real number.

2.3. Fisher Information Matrix

In this subsection, by making use of the idea of missing information principle proposed by Louis [31], we can obtain the observed Fisher information matrix. Louis [31] suggested the following relation

$$\mathbf{I}_X(\psi) = \mathbf{I}_W(\psi) - \mathbf{I}_{W|X}(\psi) \tag{2.25}$$

where $\psi = (\alpha, \beta)'$, $\mathbf{I}_X(\psi), \mathbf{I}_W(\psi)$ and $\mathbf{I}_{W|X}(\psi)$ are the observed, complete and missing information matrices respectively. Now, the complete information matrix of a complete data set following the Weibull distribution can be obtained as

$$\begin{aligned} \mathbf{I}_W(\psi) &= -E\left(\frac{\partial^2 \ln \mathcal{L}}{\partial \psi^2}\right) \\ &= E\begin{bmatrix} \frac{n}{\alpha^2} + \beta \sum_{i=1}^{n} x_i^{\alpha} & \sum_{i=1}^{n} x_i^{\alpha} \ln x_i \\ \sum_{i=1}^{n} x_i^{\alpha} \ln x_i & \frac{n}{\beta^2} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{aligned} \tag{2.26}$$

where

$$\begin{aligned} b_{11} &= \frac{n}{\alpha^2} + n\alpha\beta^2 \int_0^{\infty} \frac{x^{2\alpha-1}\ln(x)}{\exp(\beta \mathrm{x}^{\alpha})} dx \\ b_{12} &= b_{21} = n\alpha\beta \int_0^{\infty} \frac{x^{2\alpha-1}\ln(x)}{\exp(\beta \mathrm{x}^{\alpha})} dx \\ b_{22} &= \frac{n}{\beta^2} \end{aligned}$$

and $\ln\mathcal{L}(\psi) = n\ln\alpha + n\ln\beta + (\alpha-1)\sum_{i=1}^{n} x_i + \beta\sum_{i=1}^{n} x_i^{\alpha}$ is the corresponding log–likelihood equation. Moreover, the missing information matrix $\mathbf{I}_{W|X}(\psi)$ is given by

$$\mathbf{I}_{W|X}(\psi) = \sum_{i=1}^{r} R_i \mathbf{I}_{W|X}^{(i)}(\psi) + R_T \mathbf{I}_{W|X}^{*}(\psi) \tag{2.27}$$

where $\mathbf{I}_{W|X}^{(i)}(\psi)$ and $\mathbf{I}_{W|X}^{*}(\psi)$ are the information matrices of a single observation from a truncated Weibull distribution from left at $x_{(i)}$ and T respectively, such that

$$\mathbf{I}_{W|X}^{(i)}(\psi) \; = \; -E\left(\frac{\partial^2 \ln\mathcal{L}}{\partial\psi^2}\ln\left\{f\left(z_{ij};\psi|z_{ij} > x_{(i)}\right)\right\}\right).$$

Now to calculate the missing information matrix $\mathbf{I}_{W|X}^{(i)}(\psi)$, the conditional distribution given in Equation (2.17) is used to obtain the following

$$L_f \; = \; \ln\left(f(z_{ij} \mid z_{ij} > x_{(i)})\right) = \ln(\alpha) + \ln(\beta) + (\alpha-1)\ln(z_{ij}) - \beta z_{ij}^{\alpha} + \beta x_{(i)}^{\alpha}.$$

The second partial derivatives of L_f are obtained as follows

$$\begin{aligned}
\frac{\partial^2 L_f}{\partial\alpha^2} &= -\frac{1}{\alpha^2} - \beta z_{ij}^{\alpha}\ln(z_{ij})^2 + \beta x_{(i)}^{\alpha}\ln(x_{(i)})^2 \\
\frac{\partial^2 L_f}{\partial\alpha\partial\beta} &= -z_{ij}^{\alpha}\ln(z_{ij}) + x_{(i)}^{\alpha}\ln(x_{(i)}) \\
\frac{\partial^2 L_f}{\partial\beta^2} &= -\frac{1}{\beta^2}.
\end{aligned}$$

Now, in order to obtain the information matrices, the negative expected values of the quantities above are computed respectively as follows

$$\begin{aligned}
E\left(-\frac{\partial^2 L_f}{\partial\alpha^2}\right) &= \frac{1}{\alpha^2} + \beta\mathcal{E}_4\left(x_{(i)},\alpha,\beta\right) - \beta x_{(i)}^{\alpha}\ln(x_{(i)})^2 \\
E\left(-\frac{\partial^2 L_f}{\partial\alpha\partial\beta}\right) &= \mathcal{E}_3\left(x_{(i)},\alpha,\beta\right) - x_{(i)}^{\alpha}\ln(x_{(i)}) \\
E\left(-\frac{\partial^2 L_f}{\partial\beta^2}\right) &= \frac{1}{\beta^2}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{E}_3\left(\mathcal{C}^*,\alpha,\beta\right) = E\left(Z^{\alpha}\ln(Z) \mid Z > \mathcal{C}^*\right) &= \frac{1}{1-F\left(\mathcal{C}^*,\alpha,\beta\right)}\int_{\mathcal{C}^*}^{\infty} t^{\alpha}\ln(t)f(t)dt \\
\mathcal{E}_4\left(\mathcal{C}^*,\alpha,\beta\right) = E\left(Z^{\alpha}\ln(Z)^2 \mid Z > \mathcal{C}^*\right) &= \frac{1}{1-F\left(\mathcal{C}^*,\alpha,\beta\right)}\int_{\mathcal{C}^*}^{\infty} t^{\alpha}\ln(t)^2 f(t)dt.
\end{aligned}$$

Using similar arguments, the information matrix $\mathbf{I}_{W|X}^{*}(\psi)$ can also be computed easily. Then, using (2.25)–(2.26), the asymptotic variance-covariance matrix of $\widehat{\psi}$ can be computed by inverting the observed information matrix $\mathbf{I}_X\left(\widehat{\psi}\right)$. Note that $\widehat{\psi}$ is computed using the NR estimates.

