CONFIDENCE INTERVALS AND REGIONS FOR THE GENERALIZED INVERTED EXPONENTIAL DISTRIBUTION BASED ON PROGRESSIVELY CENSORED AND UPPER RECORDS DATA

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Abstract:

- In this paper, we study the estimation problems for the generalized inverted exponential distribution based on progressively type-II censored order statistics and record values. We establish some theorems to construct the exact confidence intervals and regions for the parameters. Monte Carlo simulation studies are used to assess the performance of our proposed methods. Simulation results show that the coverage probabilities of the exact confidence interval and the exact confidence region are all close to the desired level. Finally, two numerical examples are presented to illustrate the methods developed here.

Key-Words:

- *Confidence interval; joint confidence region; pivot; progressive type-II censoring; record values.*

AMS Subject Classification:

- 62F25, 62N01, 62N05.

1. INTRODUCTION

The exponential distribution was the first widely discussed lifetime distribution in the literature. This is because of its simplicity and mathematical feasibility. If the random variable T has an exponential distribution, then the random variable $Y = 1/X$ has an inverted exponential distribution. The exponential distribution was generalized, by introducing a shape parameter, and discussed by several researchers such as Gupta and Kundu [11, 12] and Raqab and Madi [19]. By introducing a shape parameter in the inverted exponential distribution, Abouammoh and Alshingiti [1] proposed a generalized inverted exponential (GIE) distribution. The probability density function and cumulative distribution function of the generalized inverted exponential distribution are given, respectively, by

$$f(x;\beta,\lambda) = \frac{\lambda\beta}{x^2}\exp(-\lambda/x)\left(1-\exp(-\lambda/x)\right)^{(\beta-1)}, \quad x>0,$$

and

$$F(x;\beta,\lambda) = 1-\left(1-\exp(-\lambda/x)\right)^{\beta}, \quad x>0,$$

where $\beta > 0$ is the shape parameter and $\lambda > 0$ is the scale parameter.

The properties and inferences for the GIE distribution were investigated by several authors. Abouammoh and Alshingiti [1] derived some distributional properties and reliability characteristics as well as maximum likelihood estimators (MLEs) based on complete sample. Krishna and Kumar [14] obtained the MLEs and least squares estimators of the parameters of the GIE distribution under progressively type-II censored sample. Dey and Dey [8] discussed the necessary and sufficient conditions for existence, uniqueness and finiteness of the MLEs of the parameters based on progressively type-II censored sample data. Recently, Dey and Pradhan [9] made Bayesian inference for the GIE parameters under hybrid random censoring. Ghitany *et al.* [10] established the existence and uniqueness of the MLEs of the parameters for a general class of inverse exponentiated distributions based on complete as well as progressively type-I and type-II censored data.

In this study, statistical inference for both progressive type-II right censored sample and record values from the GIE distribution are investigated. Dey and Dey [8] obtained approximate confidence intervals for the GIE parameters based on progressive censored sample. However, if the sample size is small, the approximate confidence interval may not be adequate. Thus, exact confidence intervals and regions become important when the sample size is small. The method of pivotal quantity are used to construct the confidence intervals and regions for the model parameters. The rest of this paper is organized as follows. In Section 2, an exact confidence interval and an exact confidence region for the parameters are constructed based on progressive type-II right censored sample. In Section 3, two theorems are proposed to obtain the exact confidence interval and region for the parameters based on upper record values. Two numerical examples are presented in Section 4. Some conclusions are made in Section 5.

2. INTERVAL ESTIMATION UNDER PROGRESSIVE TYPE-II CENSORING

Progressive type-II right censoring is of importance in the field of reliability and life testing. Suppose n identical units are placed on a lifetime test. At the time of the i-th failure, r_i surviving units are randomly withdrawn from the experiment, $1 \leq i \leq m$. Thus, if m failures are observed then $r_1 + \cdots + r_m$ units are progressively censored; hence, $n = m + r_1 + \cdots + r_m$. Let $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ be the progressively censored failure times, where $\mathbf{r} = (r_1, \ldots, r_m)$ denotes the censoring scheme. As a special case, if $\mathbf{r} = (0, \ldots, 0)$ where no withdrawals are made, we obtain the ordinary order statistics (Bairamov and Eryılmaz [5]). If $\mathbf{r} = (0, \ldots, 0, n-m)$, the progressive type-II censoring becomes type-II censoring. For more details see Balakrishnan and Aggarwala [6].

In this section, we will construct the exact confidence interval and region for model parameters by using pivotal quantity method. We will also conduct a simulation study to assess the performance of proposed interval and region.

2.1. Exact confidence interval and region

Suppose that $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ denote progressively type-II right censored order statistics from a GIE distribution. Let

$$Y^{\mathbf{r}}_{i:m:n} = -\beta \log\left(1 - \exp\left(-\lambda / X^{\mathbf{r}}_{i:m:n}\right)\right), \quad i = 1, 2, \ldots, m.$$

