BLOCK BOOTSTRAP PREDICTION INTERVALS FOR GARCH PROCESSES

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Abstract:

- In this paper, we propose a new resampling algorithm based on block bootstrap to obtain prediction intervals for future returns and volatilities of GARCH processes. The finite sample properties of the proposed methods are illustrated by an extensive simulation study and they are applied to Japan Yen (JPY) / U.S. dollar (USD) daily exchange rate data. Our results indicate that: (*i*) the proposed algorithm is a good competitor or even better and (*ii*) computationally more efficient than traditional method(s).

Key-Words:

- *Financial time series; Prediction; Resampling methods; Exchange rate.*

AMS Subject Classification:

- 62F40, 92B84, 62M20.

1. INTRODUCTION

Many macroeconomic and financial time series vary over wide range around mean, and very large or small prediction errors may occur in practice. Since financial markets are sensitive to political events, speculations, changes in monetary policy etc., this variability in the error terms may occur. This implies that the variance of the errors may not be constant and it changes over time so the errors can be serially correlated in financial data. Additionally, one of the uncertain and decisive factors in financial time series analysis is the volatility as a measure of dispersion and an indicator of magnitude of fluctuations of the asset price series. Hence, measuring volatility as well as construction of valid predictions for future returns and volatilities have an important role in assessing risk and uncertainty in the financial market. Since volatility is the unobservable component of financial time series, it should be modeled correctly to obtain efficient parameter estimation and improve the accuracy of prediction intervals for assessing uncertainty in risk management. In this context, the generalized autoregressive conditionally heteroscedastic (GARCH) model proposed by [7] is one of the most commonly used technique for modeling volatility and obtaining dynamic prediction intervals for returns as well as volatilities. See [4], [22], [13] and [28] for recent studies on GARCH model in modelling volatility. Also see [1], [2], [3], and [15] for detailed information about construction of prediction intervals for future returns in financial time series analysis. However, those works only consider point forecast of volatility even though prediction intervals provide better inference taking into account uncertainty of unobservable sequence of volatilities. On the other hand, construction of prediction intervals requires some distributional assumptions which are generally unknown in practice. Moreover, they can be affected due to any departure from the assumptions and may lead us to unreliable results. One remedy to construct prediction intervals without considering distributional assumptions is to apply the well known resampling methods, such as the bootstrap.

For the serially correlated data, the method of block bootstrap is one of the most general tool to approximate the properties of estimators. In this technique the underlying idea is to construct a resample of the data of size n by dividing the data into several blocks with a sufficiently large block length ℓ and choosing among them till the bootstrap sample is obtained. Then, the dependence structure of the original data is attempted to be captured by these ℓ consecutive observations in each block drawn independently. The commonly used block bootstrap procedures called "non-overlapping" and "overlapping" are first proposed by [16] in the context of spatial data. Then [10] and [20], respectively, adapted the non-overlapping block bootstrap (NBB) and moving block bootstrap (MBB) approaches to the univariate time series context. In addition to these methods, [25] introduced the circular block bootstrap (CBB) method by wrapping the data around a circle before blocking them. Also, the stationary bootstrap (SB) method which deals with random block lengths is proposed by [26]. Moreover, Ordered non-overlapping block bootstrap (ONBB), which orders the bootstrapped blocks according to given labels to each original block, was suggested by [5] to improve the performance of the block bootstrap technique by taking into account the correlations between the blocks.

Bootstrap-based prediction intervals of autoregressive conditionally heteroscedastic (ARCH) model for future returns and volatilities are proposed by [23] and [27]. [24] further extends the previous works to GARCH(1,1) model. Later, [11] suggests computationally efficient bootstrap prediction intervals for ARCH and GARCH processes in the context for financial time series. All of these methods are based on resampling the residuals. The block bootstrap methods are not suitable for construction of prediction intervals in conditionally heteroskedastic time series models because of their poor finite sample performances. On the other hand, it is possible to construct valid block bootstrap based prediction intervals for GARCH processes by using the autoregressive-moving average (ARMA) representation of the GARCH models. For instance, [6] proposed to use the ONBB method to obtain prediction intervals for GARCH process and they obtained better prediction intervals for returns and volatilities compared to the existing residual based bootstrap method(s). Also, [19] introduced a stationary bootstrap prediction interval for GARCH models. In this paper, following the idea of [19], we propose a new bootstrap algorithm to obtain prediction intervals for future returns and volatilities under GARCH processes. In summary, our extension works as follows: First, we use the squares of the GARCH process, which have the ARMA representation, to make the parameter estimation process linear. The ordinary least squares estimators of the ARMA model are calculated by a high order autoregressive model of order m, and the residuals are computed. Then the block bootstrap methods are applied to the data to obtain the bootstrap sample of the returns which are used to calculate the bootstrap estimators of the ARMA coefficients and the bootstrap sample of the volatilities. Finally, the future values of the returns and volatilities of the GARCH process are obtained by means of bootstrap replicates and quantiles of the Monte Carlo estimates of the generated bootstrap distribution.

The rest of the paper is organized as follows. We describe our proposed methods in Section 2. An extensive Monte Carlo simulation is conducted to examine the finite sample performance of the proposed methods and the results are presented in Section 3. In Section 4, the JPY/USD daily exchange rate data is analyzed using the new methods and the results are presented. Section 5 concludes the paper.

2. METHODOLOGY

We use ARMA parameterization of a GARCH model and its least squares (LS) estimators in order to employ block bootstrap methods for constructing prediction intervals.