3. Bayesian Estimation

In this section, following Kundu [26], we consider the Bayesian estimation for the parameters of the Weibull distribution under the assumption that the random variables α and β have independent gamma priors such that $\alpha \sim Gamma(a,b)$ and $\beta \sim Gamma(c,d)$. Therefore, the joint prior density of α and β can be written as

$$\pi\left(\alpha,\beta\right) \propto \alpha^{a-1}\beta^{c-1}\exp\{-(b\alpha+d\beta)\},\ a,b,c,d>0.$$

Now, the posterior distribution of α and β can be obtained as follows

$$\begin{aligned}\pi\left(\alpha,\beta\mid\mathbf{x}\right) &= \frac{L\left(\alpha,\beta\mid\mathbf{x}\right)\pi\left(\alpha,\beta\right)}{\int_0^\infty\int_0^\infty L\left(\alpha,\beta\mid\mathbf{x}\right)\pi\left(\alpha,\beta\right)d\alpha d\beta}\\ &= \frac{\left(\prod_{i=1}^r x_{(i)}^{\alpha-1}\right)\beta^{c+r-1}\alpha^{a+r-1}}{\Gamma(c+r)\Psi(a,c,\mathbf{x})}\exp\left\{d-b\alpha+\sum_{i=1}^r(1+R_i)x_{(i)}^\alpha+C^\alpha R_T\right\} \qquad (3.1)\end{aligned}$$

where

$$\Psi(a,c,\mathbf{x})=\int_0^\infty\frac{\alpha^{a+r-1}\exp\left\{-b\alpha\right\}\left(\prod_{i=1}^r x_{(i)}^{\alpha-1}\right)}{\left[d+\sum_{i=1}^r(1+R_i)x_{(i)}^\alpha+C^\alpha R_T\right]^{a+c+r}}d\alpha.$$

In this paper, three different loss functions are considered. One of them is the most commonly used squared error loss function (SEL) which is defined as follows:

$$L_S\left(\widehat{t}(\psi),t(\psi)\right)=\left(\widehat{t}(\psi)-t(\psi)\right)^2$$

where $\widehat{t}(\psi)$ is an estimator of $t(\psi)$. SEL is a symmetric loss function which gives equal weights to both underestimation and overestimation. However, in certain situation overestimation and underestimation may have serious consequences ([37]). In such cases using SEL may not be appropriate. One remedy is to use linear-exponential (LINEX) loss function. LINEX is an asymmetric loss function introduced by Varian [45] as follows

$$L_L\left(\widehat{t}(\psi),t(\psi)\right)=e^{\nu\left(\widehat{t}(\psi)-t(\psi)\right)}-\nu\left(\widehat{t}(\psi)-t(\psi)\right)-1,\ \nu\neq 0.$$

The LINEX loss function is a convex function whose shape is determined by the value of ν. The negative (positive) value of ν gives more weight to overestimation (underestimation) and its magnitude reflects the degree of asymmetry. It is seen that, for $\nu=1$, the function is quite asymmetric with overestimation being costlier than underestimation. If $\nu<0$, it rises almost exponentially when the estimation error $\widehat{t}(\psi)-t(\psi)<0$ and almost linearly if $\widehat{t}(\psi)-t(\psi)>0$. For small values of $|\nu|$, the LINEX loss function is almost symmetric and not far from squared error loss function.

Under the SEL function, the Bayes estimators of α and β which are the expected values of the corresponding posterior distributions are computed respectively as follows

$$\widehat{\alpha}_S = E\left(\pi\left(\alpha \mid \mathbf{x}\right)\right) = \frac{\Psi(a+1, c-1, \mathbf{x})}{\Psi(a, c, \mathbf{x})} \tag{3.2}$$

and

$$\widehat{\beta}_S = E\left(\pi\left(\beta \mid \mathbf{x}\right)\right) = (a+c+r)\frac{\Psi(a, c+1, \mathbf{x})}{\Psi(a, c, \mathbf{x})}. \tag{3.3}$$

Since the Bayes estimators given above includes the complicated integral function $\Psi(a, c+1, \mathbf{x})$ we also consider using the Bayes estimate of $t(\psi)$ under the LINEX loss function is given by

$$\widehat{t}_L(\psi) = -\frac{1}{\nu}\ln\left[E_t\left(e^{-\nu t(\psi)} \mid \mathbf{x}\right)\right] = -\frac{1}{\nu}\ln\left[\int_0^\infty\int_0^\infty e^{-\nu t(\psi)}\pi(\alpha, \beta \mid \mathbf{x})d\alpha d\beta\right].$$

Another asymmetric loss function that gained more attention is the general entropy loss (GEL) function given by

$$L_{GEL}\left(\widehat{t}(\psi), t(\psi)\right) = \left(\frac{\widehat{t}(\psi)}{t(\psi)}\right)^{\kappa} - \kappa\ln\left(\frac{\widehat{t}(\psi)}{t(\psi)}\right) - 1,\ \kappa \neq 0$$

where κ is the shape parameter showing the departure from symmetry. When $\kappa > 0$, the overestimation is considered to be more serious than underestimation and for $\kappa < 0$ vice versa. The Bayes estimator under GEL function is given by

$$\widehat{t}_{GEL}(\psi) = \left[E_t\left(t(\psi)^{-\kappa} \mid \mathbf{x}\right)\right]^{-1/\kappa} = \left[\int_0^\infty\int_0^\infty t(\psi)^{-\kappa}\pi(\alpha, \beta \mid \mathbf{x})d\alpha\, d\beta\right]^{-1/\kappa}.$$

3.1. Tierney-Kadane Approximation

In this subsection, the approximation method of Tierney and Kadane [44] is used to obtain the approximate Bayes estimators under SEL, LINEX and GEL loss functions. Now, we consider the following functions

$$\Delta(\alpha, \beta) = \frac{1}{n}\ln[L(\alpha, \beta \mid \mathbf{x})\pi(\alpha, \beta)], \tag{3.4}$$

$$\Delta^*(\alpha, \beta) = \frac{1}{n}\ln[L(\alpha, \beta \mid \mathbf{x})\pi(\alpha, \beta)t(\psi)]. \tag{3.5}$$

Now assume that $(\widetilde{\alpha}_\Delta, \widetilde{\beta}_\Delta)$ and $(\widetilde{\alpha}_{\Delta^*}, \widetilde{\beta}_{\Delta^*})$ respectively maximize the functions $\Delta(\alpha, \beta)$ and $\Delta^*(\alpha, \beta)$. Then the approximation method of Tierney and Kadane [44] is given by

$$\widetilde{t}_{\mathrm{SEL}}(\alpha, \beta) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[n\left(\Delta_1^*\left(\widetilde{\alpha}_{\Delta^*}, \widetilde{\beta}_{\Delta^*}\right) - \Delta\left(\widetilde{\alpha}_\Delta, \widetilde{\beta}_\Delta\right)\right)\right]$$

where $|\Sigma|$ and $|\Sigma^*|$ are the negative of inverses the second derivative matrices of $\Delta(\alpha,\beta)$ and $\Delta_1^*(\alpha,\beta)$ respectively obtained at $(\widetilde{\alpha}_\Delta, \widetilde{\beta}_\Delta)$ and $(\widetilde{\alpha}_{\Delta^*}, \widetilde{\beta}_{\Delta^*})$. The function $\Delta(\alpha,\beta)$ can be easily obtained by using the Equation (3.4) as follows

$$\begin{aligned}\Delta(\alpha,\beta) &= \frac{1}{n}\left[\ln(M) + (\alpha-1)\sum_{i=1}^{r}\ln(x_{(i)}) - \beta\left(d + b\alpha + \sum_{i=1}^{r}(1+R_i)x_{(i)}^{\alpha} + C^{\alpha}R_T\right)\right. \\ &\left. +(a+c+r-1)\ln(\beta) + (a+r-1)\ln(\alpha)\right] \qquad (3.6)\end{aligned}$$

where $M = \frac{d^c b^a}{\Gamma(c)\Gamma(a)}$. Now, differentiating Equation (3.6) with respect to α and β solving for these parameters, one gets the following equations

$$\begin{aligned}\widetilde{\alpha}_\Delta &= (a+r-1)\left[\beta\left(b + \sum_{i=1}^{r}(1+R_i)x_{(i)}^{\alpha} + C^{\alpha}R_T\right) - \sum_{i=1}^{r}\ln(x_{(i)})\right]^{-1}, \\ \widetilde{\beta}_\Delta &= (a+c+r-1)\left[\sum_{i=1}^{r}(1+R_i)x_{(i)}^{\alpha} + C^{\alpha}R_T + d + b\alpha\right]^{-1}.\end{aligned}$$