It can be seen that $Y^{\mathbf{r}}_{1:m:n} < Y^{\mathbf{r}}_{2:m:n} < \cdots < Y^{\mathbf{r}}_{m:m:n}$ are progressively type-II right censored order statistics from a standard exponential distribution. It is well known that, from Thomas and Wilson [21],

$$\begin{aligned}
\pi_1 &= nY^{\mathbf{r}}_{1:m:n} \\
\pi_2 &= (n - r_1 - 1)(Y^{\mathbf{r}}_{2:m:n} - Y^{\mathbf{r}}_{1:m:n}) \\
&\vdots \\
\pi_m &= (n - r_1 - \cdots - r_{m-1} - m + 1)\left(Y^{\mathbf{r}}_{m:m:n} - Y^{\mathbf{r}}_{m-1:m:n}\right)
\end{aligned}$$

are independent and identically distributed as a standard exponential distribution. Hence,

$$\kappa_1 = 2\pi_1 = 2nY^{\mathbf{r}}_{1:m:n}$$

has a chi-squared distribution with 2 degrees of freedom and

$$\varepsilon_1 = 2\sum_{i=2}^{m}\pi_i = 2\left\{\sum_{i=1}^{m}(r_i+1)\,Y^{\mathbf{r}}_{i:m:n} - nY^{\mathbf{r}}_{1:m:n}\right\}$$

has a chi-squared distribution with $2m-2$ degrees of freedom. It is also clear that ε_1 and κ_1 are independent random variables. Let

$$\xi_1 = \frac{\varepsilon_1}{(m-1)\,\kappa_1} = \frac{\sum_{i=1}^{m}(r_i+1)\,Y^{\mathbf{r}}_{i:m:n} - nY^{\mathbf{r}}_{1:m:n}}{n\,(m-1)\,Y^{\mathbf{r}}_{1:m:n}} \tag{2.1}$$

and

$$\eta_1 = \varepsilon_1 + \kappa_1 = 2\sum_{i=1}^{m}(r_i+1)\,Y^{\mathbf{r}}_{i:m:n}. \tag{2.2}$$

It is easy to show that ξ_1 has an F distribution with $2m-2$ and 2 degrees of freedom and η_1 has a chi-squared distribution with $2m$ degrees of freedom. Furthermore, ξ_1 and η_1 are independent (see Johnson *et al.* [13]).

The following lemma helps us to construct the exact confidence interval for λ and exact joint confidence region for (λ,β).

Lemma 2.1. *Suppose that* $0 < a_1 < a_2 < \cdots < a_m$. *Let*

$$\xi_1(\lambda) = \frac{1}{n\,(m-1)}\sum_{i=1}^{m}(r_i+1)\,\frac{\log\left(1-\exp\left(-\lambda/a_i\right)\right)}{\log\left(1-\exp\left(-\lambda/a_1\right)\right)} - \frac{1}{m-1},$$

where $r_i \geq 0$, $i = 1,2,\ldots,m$, *and* $\sum_{i=1}^{m} r_i = n-m$. *Then,* $\xi_1(\lambda)$ *is strictly increasing in* λ *for any* $\lambda > 0$.

Proof: To prove $\xi_1(\lambda)$ is strictly increasing, it suffices to show that the function

$$g(\lambda) = \frac{\log\left(1-\exp\left(-\lambda/a_i\right)\right)}{\log\left(1-\exp\left(-\lambda/a_1\right)\right)}$$

is strictly increasing in λ. The derivative of $g(\lambda)$ is given by

$$g'(\lambda) = \left(\frac{h_1(a_1)}{h_2(a_i)} - \frac{h_1(a_i)}{h_2(a_1)}\right)\left(\frac{1}{h_1(a_1)}\right)^2,$$

where

$$h_1(x) = \log\left(1-\exp\left(-\lambda/x\right)\right)$$

and

$$h_2(x) = x\left(\exp\left(\lambda/x\right)-1\right).$$

If both $h_1(x)$ and $h_2(x)$ are decreasing, it can be said that $\left(\frac{h_1(a_1)}{h_2(a_i)} - \frac{h_1(a_i)}{h_2(a_1)}\right) > 0$ for $a_i > a_1$ and hence $g'(\lambda) > 0$.

It is clear that $h_1(x)$ is strictly decreasing in x. From the second order Taylor polynomial of $\exp(a)$ at $a = 0$, one has the following inequality, for $a < 0$,

$$\exp(a) > a + 1. \tag{2.3}$$

Let $a = -\lambda/x$. Equation (2.3) can be written as

$$1 - \lambda/x - \exp(-\lambda/x) < 0, \quad \text{for} \quad x > 0. \tag{2.4}$$

Note that the first derivative of $h_2(x)$ is

$$h_2'(x) = \exp(\lambda/x)\left[1 - \lambda/x - \exp(-\lambda/x)\right].$$

From Equation (2.4), it is easy to see that $h_2'(x) < 0$ for $x > 0$. That is, $h_2(x)$ is strictly decreasing in x. Hence, $g'(\lambda)$ is positive. This completes the proof. □

Let $F_{\alpha(\delta_1,\delta_2)}$ be the upper α percentile of F distribution with δ_1 and δ_2 degrees of freedom. The following theorem gives an exact confidence interval for the parameter λ.

Theorem 2.1. *Suppose that $X^{\mathbf{r}}_{1:m:n} < X^{\mathbf{r}}_{2:m:n} < \cdots < X^{\mathbf{r}}_{m:m:n}$ is a progressively type-II censored sample from the GIE distribution. Then, for any $0 < \alpha < 1$,*

$$\Big(\varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, F_{1-\alpha/2;2m-2,2}\right), \\ \varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, F_{\alpha/2;2m-2,2}\right)\Big)$$

is a $100(1-\alpha)\%$ confidence interval for λ, where $\varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, t\right)$ is the solution of λ for the equation

$$\frac{1}{n(m-1)} \sum_{i=1}^{m} (r_i + 1) \frac{\log\left(1 - \exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)}{\log\left(1 - \exp\left(-\lambda/X^{\mathbf{r}}_{1:m:n}\right)\right)} - \frac{1}{m-1} = t. \tag{2.5}$$