Since it is easy to obtain the second derivatives and the related Hessian matrices, we skip this part. Thus under the SEL function, the approximate Bayes estimators are computed by

$$\begin{aligned}\widetilde{\alpha}_{\text{SEL}} &= \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[n\left(\Delta_{1\alpha}^*\left(\widetilde{\alpha}_{\Delta^*},\widetilde{\beta}_{\Delta^*}\right) - \Delta\left(\widetilde{\alpha}_\Delta,\widetilde{\beta}_\Delta\right)\right)\right], \\ \widetilde{\beta}_{\text{SEL}} &= \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[n\left(\Delta_{1\beta}^*\left(\widetilde{\alpha}_{\Delta^*},\widetilde{\beta}_{\Delta^*}\right) - \Delta\left(\widetilde{\alpha}_\Delta,\widetilde{\beta}_\Delta\right)\right)\right]\end{aligned}$$

where $\Delta_{1\alpha}^*(\alpha,\beta) = \Delta(\alpha,\beta) + \frac{1}{n}\ln(\alpha)$ for $t(\alpha,\beta) = \alpha$ and $\Delta_{1\beta}^*(\alpha,\beta) = \Delta(\alpha,\beta) + \frac{1}{n}\ln(\beta)$ for $t(\alpha,\beta) = \beta$.

One can also compute the Bayes estimators under the LINEX loss and get

$$\widetilde{t}_{\text{LINEX}}(\alpha,\beta) = \sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[n\left\{\Delta_2^*\left(\widetilde{\alpha}_{\Delta^*},\widetilde{\beta}_{\Delta^*}\right) - \Delta\left(\widetilde{\alpha}_\Delta,\widetilde{\beta}_\Delta\right)\right\}\right].$$

Letting $t(\alpha,\beta) = e^{-\nu\alpha}$, one gets $\Delta_{2\alpha}^*(\alpha,\beta) = \Delta(\alpha,\beta) - \frac{1}{n}\nu\alpha$ and letting $t(\alpha,\beta) = e^{-\nu\beta}$, $\Delta_{2\beta}^*(\alpha,\beta) = \Delta(\alpha,\beta) - \frac{1}{n}\nu\beta$. Thus, approximate Bayes estimators under LINEX function are computed as

$$\begin{aligned}\widetilde{\alpha}_{\text{LINEX}} &= -\frac{1}{\nu}\ln\left(\sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[n\left(\Delta_{2\alpha}^*\left(\widetilde{\alpha}_{\Delta^*},\widetilde{\beta}_{\Delta^*}\right) - \Delta\left(\widetilde{\alpha}_\Delta,\widetilde{\beta}_\Delta\right)\right)\right]\right), \\ \widetilde{\beta}_{\text{LINEX}} &= -\frac{1}{\nu}\ln\left(\sqrt{\frac{|\Sigma^*|}{|\Sigma|}}exp\left[n\left(\Delta_{2\beta}^*\left(\widetilde{\alpha}_{\Delta^*},\widetilde{\beta}_{\Delta^*}\right) - \Delta\left(\widetilde{\alpha}_\Delta,\widetilde{\beta}_\Delta\right)\right)\right]\right).\end{aligned}$$

Finally, letting $t(\alpha,\beta) = \alpha^{-\kappa}$, one gets $\Delta_{3\alpha}^*(\alpha,\beta) = \Delta(\alpha,\beta) - \frac{\kappa}{n}\ln(\alpha)$ and letting $t(\alpha,\beta) = \beta^{-\kappa}$, $\Delta_{3\beta}^*(\alpha,\beta) = \Delta(\alpha,\beta) - \frac{\kappa}{n}\ln(\beta)$. Thus, approximate Bayes estimators under GEL function are obtained by

$$\widetilde{\alpha}_{\text{GEL}} = \left(\sqrt{\frac{|\Sigma^*|}{|\Sigma|}} exp\left[n\left(\Delta^*_{3\alpha}\left(\widetilde{\alpha}_{\Delta^*}, \widetilde{\beta}_{\Delta^*}\right) - \Delta\left(\widetilde{\alpha}_{\Delta}, \widetilde{\beta}_{\Delta}\right)\right)\right]\right)^{-1/\kappa},$$

$$\widetilde{\beta}_{\text{GEL}} = \left(\sqrt{\frac{|\Sigma^*|}{|\Sigma|}} exp\left[n\left(\Delta^*_{3\beta}\left(\widetilde{\alpha}_{\Delta^*}, \widetilde{\beta}_{\Delta^*}\right) - \Delta\left(\widetilde{\alpha}_{\Delta}, \widetilde{\beta}_{\Delta}\right)\right)\right]\right)^{-1/\kappa}.$$

3.2. MCMC Method

Metropolis–Hastings (MH) algorithm, a method for generating random samples from the posterior distribution using a proposal density, is considered in this subsection. A symmetric proposal density of type $q(\theta'|\theta) = q(\theta|\theta')$ may be considered generally, where θ is the parameter vector of the distribution considered. Following Dey et al. [18], we consider a bivariate normal distribution as the proposal density such that $q(\theta'|\theta) = N(\theta|\mathbf{V}_\theta)$ where $\mathbf{V}_\theta$ is the covariance matrix and $\theta = (\alpha, \beta)$. Although, the bivariate normal distribution may generate negative observations, the domain of both shape and scale parameters of Weibull distribution is positive. Therefore, the following steps of MH algorithm is used to generate MCMC sample from the posterior density given by (3.1)

(1) Set the initial parameter values as $\theta = \theta_0$.

(2) For $j = 1, 2, ..., N$, repeat the following steps:

   (i) Set $\theta = \theta_{j-1}$
   (ii) Generate new parameters λ from bivariate normal $N_2(\ln(\theta), \mathbf{V}_\theta)$
   (iii) Compute $\theta_{new} = \exp(\lambda)$
   (iv) Calculate $\gamma = \min\left(1, \frac{\pi(\theta_{new}|\mathbf{x})\theta_{new}}{\pi(\theta|\mathbf{x})\theta}\right)$
   (v) Set $\theta_j = \theta_{new}$ with probability λ, otherwise $\theta_j = \theta$.