Proof: From Equation (2.1), we know that the pivot

$$\begin{aligned}\xi_1(\lambda) &= \frac{\sum_{i=1}^{m}(r_i+1)\, Y^{\mathbf{r}}_{i:m:n} - nY^{\mathbf{r}}_{1:m:n}}{n(m-1)\, Y^{\mathbf{r}}_{1:m:n}} \\ &= \frac{1}{n(m-1)} \sum_{i=1}^{m} (r_i + 1) \frac{\log\left(1 - \exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)}{\log\left(1 - \exp\left(-\lambda/X^{\mathbf{r}}_{1:m:n}\right)\right)} - \frac{1}{m-1}\end{aligned}$$

has an F distribution with $2m-2$ and 2 degrees of freedom. By Lemma 2.1, $\xi_1(\lambda)$ is strictly increasing function of λ, and hence, $\xi_1(\lambda) = t$ has a unique solution for any $\lambda > 0$. Thus, for $0 < \alpha < 1$, the event

$$F_{1-\alpha/2;2m-2,2} < \frac{1}{n(m-1)} \sum_{i=1}^{m} (r_i + 1) \frac{\log\left(1 - \exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)}{\log\left(1 - \exp\left(-\lambda/X^{\mathbf{r}}_{1:m:n}\right)\right)} - \frac{1}{m-1} \\ < F_{\alpha/2;2m-2,2}$$

is equivalent to the event

$$\varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, F_{1-\alpha/2;2m-2,2}\right) < \lambda < \varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, F_{\alpha/2;2m-2,2}\right).$$

Then, the proof follows. □

Let us now discuss the joint confidence region for (λ, β). Let $\chi^2_{\alpha;\delta}$ denote the upper α percentile of a chi-squared distribution with δ degrees of freedom. An exact joint confidence region for (λ, β) is given in the following theorem.

Theorem 2.2. *Suppose that $X^{\mathbf{r}}_{i:m:n}$, $i = 1, 2, \ldots, m$, are progressive type-II right censored order statistics from the GIE distribution with censoring scheme $\mathbf{r}$. Then for any $0 < \alpha < 1$, a $100(1-\alpha)\%$ joint confidence region for (λ, β) is determined by the following inequalities:*

$$\begin{cases} \varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, F_{(1+\sqrt{1-\alpha})/2;2m-2,2}\right) < \lambda \\ \qquad < \varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, F_{(1-\sqrt{1-\alpha})/2;2m-2,2}\right) \\ -\dfrac{\chi^2_{(1+\sqrt{1-\alpha})/2;2m}}{2\sum_{i=1}^{m}(r_i+1)\log\left(1-\exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)} < \beta \\ \qquad < -\dfrac{\chi^2_{(1-\sqrt{1-\alpha})/2;2m}}{2\sum_{i=1}^{m}(r_i+1)\log\left(1-\exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)}, \end{cases}$$

where $\varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, t\right)$ is defined in Equation (2.5).

Proof: From Equation (2.1), we know that the pivot

$$\begin{aligned} \xi_1(\lambda) &= \frac{\sum_{i=1}^{m}(r_i+1)\,Y^{\mathbf{r}}_{i:m:n} - nY^{\mathbf{r}}_{1:m:n}}{n(m-1)\,Y^{\mathbf{r}}_{1:m:n}} \\ &= \frac{1}{n(m-1)}\sum_{i=1}^{m}(r_i+1)\frac{\log\left(1-\exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)}{\log\left(1-\exp\left(-\lambda/X^{\mathbf{r}}_{1:m:n}\right)\right)} - \frac{1}{m-1} \end{aligned}$$

has an F distribution with $2m-2$ and 2 degrees of freedom. From Equation (2.2), we also know that

$$\eta_1 = 2\sum_{i=1}^{m}(r_i+1)\,Y^{\mathbf{r}}_{i:m:n} = -2\beta\sum_{i=1}^{m}(r_i+1)\log\left(1-\exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)$$

has a chi-squared distribution with $2m$ degrees of freedom, and it is independent of $\xi_1(\lambda)$. Thus, for $0 < \alpha < 1$, we have

$$\begin{aligned}
P\Bigg\{ & \varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, F_{(1+\sqrt{1-\alpha})/2;2m-2,2}\right) < \lambda \\
& \qquad < \varphi_1\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, F_{(1-\sqrt{1-\alpha})/2;2m-2,2}\right), \\
& -\frac{\chi^2_{(1+\sqrt{1-\alpha})/2;2m}}{2\sum_{i=1}^{m}(r_i+1)\log\left(1-\exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)} < \beta \\
& \qquad < -\frac{\chi^2_{(1-\sqrt{1-\alpha})/2;2m}}{2\sum_{i=1}^{m}(r_i+1)\log\left(1-\exp\left(-\lambda/X^{\mathbf{r}}_{i:m:n}\right)\right)}\Bigg\} \\
= \; & P\left(F_{(1+\sqrt{1-\alpha})/2;2m-2,2} < \xi_1 < F_{(1-\sqrt{1-\alpha})/2;2m-2,2}\right) \\
& \qquad P\left(\chi^2_{(1+\sqrt{1-\alpha})/2;2m} < \eta_1 < \chi^2_{(1-\sqrt{1-\alpha})/2;2m}\right) \\
= \; & \sqrt{1-\alpha}\sqrt{1-\alpha} \\
= \; & 1-\alpha.
\end{aligned}$$

The proof is completed. □

2.2. Simulation study

The simulation study is performed with 5000 trials to investigate the performance of exact and approximate confidence intervals and confidence regions under progressive censoring. We consider the values of parameters $(\lambda, \beta) = (2, 0.5), (0.5, 2)$ and different combinations of n, m, and censoring schemes $\mathbf{r}$. The approximate intervals are considered as in Dey and Dey [8]. The nominal confidence level is chosen as 95%. The results are given in Table 1 and Table 2. From these tables, one can conclude that both the coverage probabilities of approximate and exact confidence intervals are close to the desired level. The coverage probabilities of exact confidence regions are also close to the nominal level. However, the coverage probabilities of the approximate confidence regions are lower than the nominal level. When the sample size increases, the coverage probability of approximate confidence region reaches to nominal level 95%. During simulation, the authors observed that the MLEs of parameters are not obtained uniquely for different initials values. However, this problem disappeared for the large sample size. In this regards, coverage probability of approximate confidence region works for only large sample. As a conclusion, exact confidence region should be used for the small sample size.

Table 1: Coverage probabilities for the proposed methods and the approximations under progressive censoring when $(\lambda, \beta) = (2, 0.5)$

			λ		(λ, β)	
n	m	**r**	approx.	exact	approx.	exact
20	10	(1,1,1,1,1,1,1,1,1,1)	0.9482	0.9494	0.8122	0.9484
		(5,0,0,0,0,0,0,0,0,5)	0.9532	0.9500	0.8966	0.9468
		(5,5,0,0,0,0,0,0,0,0)	0.9476	0.9500	0.8966	0.9468
		(0,0,0,0,0,0,0,0,5,5)	0.9540	0.9480	0.8640	0.9420
		(0,0,0,0,5,5,0,0,0,0)	0.9422	0.9492	0.9130	0.9474
		(2,2,1,0,0,0,0,1,2,2)	0.9498	0.9456	0.9504	0.9456
40	20	(1,1,1,1,1,. . . ,1,1)	0.9462	0.9526	0.9294	0.9536
		(10,0,0,0,. . . ,0,10)	0.9556	0.9518	0.9344	0.9460
		(10,10,0,. . . ,0,0)	0.9480	0.9510	0.9380	0.9530
		(0,0,0,0,. . . ,10,10)	0.9586	0.9512	0.9156	0.9534
		(0,. . . ,0,10,10,0,. . . ,0,0)	0.9508	0.9550	0.9432	0.9526
		(2,2,2,2,2,0,. . . ,0,2,2,2,2,2)	0.9520	0.9568	0.9584	0.9562
100	50	(1,1,1,1,1,. . . ,1,1)	0,9506	0.9552	0.8416	0.9574
		(25,0,0,0,. . . ,0,25)	0,9544	0.9456	0.9528	0.9496
		(25,25,0,0,. . . ,0,0)	0,9530	0.9516	0.9524	0.9488
		(0,0,0,0,,. . . ,25,25)	0,9508	0.9528	0.9404	0.9540
		(0,. . . ,0,25,25,0,. . . ,0,0)	0,9464	0.9496	0.9484	0.9500
		(2,. . . ,2,1,0,. . . ,0,1,2,. . . ,2)	0.9484	0.9512	0.9594	0.9534
10	5	(1,1,1,1,1)	0.9452	0.9468	0.8356	0.9446
		(2,1,0,0,2)	0.9486	0.9484	0.9202	0.9468
		(2,2,1,0,0)	0.9436	0.9510	0.9352	0.9484
		(0,0,1,2,2)	0.9530	0.9486	0.9016	0.9458
		(0,2,1,2,0)	0.9384	0.9462	0.9168	0.9468

3. INTERVAL ESTIMATION UNDER RECORD VALUES

Record values were first introduced by Chandler [7]. A record value is either the largest or the smallest value obtained from a sequence of random variables. Ahsanullah and Nevzorov [3] pointed out that records are very popular because they arise naturally in many fields of studies such as climatology, sports, medicine, traffic, industry and so on. In reliability studies, Lee *et al.* [16] indicated that there are some situations in lifetime testing experiments in which a failure time of a product is recorded if it exceeds all preceding failure times. These recorded failure times are the upper record value sequence. An account on record values can be found in the books by Ahsanullah [2] and Arnold *et al.* [4].

Table 2: Coverage probabilities for the proposed methods and the approximations under progressive censoring when $(\lambda, \beta) = (0.5, 2)$