After generating the MCMC sample, some of the initial samples, say N_0, can be discarded as burn-in process and the estimations can be computed via the remaining ones $(M = N - N_0)$ under SEL, LINEX and GEL loss functions as follows

$$\widehat{t}_{\text{SEL}}(\psi) = \frac{1}{M}\sum_{i=1}^{M} t(\psi_i),$$

$$\widehat{t}_{\text{LINEX}}(\psi) = -\frac{1}{\nu}\ln\left(\frac{1}{M}\sum_{i=1}^{M} exp\left(-\nu t(\psi_i)\right)\right),$$

$$\widehat{t}_{\text{GEL}}(\psi) = \left(\frac{1}{M}\sum_{i=1}^{M}\left(t(\psi_i)^{-\kappa}\right)\right)^{-1/\kappa}.$$

The main advantage of MCMC method over Tierney–Kadane method is that the MCMC samples can also be used to compute highest posterior density (HPD) intervals. Chen and Shao [14] proposed a method to compute the HPD intervals using MCMC samples. This method has been used in the literature extensively. Now, consider the posterior density $\pi(\theta|\mathbf{x})$. Assume that the pth quantile of the distribution is given by $\theta^{(p)} = \inf\{\theta : \Pi(\theta|\mathbf{x}) \geq p; 0 < p < 1\}$ where $\Pi(\theta|\mathbf{x})$ denotes the posterior distribution function of θ. Now, for a given θ^*, a simulation consistent estimator of $\Pi(\theta^*|\mathbf{x})$ can be computed as

$$\Pi(\theta^*|\mathbf{x}) = \frac{1}{M}\sum_{i=1}^{M} I(\theta \leq \theta^*)$$

where $I(\theta \leq \theta^*)$ is an indicator function. Then, the estimate of $\Pi(\theta^*|\mathbf{x})$ is given as

$$\widehat{\Pi}(\theta^*|\mathbf{x}) = \begin{cases} 0 & \text{if } \theta^* < \theta_{(N_0)} \\ \sum_{j=N_0}^{i} \gamma_j & \text{if } \theta_{(i)} < \theta^* < \theta_{(i+1)} \\ 1 & \text{if } \theta_{(M)} \end{cases}$$

where $\gamma_j = 1/M$ and $\theta_{(j)}$ is the jth ordered value of θ_j. $\theta^{(p)}$ can be approximated by the following

$$\theta^{(p)} = \begin{cases} \theta_{(N_0)} & \text{if } p = 0 \\ \theta_{(j)} & \text{if } \sum_{j=N_0}^{i-1} \gamma_j < p < \sum_{j=N_0}^{i} \gamma_j \end{cases}$$

Now, one can construct the $100(1-p)\%$ confidence intervals where $0 < p < 1$ as $\left(\widehat{\theta}^{j/s}, \widehat{\theta}^{(j+[(1-p)s])/s}\right)$, $j = 1, 2, ..., s - [(1-p)s]$ such that $[v]$ denotes the greatest integer less than or equal to v. At the end, the HPD credible interval of θ is the one having the shortest length.

4. Shrinkage Estimation

In the problem of statistical inference there may be some non-sample prior information that practitioner may have from previous experiences or knowledge Saleh [39]. For example, medical experts may know the average time of that a vaccine may take to relief a pain according their medical knowledge. This non-sample Prior information on the parameters in a statistical model generally leads to an improved inference procedure in problems of statistical inference. Restricted models arise from the incorporation of the known prior information in the model in the form of a constraint. The estimators obtained from restricted (unrestricted) model is known as the restricted (unrestricted) estimators. The results of an analysis of the restricted and unrestricted models can be weighted against loss of efficiency and validity of the constraints in deciding a choice between these two extreme inference methods, when a full confidence may not be in the prior information (see [2]).

Bancroft [12] was the first to consider a pre–test procedure when there is doubt that the prior information is not certain (uncertain prior information). After the pioneering study [12], pre–test estimators has gained much attention. Thompson [43] defined an efficient shrinkage estimator. Following [43], shrinkage estimation of the Weibull parameters has been discussed by a number of authors, including [41], [35], [36] and [42]. We also refer to the following book and papers among others: [22], [40], [39], [23].

Now suppose that there is an uncertain prior information in the form of $\theta = \theta_0$ where θ is the parameter of a distribution of interest. Our aim is to estimate θ using a pre–test estimation strategy and this prior information. Therefore, we consider the following hypothesis to check the validity of this information

$$H_0 : \theta = \theta_0$$
$$H_0 : \theta \neq \theta_0$$

It is known that under H_0, the asymptotic distribution of $\sqrt{D}(\widehat{\theta} - \theta_0)$ is normal with $N(0, \sigma^2_{\widehat{\theta}})$ and the related test statistics can be defined as follows

$$W_D = \left(\frac{\sqrt{D}(\widehat{\theta} - \theta_0)}{\sigma_{\widehat{\theta}}} \right)^2 .$$

One can reject the null hypothesis when $W_D > \chi^2_1(\lambda)$ based on the distribution of W_D where λ can be treated as the degree of trust in the prior information about the parameter such that $\theta = \theta_0$, see [39] and [1]. Thus, the shrinkage pre–test estimator (SPT) can be defined as

$$\widehat{\theta}_{\mathrm{SPT}} = \lambda\theta_0 + (1 - \lambda)\widehat{\theta} I \left(W_D < \chi^2_1(\lambda) \right)$$

where $I(A)$ is the indicator of the set A.

5. Monte Carlo Simulation Experiments

In this section, we conduct a simulation study to illustrate the performance of the different estimation techniques discussed in this paper by considering $(n, m) = (30, 15)$, different values of predetermined time $T = 1.0, 2.0$, and the real values of the parameters are chosen as $\alpha = 0.5$ and $\beta = 1.5$ in all cases. The following three schemes are considered in the simulation

- Scheme 1: $R = (0^{m-1}, n - m)$
- Scheme 2: $R = (n - m, 0^{m-1})$
- Scheme 3: $R = (2^5, 0^{m-6}, n - m - 10)$

It is noted that Scheme 1 is the type-II censoring such that $n-m$ units are removed from the experiment at the time of the m-th failure; in Scheme 2, $n-m$ units are removed at the time of the first failure. However, in Scheme 3, a progressive type-II censoring scheme allowing different numbers of censoring within the experiment is considered. The progressive type–II censored data from Weibull distribution is generated using algorithm proposed by Balakrishnan and Aggarwala [7]. The maximum likelihood estimators of α and β are obtained using NR, EM and SEM algorithms. In computing the Bayes estimates, two different priors are used such as the non–informative priors as $a=b=c=d=0$ and the informative priors where we assume that we have past samples from Weibull(α,β) distribution, say K samples and their corresponding MLEs as $\left(\widehat{\alpha}_j,\widehat{\beta}_j\right)$, $j=1,2,...,K$. Now, equating the sample means and variances of these values to the means and variances of gamma priors respectively and solving the equations for $K=1000$, and $n=30$ being the sample size of past samples, we obtain the following informative prior values, $a=43.77, b=83.45, c=24.24, d=15.47$.