			λ		(λ, β)	
n	m	**r**	approx.	exact	approx.	exact
20	10	(1,1,1,1,1,1,1,1,1,1)	0.9538	0.9556	0.8904	0.9522
		(5,0,0,0,0,0,0,0,0,5)	0.9534	0.9502	0.9570	0.9514
		(5,5,0,0,0,0,0,0,0,0)	0.9540	0.9588	0.9570	0.9514
		(0,0,0,0,0,0,0,0,5,5)	0.9526	0.9530	0.9396	0.9510
		(0,0,0,0,5,5,0,0,0,0)	0.9474	0.9474	0.9530	0.9534
		(2,2,1,0,0,0,0,1,2,2)	0.9452	0.9482	0.9440	0.9482
40	20	(1,1,1,1,1,. . . ,1,1)	0.9460	0.9546	0.8946	0.9482
		(10,0,0,0,. . . ,0,10)	0.9534	0.9502	0.9570	0.9514
		(10,10,0,. . . ,0,0)	0.9540	0.9488	0.9576	0.9538
		(0,0,0,0,. . . ,10,10)	0.9526	0.9472	0.9522	0.9486
		(0,. . . ,0,10,10,0,. . . ,0,0)	0.9504	0.9534	0.9534	0.9508
		(2,2,2,2,2,0,. . . ,0,2,2,2,2,2)	0.9468	0.9422	0.9330	0.9478
100	50	(1,1,1,1,1,. . . ,1,1)	0.9486	0.9488	0.8988	0.9488
		(25,0,0,0,. . . ,0,25)	0.9490	0.9538	0.9530	0.9508
		(25,25,0,0,. . . ,0,0)	0.9510	0.9500	0.9514	0.9470
		(0,0,0,0,,. . . ,25,25)	0.9486	0.9504	0.9560	0.9494
		(0,. . . ,0,25,25,0,. . . ,0,0)	0.9510	0.9492	0.9534	0.9490
		(2,. . . ,2,1,0,. . . ,0,1,2,. . . ,2)	0.9496	0.9514	0.9418	0.9516
10	5	(1,1,1,1,1)	0.9466	0.9524	0.8194	0.9528
		(2,1,0,0,2)	0.9510	0.9474	0.8292	0.9482
		(2,2,1,0,0)	0.9420	0.9538	0.8694	0.9552
		(0,0,1,2,2)	0.9656	0.9544	0.7712	0.9516
		(0,2,1,2,0)	0.9390	0.9504	0.8226	0.9464

In this section, we will establish the exact confidence interval and region for model parameters based on pivotal quantity method. A simulation study is also conducted to investigate the performance of proposed interval and region.

3.1. Exact confidence interval and region

Let $X_{U(1)} < X_{U(2)} < \cdots < X_{U(m)}$ be the first m upper record values from the GIE distribution. Set

$$W_i = -\beta \log\left(1 - \exp\left(-\lambda / X_{U(i)}\right)\right), \quad i = 1, 2, \ldots, m.$$

Then, it is easily seen that $W_1 < W_2 < \cdots < W_m$ are the first m upper record values from a standard exponential distribution. Moreover, Arnold *et al.* [4]

showed that

$$\begin{aligned}\rho_1 &= W_1\\ \rho_2 &= W_2 - W_1\\ &\vdots\\ \rho_n &= W_m - W_{m-1}\end{aligned}$$

are independent and identically distributed random variables from a standard exponential distribution. Hence,

$$\kappa_2 = 2\rho_1 = 2W_1$$

has a chi-squared distribution with 2 degrees of freedom and

$$\varepsilon_2 = 2\sum_{i=2}^{m} \rho_i = 2\left(W_m - W_1\right)$$

has a chi-squared distribution with $2m - 2$ degrees of freedom. We can also find that ε_2 and κ_2 independent. Let

$$\xi_2 = \frac{\varepsilon_2}{(m-1)\,\kappa_2} = \frac{1}{m-1}\frac{W_m - W_1}{W_1} \tag{3.1}$$

and

$$\eta_2 = \varepsilon_2 + \kappa_2 = 2W_m. \tag{3.2}$$

It is easy to show that ξ_2 has an F distribution with $2m - 2$ and 2 degrees of freedom and η_2 has a chi-squared distribution with $2m$ degrees of freedom. Furthermore, ξ_2 and η_2 are independent.

Lemma 3.1. *Suppose that $0 < a_1 < a_2 < \cdots < a_m$. Let*

$$\begin{aligned}\xi_2\left(\lambda\right) &= \frac{1}{m-1}\frac{W_m - W_1}{W_1}\\ &= \frac{1}{m-1}\left(\frac{\log\left(1 - \exp\left(-\lambda/a_m\right)\right)}{\log\left(1 - \exp\left(-\lambda/a_1\right)\right)} - 1\right).\end{aligned}$$

Then, $\xi_2(\lambda)$ is strictly increasing in λ for any $\lambda > 0$.

Proof: The proof is analogous to that of Lemma 2.1. □

To construct the exact confidence interval for λ based on record values, we have the following theorem.

Theorem 3.1. *Suppose that $X_{U(1)} < X_{U(2)} < \cdots < X_{U(m)}$ are first m upper record values from the GIE distribution. Then, for any $0 < \alpha < 1$,*

$$\Big(\varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{1-\alpha/2;2m-2,2}\right), \quad \varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{\alpha/2;2m-2,2}\right)\Big)$$

is a $100(1-\alpha)\%$ confidence interval for λ, where $\varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, t\right)$ is the solution of λ for the equation

$$\frac{1}{m-1}\left(\frac{\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right)}{\log\left(1-\exp\left(-\lambda/X_{U(1)}\right)\right)} - 1\right) = t. \tag{3.3}$$

Proof: From Equation (3.1), we know that the pivot

$$\begin{aligned}\xi_2(\lambda) &= \frac{1}{m-1}\frac{W_m - W_1}{W_1} \\ &= \frac{1}{m-1}\left(\frac{\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right)}{\log\left(1-\exp\left(-\lambda/X_{U(1)}\right)\right)} - 1\right)\end{aligned}$$

has an F distribution with $2m-2$ and 2 degrees of freedom. By Lemma 3.1, $\xi_2(\lambda)$ is strictly increasing function of λ, and hence, $\xi_2(\lambda) = t$ has a unique solution for any $\lambda > 0$. Thus, for $0 < \alpha < 1$, the event

$$F_{1-\alpha/2;2m-2,2} < \frac{1}{m-1}\left(\frac{\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right)}{\log\left(1-\exp\left(-\lambda/X_{U(1)}\right)\right)} - 1\right) < F_{\alpha/2;2m-2,2}$$

is equivalent to the event

$$\begin{aligned}&\varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{1-\alpha/2;2m-2,2}\right) < \lambda \\ &\qquad < \varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{\alpha/2;2m-2,2}\right).\end{aligned}$$

Then, the proof follows. □

For the joint confidence region for (λ, β) based on record values, we have the following result.

Theorem 3.2. *Suppose that $X_{U(i)}$, $i = 1, 2, \ldots, m$ are first i-th upper record values from the GIE distribution. Then, for any $0 < \alpha < 1$, a $100(1-\alpha)\%$ joint confidence region for (λ, β) is determined by the following inequalities:*

$$\begin{cases}\varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{(1+\sqrt{1-\alpha})/2;2m-2,2}\right) < \lambda \\ \qquad < \varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{(1-\sqrt{1-\alpha})/2;2m-2,2}\right) \\ -\dfrac{\chi^2_{(1+\sqrt{1-\alpha})/2;2m}}{2\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right)} < \beta \\ \qquad < -\dfrac{\chi^2_{(1-\sqrt{1-\alpha})/2;2m}}{2\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right)},\end{cases}$$

where $\varphi_2\left(X^{\mathbf{r}}_{1:m:n}, X^{\mathbf{r}}_{2:m:n}, \ldots, X^{\mathbf{r}}_{m:m:n}, t\right)$ *is defined in Equation (3.3).*

Proof: From Equation (3.1), we know that the pivot

$$\xi_2(\lambda) = \frac{1}{m-1}\left(\frac{\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right)}{\log\left(1-\exp\left(-\lambda/X_{U(1)}\right)\right)} - 1\right)$$

has an F distribution with $2m-2$ and 2 degrees of freedom. From Equation (3.2), we know that

$$\eta_2 = -2\beta\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right).$$

has a chi-square distribution with $2m$ degrees of freedom, and it is independent of $\xi_2(\lambda)$. For $0<\alpha<1$, we have

$$\begin{aligned}
&P\Bigg\{\varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{(1+\sqrt{1-\alpha})/2;2m-2,2}\right) < \lambda \\
&\qquad < \varphi_2\left(X_{U(1)}, X_{U(2)}, \ldots, X_{U(m)}, F_{(1-\sqrt{1-\alpha})/2;2m-2,2}\right), \\
&\qquad -\frac{\chi^2_{(1+\sqrt{1-\alpha})/2;2m}}{2\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right)} < \beta < -\frac{\chi^2_{(1-\sqrt{1-\alpha})/2;2m}}{2\log\left(1-\exp\left(-\lambda/X_{U(m)}\right)\right)}\Bigg\} \\
=&P\left(F_{(1+\sqrt{1-\alpha})/2;2m-2,2} < \xi_2 < F_{(1-\sqrt{1-\alpha})/2;2m-2,2}\right) \\
&\qquad P\left(\chi^2_{(1+\sqrt{1-\alpha})/2;2m} < \eta_2 < \chi^2_{(1-\sqrt{1-\alpha})/2;2m}\right) \\
=&\sqrt{1-\alpha}\sqrt{1-\alpha} \\
=&1-\alpha.
\end{aligned}$$

□

3.2. Simulation study

It is important to examine how well our proposed method works for constructing confidence interval and region. We consider the values of parameters $(\lambda, \beta) = (2, 0.5), (0.5, 2)$ and different values of m. For each case, we simulated 5000 upper record samples from the GIE distribution. The nominal confidence level is chosen as 95%. The results are given in Table 3. From this table, one can see that the exact confidence intervals and regions have desired coverage probability for small and large sample sizes. As a conclusion, the proposed methods work well.

Table 3: Coverage probability of exact confidence interval and confidence region based on upper record values when $(\lambda, \beta) = (2, 0.5), (0.5, 2)$

	$(\lambda,\beta)=(2,0.5)$		$(\lambda,\beta)=(0.5,2)$	
m	λ	(λ,β)	λ	(λ,β)
2	0.9502	0.9520	0.9566	0.9540
3	0.9502	0.9488	0.9466	0.9446
4	0.9474	0.9546	0.9548	0.9504
5	0.9510	0.9500	0.9454	0.9498
6	0.9476	0.9526	0.9546	0.9528
7	0.9548	0.9606	0.9502	0.9512
8	0.9522	0.9606	0.9540	0.9548
9	0.9518	0.9604	0.9514	0.9498
10	0.9498	0.9578	0.9512	0.9516
11	0.9476	0.9570	0.9522	0.9526
12	0.9532	0.9600	0.9478	0.9488
13	0.9478	0.9560	0.9472	0.9468
14	0.9494	0.9524	0.9488	0.9452
15	0.9498	0.9490	0.9488	0.9520

4. ILLUSTRATIVE EXAMPLES

To illustrate the use of our proposed estimation method, the following two examples are discussed.

Example 4.1. (*Progressively Type-II Censored Data*) We apply the proposed interval estimation methods to the polished window strengths data set presented in Abouammoh and Alshingiti [1]. Dey and Dey [8] indicated that the GIE distribution is acceptable for these data. For the purposes of illustrating the estimation methods discussed in this paper, we adopt the progressively type-II censored sample with $n = 31$ and $m = 11$ which was generated from this data set by Dey and Dey [8]. The progressively censored data are reported in Table 4.