Bayes estimates are computed under SEL, LINEX, GEL loss functions. Notice that for the LINEX loss function, we considered two values of ν as $\nu=-0.5, 0.5$ giving more weight to underestimation and overestimation respectively. Similarly, two choices of κ such as $\kappa=-0.5, 0.5$ are taken into account under GEL function. Moreover, 6000 MCMC samples are generated and MCMC estimations are computed under the listed loss function and respective parameter values. The first 1000 MCMC samples are considered as a burn–in sample so that the average values and MSEs are computed via the remaining 5000 samples for each replicate in the simulation.

For the shrinkage estimators, the test statistic W_D is calculated and then shrinkage pre–test (SPT) estimators are obtained. The distribution of the test statistic W_D is computed under the null hypothesis, that is, $H_0:\theta=\theta_0$. Moreover, we take $\lambda=0.5$ giving equal weight to both restricted and unrestricted estimators and the type one test error is set to 0.05 in testing the hypothesis, prior values of the parameters are taken as $\alpha_0=0.7, \beta_0=1.7$ for practical purposes. The MLE shrinkage pre–test estimators are obtained using NR algorithm and also the Bayes estimator with T–K method under different loss functions.

Totally, 5000 repetitions are carried out and average values (Avg), mean squared errors (MSE), confidence/ credible interval lengths (IL) and coverage probabilities (CP) are obtained for the purpose of comparison. MSEs of the estimators are computed as follows

$$\mathrm{MSE}\left(\widehat{\theta}\right)=\frac{1}{5000}\sum_{i=1}^{5000}\left(\widehat{\theta}_i-\theta\right)^2$$

where $\widehat{\theta}_i$ is NR, EM, SEM, SPT estimators and Bayes estimators under SEL loss function in the ith replication. However, the MSEs of Bayes estimators under

Table 1: Average values (Avg) and the corresponding MSEs of the estimators NR, EM and SEM.

			NR		EM		SEM	
T	R		α	β	α	β	α	β
1	1	Avg	0.5559	1.8433	0.5276	1.6415	0.5352	1.6669
		MSE	0.0224	0.6931	0.0112	0.2207	0.0139	0.2788
	2	Avg	0.5279	1.6480	0.5239	1.5958	0.5294	1.6065
		MSE	0.0158	0.3385	0.0135	0.1772	0.0141	0.1946
	3	Avg	0.5435	1.7540	0.5315	1.6490	0.5330	1.6485
		MSE	0.0175	0.5108	0.0127	0.2552	0.0131	0.2647
2	1	Avg	0.5559	1.8433	0.5276	1.6416	0.5353	1.6670
		MSE	0.0224	0.6930	0.0112	0.2206	0.0139	0.2788
	2	Avg	0.5280	1.6412	0.5233	1.5947	0.5287	1.6020
		MSE	0.0137	0.3045	0.0124	0.1723	0.0129	0.1869
	3	Avg	0.5476	1.7676	0.5339	1.6578	0.5353	1.6567
		MSE	0.0172	0.5001	0.0126	0.2494	0.0130	0.2593

Table 2: Average values (Avg) and the corresponding MSEs of the Bayes estimators with T-K approximation.

Informative Priors

			SEL		LINEX				GEL			
					$\nu=-0.5$		$\nu=0.5$		$\kappa=-0.5$		$\kappa=0.5$	
T	R		α	β	α	β	α	β	α	β	α	β
1	1	Avg	0.5210	1.5773	0.5220	1.5946	0.5200	1.5600	0.5192	1.5665	0.5155	1.5449
		MSE	0.0018	0.0257	0.0002	0.0036	0.0002	0.0029	0.0008	0.0012	0.0007	0.0011
	2	Avg	0.5199	1.5632	0.5209	1.5807	0.5189	1.5460	0.5180	1.5522	0.5141	1.5302
		MSE	0.0018	0.0252	0.0002	0.0034	0.0002	0.0029	0.0008	0.0012	0.0007	0.0012
	3	Avg	0.5206	1.5702	0.5215	1.5869	0.5196	1.5537	0.5188	1.5598	0.5151	1.5389
		MSE	0.0019	0.0273	0.0002	0.0037	0.0002	0.0031	0.0008	0.0013	0.0008	0.0012
2	1	Avg.3	0.5210	1.5773	0.5220	1.5946	0.5200	1.5600	0.5192	1.5665	0.5155	1.5449
		MSE	0.0018	0.0257	0.0002	0.0036	0.0002	0.0029	0.0008	0.0012	0.0007	0.0011
	2	Avg	0.5193	1.5605	0.5203	1.5772	0.5183	1.5442	0.5175	1.5501	0.5137	1.5291
		MSE	0.0018	0.0268	0.0002	0.0036	0.0002	0.0031	0.0008	0.0013	0.0008	0.0013
	3	Avg	0.5210	1.5719	0.5220	1.5885	0.5200	1.5554	0.5192	1.5615	0.5156	1.5408
		MSE	0.0019	0.0271	0.0002	0.0037	0.0002	0.0031	0.0008	0.0013	0.0008	0.0012

Non-Informative Priors

			SEL		LINEX				GEL			
					$\nu=-0.5$		$\nu=0.5$		$\kappa=-0.5$		$\kappa=0.5$	
T	R		α	β	α	β	α	β	α	β	α	β
1	1	Avg	0.5519	1.8793	0.5441	1.9056	0.5353	1.6135	0.5560	1.9979	0.5397	1.8118
		MSE	0.0211	0.7592	0.0022	0.0746	0.0022	0.0329	0.0083	0.0243	0.0086	0.0250
	2	Avg	0.5298	1.6345	0.5322	1.7031	0.5248	1.5557	0.5234	1.5990	0.5100	1.5181
		MSE	0.0159	0.3310	0.0020	0.0425	0.0019	0.0355	0.0065	0.0140	0.0067	0.0144
	3	Avg	0.5411	1.7500	0.5408	1.8062	0.5341	1.6223	0.5384	1.7550	0.5262	1.6597
		MSE	0.0170	0.5159	0.0020	0.0582	0.0020	0.0402	0.0065	0.0166	0.0067	0.0172
2	1	Avg	0.5519	1.8793	0.5441	1.9057	0.5353	1.6136	0.5560	1.9979	0.5397	1.8118
		MSE	0.0211	0.7591	0.0022	0.0746	0.0021	0.0329	0.0083	0.0243	0.0086	0.0250
	2	Avg	0.5290	1.6203	0.5315	1.6773	0.5254	1.5573	0.5237	1.5920	0.5125	1.5253
		MSE	0.0137	0.2934	0.0017	0.0366	0.0017	0.0322	0.0056	0.0118	0.0057	0.0122
	3	Avg	0.5453	1.7632	0.5451	1.8196	0.5384	1.6361	0.5427	1.7685	0.5307	1.6741
		MSE	0.0167	0.5052	0.0020	0.0568	0.0020	0.0389	0.0062	0.0152	0.0064	0.0159