To obtain a 95% confidence interval for λ, we need the percentiles

$$F_{0.025;22,2} = 39.4479 \qquad \text{and} \qquad F_{0.975;22,2} = 0.2242.$$

Then, we can solve Equation (2.5) and get the following values

$$\varphi_1\left(x^{\mathbf{r}}_{1:m:n}, x^{\mathbf{r}}_{2:m:n}, \ldots, x^{\mathbf{r}}_{m:m:n}, F_{0.975;22,2}\right) = 81.8086,$$

and

$$\varphi_1\left(x^{\mathbf{r}}_{1:m:n}, x^{\mathbf{r}}_{2:m:n}, \ldots, x^{\mathbf{r}}_{m:m:n}, F_{0.025;22,2}\right) = 401.0639.$$

Table 4: Progressively type-II censored data based on window strength data

i	1	2	3	4	5	6
r_i	0	0	0	0	0	0
$x^{\mathbf{r}}_{i:m:n}$	18.83	20.8	21.657	23.03	23.23	24.05
i	7	8	9	10	11	
r_i	0	0	0	0	20	
$x^{\mathbf{r}}_{i:m:n}$	24.321	25.5	25.52	25.8	26.69	

By Theorem 2.1, the 95% confidence interval for λ is obtained as (81.8086, 401.0639).

Furthermore, to obtain a 95% joint confidence region for (λ, β), we need the percentiles

$$F_{0.9873;22,2} = 0.1825, \qquad F_{0.0127;22,2} = 78.4361,$$

$$\chi^2_{.9873;24} = 9.8824, \qquad \text{and} \qquad \chi^2_{.0127;24} = 39.4099.$$

By Theorem 2.2, the 95% confidence region for (λ, β) is determined by the following two inequalities:

$$71.9165 < \lambda < 458.4111$$

and

$$-\frac{9.8824}{2\sum_{i=1}^{11}(r_i+1)\log\left(1-\exp\left(-\lambda/x^{\mathbf{r}}_{i:m:n}\right)\right)} < \beta < -\frac{39.4099}{2\sum_{i=1}^{11}(r_i+1)\log\left(1-\exp\left(-\lambda/x^{\mathbf{r}}_{i:m:n}\right)\right)}.$$

Figure 1 shows the 95% joint confidence region for (λ, β) based on progressively type-II censored data given in Table 1. It can be seen that the region is large when λ is large.

Example 4.2. (*Record Value Data*) To illustrate the use of the interval estimation based on records, we analyze one real data set. Lawless [15, p.3] presented 11 times to breakdown of electrical insulating fluid subjected to 30 kilovolts. The data, under a logarithm transformation, is 2.836, 3.120, 3.045, 5.169, 4.934, 4.970, 3.018, 3.770, 5.272, 3.856, 2.046. Luckett [18] extracted the $m = 4$ upper record values from this data set and indicated that the GIE distribution is acceptable for this data set. The upper record value data are presented in Table 5.

To obtain a 95% confidence interval for λ, we need the percentiles

$$F_{0.025;6,2} = 39.3315 \qquad \text{and} \qquad F_{0.975;6,2} = 0.1377.$$

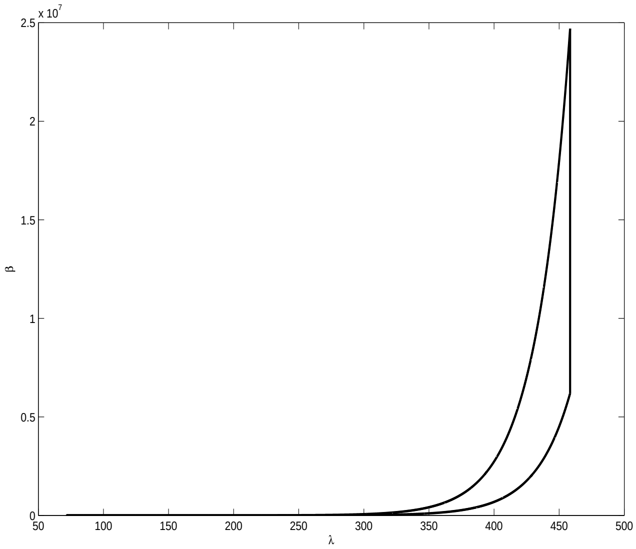

Figure 1: A 95% joint confidence region for (λ, β) based on progressively type-II censored data given in Table 4.

Table 5: Upper record values based on breakdown of electrical insulating fluid data

i	1	2	3	4
$x_{u(i)}$	2.836	3.120	5.169	5.272

By Theorem 3.1, we have the following results.

$$\varphi_2\left(x_{u(1)}, x_{u(2)}, \ldots, x_{u(10)}, F_{0.975;6,2}\right) = 0.8644,$$

and

$$\varphi_2\left(x_{u(1)}, x_{u(2)}, \ldots, x_{u(10)}, F_{0.025;6,2}\right) = 29.3207.$$

That is, the 95% confidence interval for λ is $(0.8644, 29.3207)$.

To obtain a 95% joint confidence region for (λ, β), we need the percentiles

$$F_{0.9873;6,2} = 0.1013, \qquad F_{0.0127;6,2} = 78.3196,$$

$$\chi^2_{.9873;8} = 1.7670, \qquad \text{and} \qquad \chi^2_{.0127;8} = 19.4433.$$

By Theorem 3.2, a 95% confidence region for (λ, β) is determined by the following two inequalities:

$$0.4484 < \lambda < 33.5289$$

and

$$-\frac{1.7670}{2\log\left(1-\exp\left(-\lambda/5.272\right)\right)} < \beta < -\frac{19.4433}{2\log\left(1-\exp\left(-\lambda/5.272\right)\right)}.$$