LINEX and GEL loss functions are computed respectively by

$$\mathrm{MSE}_{\mathrm{LINEX}}\left(\widehat{\theta}\right) = \frac{1}{5000}\sum_{i=1}^{5000}\left(e^{\nu\left(\widehat{\theta}_i-\theta\right)} - \nu\left(\widehat{\theta}_i-\theta\right) - 1\right),$$

$$\mathrm{MSE}_{\mathrm{GEL}}\left(\widehat{\theta}\right) = \frac{1}{5000}\sum_{i=1}^{5000}\left(\left(\frac{\widehat{\theta}_i}{\theta}\right)^{\kappa} - \kappa\ln\left(\frac{\widehat{\theta}_i}{\theta}\right) - 1\right).$$

All of the computations are performed using the R Statistical Program [38]. All the results are presented in Tables 1–5.

Based on Table 1, we can conclude that EM and SEM estimates are quiet preferable to the NR method for all schemes and Ts. Both MSEs and Avgs for

Table 3: Average values (Avg) and the corresponding MSEs of the Bayes estimators with MCMC method.

Informative Priors												
			SEL		LINEX				GEL			
					$\nu=-0.5$		$\nu=0.5$		$\kappa=-0.5$		$\kappa=0.5$	
T	R		α	β	α	β	α	β	α	β	α	β
1	1	Avg	0.5210	1.5770	0.5220	1.5944	0.5200	1.5601	0.5192	1.5663	0.5155	1.5448
		MSE	0.0018	0.0262	0.0002	0.0039	0.0002	0.0030	0.0008	0.0015	0.0008	0.0012
	2	Avg	0.5199	1.5631	0.5209	1.5806	0.5188	1.5462	0.5179	1.5522	0.5140	1.5304
		MSE	0.0018	0.0253	0.0002	0.0037	0.0002	0.0029	0.0009	0.0014	0.0008	0.0012
	3	Avg	0.5206	1.5703	0.5216	1.5871	0.5197	1.5540	0.5188	1.5599	0.5151	1.5391
		MSE	0.0019	0.0277	0.0002	0.0041	0.0002	0.0032	0.0009	0.0016	0.0008	0.0013
2	1	Avg	0.5210	1.5770	0.5220	1.5944	0.5200	1.5601	0.5192	1.5663	0.5155	1.5449
		MSE	0.0018	0.0262	0.0002	0.0039	0.0002	0.0030	0.0008	0.0015	0.0008	0.0012
	2	Avg	0.5193	1.5604	0.5203	1.5770	0.5183	1.5442	0.5174	1.5500	0.5137	1.5292
		MSE	0.0018	0.0271	0.0002	0.0039	0.0002	0.0031	0.0009	0.0015	0.0008	0.0013
	3	Avg	0.5210	1.5720	0.5220	1.5887	0.5201	1.5558	0.5192	1.5617	0.5156	1.5410
		MSE	0.0019	0.0275	0.0002	0.0040	0.0002	0.0031	0.0009	0.0016	0.0009	0.0013
Non-Informative Priors												
			SEL		LINEX				GEL			
					$\nu=-0.5$		$\nu=0.5$		$\kappa=-0.5$		$\kappa=0.5$	
T	R		α	β	α	β	α	β	α	β	α	β
1	1	Avg	0.5411	1.7748	0.5455	1.9792	0.5368	1.6503	0.5335	1.7117	0.5180	1.5932
		MSE	0.0176	0.4644	0.0024	0.1791	0.0022	0.0412	0.0073	0.0273	0.0068	0.0121
	2	Avg	0.5286	1.6289	0.5323	1.7081	0.5249	1.5607	0.5219	1.5890	0.5086	1.5091
		MSE	0.0158	0.3208	0.0021	0.0665	0.0020	0.0370	0.0069	0.0159	0.0066	0.0127
	3	Avg	0.5380	1.7158	0.5414	1.8220	0.5346	1.6347	0.5320	1.6738	0.5199	1.5910
		MSE	0.0161	0.4175	0.0022	0.1139	0.0020	0.0445	0.0067	0.0201	0.0063	0.0127
1	1	Avg	0.5411	1.7748	0.5455	1.9793	0.5368	1.6504	0.5335	1.7117	0.5180	1.5933
		MSE	0.0176	0.4643	0.0024	0.1791	0.0022	0.0412	0.0073	0.0273	0.0068	0.0120
	2	Avg	0.5280	1.6166	0.5311	1.6804	0.5249	1.5604	0.5224	1.5839	0.5112	1.5181
		MSE	0.0136	0.2856	0.0018	0.0582	0.0017	0.0338	0.0059	0.0137	0.0057	0.0110
	3	Avg	0.5422	1.7293	0.5455	1.8353	0.5389	1.6483	0.5363	1.6876	0.5245	1.6057
		MSE	0.0159	0.4065	0.0021	0.1128	0.0020	0.0432	0.0065	0.0192	0.0061	0.0117

Table 4: Average values (Avg) and the corresponding MSEs of the SPT estimators.

			NR		SEL		LINEX		GEL	
T	R		α	β	α	β	α	β	α	β
1	1	Avg	0.5911	1.8251	0.5879	1.6367	0.5870	1.6278	0.5828	1.6198
		MSE	0.0263	0.7304	0.0113	0.0249	0.0111	0.0225	0.0106	0.0208
	2	Avg	0.5576	1.6367	0.5771	1.6301	0.5759	1.6212	0.5708	1.6131
		MSE	0.0190	0.1464	0.0104	0.0233	0.0102	0.0210	0.0097	0.0193
	3	Avg	0.5708	1.7106	0.5749	1.6329	0.5738	1.6244	0.5690	1.6166
		MSE	0.0205	0.2600	0.0102	0.0247	0.0101	0.0224	0.0096	0.0208
2	1	Avg	0.5911	1.8252	0.5879	1.6367	0.5870	1.6279	0.5828	1.6198
		MSE	0.0263	0.7304	0.0113	0.0249	0.0111	0.0225	0.0106	0.0208
	2	Avg	0.5556	1.6352	0.5698	1.6274	0.5686	1.6189	0.5631	1.6108
		MSE	0.0173	0.1323	0.0097	0.0239	0.0095	0.0218	0.0090	0.0203
	3	Avg	0.5750	1.7253	0.5750	1.6335	0.5738	1.6249	0.5689	1.6170
		MSE	0.0204	0.2508	0.0102	0.0249	0.0101	0.0227	0.0096	0.0211

EM and SEM estimates are the close to each other and they are smaller than those of NR method. We also observe that as m increase, the values of MSEs and Avgs decrease, generally.

The results of Bayes estimates based on TK and MCMC methods are reported in Tables 2–3. From these tables, it is evident that all the Bayes estimates based on informative priors have very small MSEs compared to the MLEs. We also see that the Bayes estimates based on informative priors are better than those that are based on non-informative priors in all schemes and (T, n, m)s. However, EM and SEM estimates are better than non-informative Bayes estimates based on SEL in terms of MSE and Avg. So we can conclude that Bayes estimates even with non informative priors are preferable to the NR, for all schemes and

Table 5: Confidence intervals and coverage probabilities of NR and MCMC methods. (U:upper, L:lower, IL: interval length, CP: coverage probability

			NR				MCMC:Informative				MCMC:Non-Informative			
T	R		L	U	IL	CP	L	U	IL	CP	L	U	IL	CP
1	1	α	0.2952	0.8166	0.5215	95.54	0.4063	0.6483	0.2420	99.90	0.3172	0.8193	0.5021	92.90
		β	0.4102	3.2764	2.8661	97.12	1.1124	2.1299	1.0175	99.74	0.7989	3.4874	2.6885	92.80
	2	α	0.2972	0.7585	0.4614	95.02	0.4027	0.6526	0.2498	99.88	0.3232	0.7854	0.4622	94.64
		β	0.6301	2.6658	2.0357	95.78	1.0944	2.1159	1.0215	99.98	0.7981	2.7963	1.9982	94.52
	3	α	0.3203	0.7668	0.4466	94.58	0.4067	0.6481	0.2414	99.90	0.3371	0.7796	0.4424	93.50
		β	0.6540	2.8541	2.2001	96.38	1.1111	2.1114	1.0003	99.96	0.8604	2.9774	2.1170	92.98
2	1	α	0.2952	0.8167	0.5215	95.54	0.4063	0.6483	0.2420	99.90	0.3172	0.8193	0.5021	92.90
		β	0.4103	3.2764	2.8661	97.12	1.1124	2.1299	1.0175	99.74	0.7990	3.4875	2.6885	92.80
	2	α	0.3161	0.7400	0.4239	94.92	0.4044	0.6491	0.2447	99.66	0.3375	0.7609	0.4235	94.30
		β	0.7199	2.5624	1.8426	95.68	1.1020	2.0980	0.9959	99.88	0.8484	2.6496	1.8013	93.82
	3	α	0.3258	0.7695	0.4436	94.86	0.4074	0.6481	0.2407	99.86	0.3424	0.7820	0.4396	93.72
		β	0.6702	2.8649	2.1947	97.20	1.1140	2.1110	0.9970	99.86	0.8740	2.9859	2.1119	93.46

Ts. When we compare MSEs of T–K and MCMC methods, we observed that they are generally close to each other. However, T–K is better in some of the cases and vice versa in some others. However, the MCMC has the advantage of construction of the credible intervals. Thus, we can say that MCMC is preferable since it gives more information.

The performances of SPT estimators are given in Table 4. According to Table 4, we can say that SPT estimators based on informative T–K method have better performance than SPT based on NR methods in the sense of both MSE and Avg, generally. Moreover, SPT with T–K method based on GEL function seems to have the least MSE values among others. SPT estimator based on NR method has smaller MSE values than NR estimator when we consider the parameter β, and both methods have closer MSE values for the parameter α.

Finally, the confidence intervals and coverage probabilities are summarized in Table 5. It is observed that when we use non-informative priors the estimated CPs are smaller than the nominal CPs. Moreover, the expected ILs of non-informative methods are less than that of NR method. However, the estimated CPs of NR are slightly more than the non-informative method. Further, we observe that the CIs based on informative priors are better than the ones based on the non-informative priors and the once based on NR, in terms of having smaller ILs but higher CPs.

6. Real Data Example

We consider a data set reported by [5] representing the strength measured in GigaPAscal (GPA) for single carbon fibres, and impregnated 1000-carbon fibre tows. Single fibres were tested under tension at gauge lengths of 10 mm. This data was analyzed by [3] considering a hybrid censoring scheme for the Weibull distribution. Following [3], we analyze this data set using two-parameter Weibull distribution after subtracting 1.75. The authors recorded that the validity of the Weibull model based on the Kolmogorov–Smirnov (K–S) test is full-filled, namely, K–S = 0.072 and p-value = 0.885.

To compute the Bayes estimates, since we have no prior information about the unknown parameters, we assume the non-informative priors by setting $a = b = c = d = 0$. Taking $m = 40$ and $T = 2$, we use the following schemes

- Scheme 1: $R = (0^{39}, 23)$
- Scheme 2: $R = (23, 0^{39})$
- Scheme 3: $R = (2, 0^{10}, 2^3, 0^{10}, 2^3, 0^{10}, 3^3)$

Table 6: Estimation values of listed methods for Carbon Fibre data

	Sch 1		Sch 2		Sch 3	
MLE Method	α	β	α	β	α	β
NR	2.2542	0.3980	2.3058	0.3918	2.1169	0.3884
EM	2.2641	0.3975	2.2952	0.3986	2.1128	0.3908
SEM	2.2515	0.3981	2.3046	0.3922	2.1304	0.3892
Tierney-Kadane Method						
SEL	2.2505	0.3991	2.3041	0.3942	2.1104	0.3899
LINEX($\nu = -0.5$)	2.2764	0.4005	2.3310	0.3963	2.1303	0.3914
LINEX($\nu = 0.5$)	2.2260	0.3978	2.2793	0.3924	2.0915	0.3885
GEL($\kappa = -0.5$)	2.2393	0.3958	2.2929	0.3893	2.1012	0.3863
GEL($\kappa = 0.5$)	2.2169	0.3890	2.2704	0.3793	2.0827	0.3792
MCMC Method						
SEL	2.2496	0.3980	2.3042	0.3933	2.1028	0.3915
LINEX($\nu = -0.5$)	2.2735	0.3994	2.3288	0.3953	2.1232	0.3929
LINEX($\nu = 0.5$)	2.2261	0.3967	2.2802	0.3914	2.0828	0.3900
GEL($\kappa = -0.5$)	2.2390	0.3947	2.2937	0.3885	2.0932	0.3878
GEL($\kappa = 0.5$)	2.2176	0.3880	2.2725	0.3788	2.0739	0.3805
Shrinkage Method						
NR	2.2524	0.3985	2.3049	0.3930	2.1137	0.3892
SEL	2.2505	0.3991	2.3041	0.3942	2.1104	0.3899
LINEX($\nu = 0.5$)	2.2634	0.3998	2.3175	0.3953	2.1204	0.3906
GEL($\kappa = 0.5$)	2.2449	0.3974	2.2985	0.3917	2.1058	0.3881

Table 7: Confident intervals and interval lengths of NR and MCMC methods for Carbon Fibre data (U:upper, L:lower, IL: interval length)