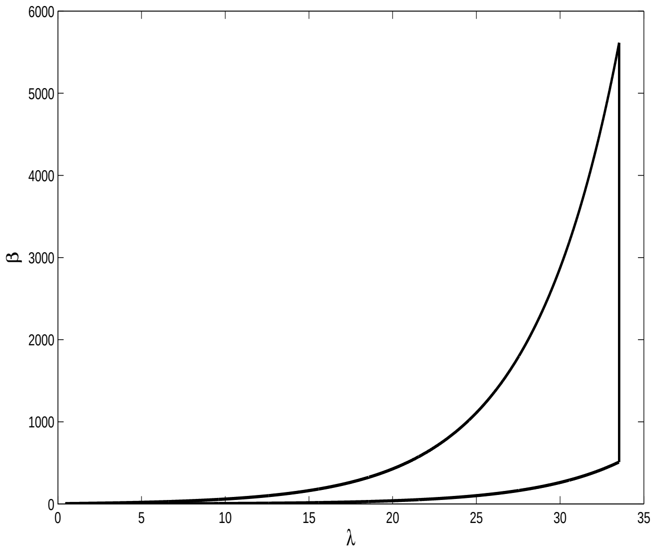

Figure 2: A 95% joint confidence region for (λ, β) based on record values given in Table 5.

Figure 2 shows the 95% joint confidence region for (λ, β) based on record data given in Table 5. It is easy to see that the region is large when λ is large.

5. CONCLUSIONS

Progressive censoring and record values have received attention in the past few decades. The GIE distribution is a new lifetime distribution and can be widely used in reliability applications. The main purpose of this study is to investigate the interval estimation of parameters of the GIE distribution based on progressive type-II censored sample and record values, respectively. We provide four theorems based on the method of pivotal quantity to construct the exact confidence intervals and regions for the parameters. The simulation results show that the proposed methods perform well. Two numerical examples are used to illustrate the proposed methods.

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REFERENCES

[1] Abouammoh, A. M. and Alshingiti, A. M. (2009). Reliability estimation of generalized inverted exponential distribution, *Journal of Statistical Computation and Simulation*, **79**, 1301–1315.

[2] Ahsanullah, M. (1995). *Introduction to Record Statistics*, NOVA Science Publishers Inc., Huntington, New York.

[3] Ahsanullah, M. and Nevzorov, V. B. (2011). Record statistics, In: *International Encyclopedia of Statistical Science*, eds M. Lovric, Springer, Berlin.

[4] Arnold, B. C.; Balakrishnan, N. and Nagaraja, H. N. (1998). *Records*, Wiley, New York.

[5] Bairamov, I. and Eryilmaz, S. (2006). Spacings, exceedances and concomitants in progressive type II censoring scheme, *Journal of Statistical Planning and Inference*, **136**, 527–536.

[6] Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods and Applications.* Birkhauser, Boston.

[7] Chandler, K. N. (1952). The distribution and frequency of record values, *Journal of the Royal Statistical Society: Series B*, **14**, 220–228.

[8] Dey, S. and Dey, T. (2014). On progressively censored generalized inverted exponential distribution, *Journal of Applied Statistics*, **41**, 2557–2576.

[9] Dey, S. and Pradhan, B. (2014). Generalized inverted exponential distribution under hybrid censoring, *Statistical Methodology*, **18**, 101-114.

[10] Ghitany, M. E.; Tuan, V. K. and Balakrishnan, N. (2014). Likelihood estimation for a general class of inverse exponentiated distributions based on complete and progressively censored data, *Journal of Statistical Computation and Simulation*, **84**, 96–106.

[11] Gupta, R. D. and Kundu, D. (1999). Generalized exponential distribution, *Australian and New Zealand Journal of Statistics*, **41**, 173–183.

[12] Gupta, R. D. and Kundu, D. (2001). Generalized exponential distribution: Different methods of estimations, *Journal of Statistical Computation and Simulation*, **69**, 315–338.

[13] Johnson, N. L.; Kotz, S. and Balakrishnan, N. (1994). *Continuous Univariate Distributions. Volume 1*, 2nd edition, Wiley, New York.

[14] Krishna, H. and Kumar, H. (2013). Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample, *Journal of Statistical Computation and Simulation*, **83**, 1007–1019.

[15] Lawless, J. F. (1982). *Statistical Models and Methods for Lifetime Data*, Wiley, New York.

[16] Lee, H. M.; Lee, W. C.; Lei, C. L. and Wu, J. W. (2011). Computational procedure of assessing lifetime performance index of Weibull lifetime products with the upper record values, *Mathematics and Computers in Simulation*, **81**, 1177–1189.

[17] Leiblein, J. and Zelen, M. (1956). Statistical investigation of the fatigue life of deep-groove ball bearings, *Journla of Research of the National Bureau of Standards*, **57**, 273–316.

[18] Luckett, D. J. (2013). *Statistical Inference Based on Upper Record Values*, Honors & Senior Thesis, The College of William and Mary, Department of Mathematics.

[19] Raqab, M. Z. and Madi, M. T. (2005). Bayesian inference for the generalized exponential distribution, *Journal of Statistical Computation and Simulation*, **75**, 841–852.

[20] Singh, S.K.; Singh, U. and Kumar, M. (2013). Estimation of parameters of generalized inverted exponential distribution for progressive type-II censored sample with binomial removals, *Journal of Probability and Statistics*, **2013**, Article ID 183652, DOI: 10.1155/2013/183652.

[21] Thomas, D. R. and Wilson, W. M. (1972). Linear order statistics estimation for the two parameter Weibull and extreme value distributions from type-II progressively censored samples, *Technometrics*, **14**, 679–691.