		α			β		
Scheme	Method	L	U	IL	L	U	IL
1	NR	1.6321	2.8764	1.2443	0.2539	0.5420	0.2880
	MCMC	1.6668	2.8725	1.2057	0.2682	0.5540	0.2857
2	NR	1.6740	2.9376	1.2636	0.2175	0.5660	0.3485
	MCMC	1.7418	2.9404	1.1986	0.2408	0.5834	0.3426
3	NR	1.5703	2.6636	1.0933	0.2417	0.5351	0.2933
	MCMC	1.5744	2.6737	1.0993	0.2584	0.5558	0.2974

We have produced 60000 MCMC samples and the first 10000 of them are considered as the burn-in sample. We have provided the histograms of the samples for each parameter in Figures 1–2 and also some diagnostics showing the efficiency

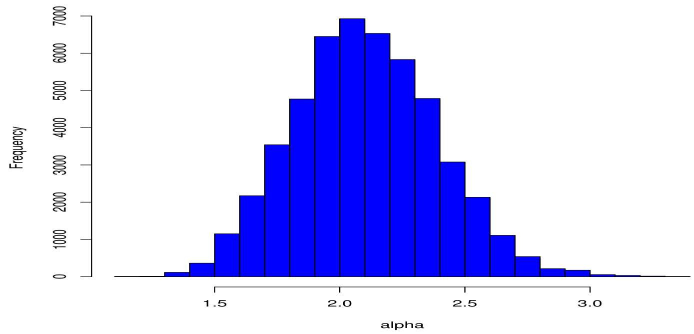

Figure 1: Histogram of the MCMC samples of the parameter α

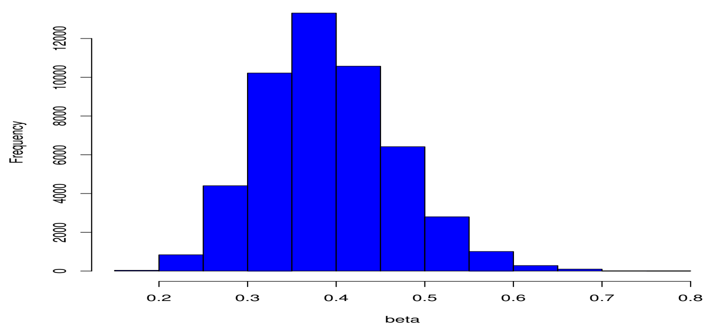

Figure 2: Histogram of the MCMC samples of the parameter β

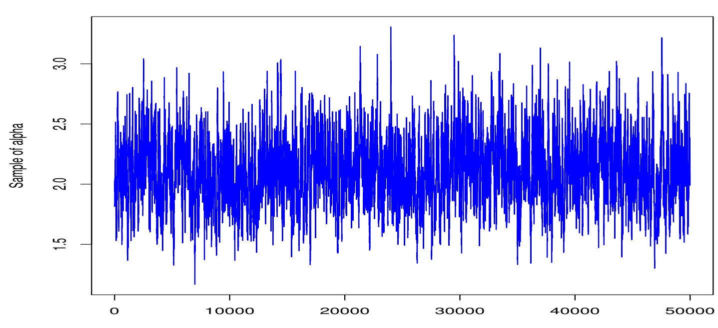

Figure 3: MCMC samples of the parameter α vs iterations

of the MCMC algorithm in Figures 3–5. The acceptance rate after the burn-in sample is close to 0.36 and it is stable. Therefore, it can be said that the MCMC algorithm works well.

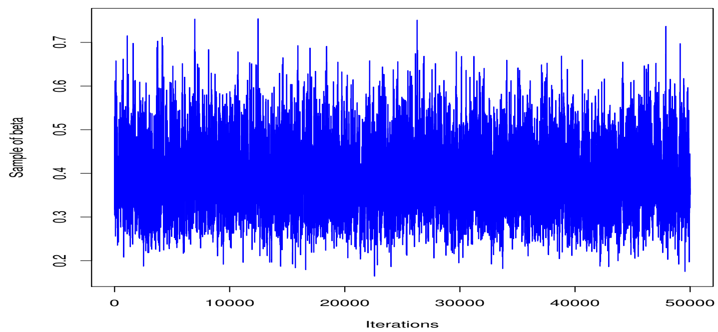

Figure 4: MCMC samples of the parameter β vs iterations

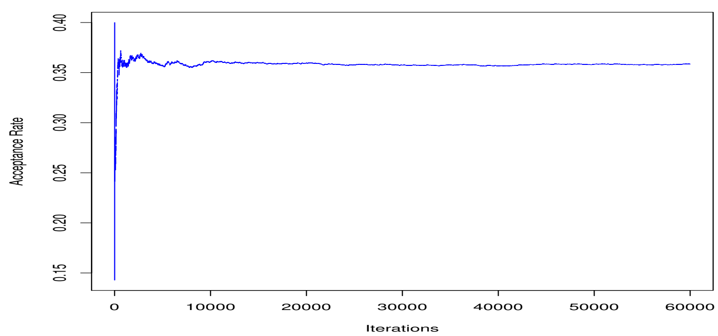

Figure 5: Acceptance rate of MCMC samples

In SPT estimates, since we don't have any prior information about parameters, we use the Bayes estimates as a an estimated prior information. Then we substitute them in the SPT formulae as $\widehat{\theta}_{\text{SPT}} = \lambda\theta_0 + (1-\lambda)\widehat{\theta}_{Bayes} I\left(W_D < \chi_1^2(\lambda)\right)$ by setting $\lambda = 0.5$ and $\alpha = 0.05$.

All the estimation methods considered in this paper are applied to this data and the estimated parameter values are reported in Table 6. We observe that the estimated values of α and β based on all the methods are closer to each other. Further, it can be seen that the Bayes estimates based on the two different methods are quite closer to each other which also show the stability of the MCMC algorithm. Moreover, asymptotic confidence intervals of NR method and HPD intervals of MCMC method are given in Table 7. According to this table, we can say that NR confidence intervals are mostly wider than the ones obtained via MCMC. This situation is also coincide with the simulation results.

7. Conclusive Remarks

In this paper, we discussed the estimation of parameters of Weibull distribution under type-I progressively hybrid censoring scheme using both classical and Bayesian strategies. Namely, MLE is obtained using NR, EM and SEM algorithms and Bayesian estimators are computed via T–K approximation and MCMC method under SEL, LINEX and GEL loss functions. We have also proposed the shrinkage preliminary test estimators based on NR and T–K with informative priors using equal weights on the prior information and the sample information. A real data application and extensive Monte Carlo simulations have been considered to compare the estimators in terms of MSE and Avg and also we compared the lengths of CIs and CPs. According to the results, EM algorithm beats the other ML estimates. However, we observed that both the T–K and MCMC methods perform quite closely. Finally, we found out that shrinkage preliminary test estimates have satisfactory performances in the presence of having proper prior information.

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