The GARCH(p,q) process considered in this study has the following representation.

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t, \\
\sigma_t^2 &= \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \quad t = 1, \ldots, T,
\end{aligned} \tag{2.1}
$$

where $\{\epsilon_t\}$ is a sequence of white noise random variables and $E(\epsilon^4) < \infty$, ω, α_i and β_j are unknown parameters satisfying $\omega > 0$, $\alpha_i \geq 0$ and $\beta_j \geq 0$ for $i = 1, \ldots, p$ and $j = 1, \ldots, q$. The stochastic process σ_t is assumed to be independent of ϵ_t. Throughout this paper, we assume that the process $\{y_t\}$ is strictly stationary, i.e., $\sum_{i=1}^{r}(\alpha_i + \beta_i) < 1$, where $r = max(p, q)$, $\alpha_i = 0$ for $i > p$ and $\beta_i = 0$ for $i > q$; see [8] and [9]. A GARCH (p,q) process $\{y_t\}$ is represented in the form of ARMA as follows.

$$
y_t^2 = \omega + \sum_{i=1}^{r}(\alpha_i + \beta_i) y_{t-i}^2 + \nu_t - \sum_{j=1}^{q} \beta_j \nu_{t-j}, \tag{2.2}
$$

where the innovation $\nu_t = y_t^2 - \sigma_t^2$ is a white noise (not i.i.d. in general) and identically distributed under the strict stationary assumption of y_t. Using the unconditional mean of the ARMA model given in 2.2, we have

$$
E(y_t^2) = \frac{\omega}{1 - \sum_{i=1}^{r}(\alpha_i + \beta_i)} \tag{2.3}
$$

According to [18], the LS estimators of an ARMA model are obtained as follows: (a) First, a high order autoregressive model of order m, AR(m), with $m > max(p, q)$, is fitted to the data by Yule-Walker method to obtain $\widehat{\nu}_t$, where m is determined from the data by using Akaike information criteria or Bayesian information criteria. (b) Then a linear regression of y_t^2 onto $y_{t-1}^2, \ldots, y_{t-r}^2, \widehat{\nu}_{t-1}, \ldots, \widehat{\nu}_{t-q}$ is fitted to estimate the parameter vector $\phi = ((\alpha_1+\beta_1), \ldots, (\alpha_r+\beta_r), -\beta_1, \ldots, -\beta_q)'$. In matrix notations, let $\mathbf{Z}_T$ and $\mathbf{X}$ are as follows.

$$
\mathbf{Z}_T = \begin{bmatrix} y_{m+1}^2 \\ \vdots \\ y_T^2 \end{bmatrix}
$$

and

$$
\mathbf{X} = \begin{bmatrix}
y_m^2 & y_{m-1}^2 & \cdots & y_{m-p+1}^2 & \hat{\nu}_m & \hat{\nu}_{m-1} & \cdots & \hat{\nu}_{m-q+1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
y_{T-1}^2 & y_{T-2}^2 & \cdots & y_{T-p}^2 & \hat{\nu}_{T-1} & \hat{\nu}_{T-2} & \cdots & \hat{\nu}_{T-q}
\end{bmatrix}
$$

Then, the LS estimator $\widehat{\phi} = ((\widehat{\alpha_1 + \beta_1}), \ldots, (\widehat{\alpha_r + \beta_r}), -\widehat{\beta}_1, \ldots, -\widehat{\beta}_q)'$ is obtained as

$$
\widehat{\phi} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}_T, \tag{2.4}
$$

given $\boldsymbol{X}'\boldsymbol{X}$ is non-singular. The corresponding $\widehat{\alpha}_i$'s are calculated as $\widehat{\alpha}_i = (\widehat{\alpha_i + \beta_i}) - \widehat{\beta}_i$, for $i = 1, \ldots, p$.

For clarity, we next describe the complete algorithm of the proposed block bootstrap prediction intervals for future returns and volatilities.

Step 1 For a realization of GARCH(p,q) process, $\{y_{1-r}, \ldots, y_0, y_1, \ldots, y_T\}$, calculate the LS estimates of ARMA coefficients as in Eq. 2.4, and the corresponding $\widehat{\omega}$ is calculated by using Eq. 2.3 such that $\widehat{\omega} = E(\hat{y}_t^2)\left[1 - \sum_{i=1}^{r}(\widehat{\alpha_i + \beta_i})\right]$, where $E(\hat{y}_t^2) = T^{-1}\sum_{t=1}^{T} y_t^2$.

Step 2 For $t = r, \ldots, T$, calculate the residuals $\widehat{\epsilon}_t = y_t/\widehat{\sigma}_t$ where $\widehat{\sigma}_t^2 = \widehat{\omega} + \sum_{i=1}^{p}\widehat{\alpha}_i y_{t-i}^2 + \sum_{j=1}^{q}\widehat{\beta}_j\widehat{\sigma}_{t-j}^2$ and $\widehat{\sigma}_0^2 = \widehat{\omega}/(1 - \sum_{i=1}^{r}(\widehat{\alpha}_i + \widehat{\beta}_i))$. Let $\widehat{F}_\epsilon$ be the empirical distribution function of the centered and rescaled residuals.

Step 3 Compute the error term as $\hat{\xi} = \mathbf{Z}_T - \mathbf{X}\hat{\phi}$ and construct the design matrix $\mathbf{Y} = (\mathbf{X}, \xi)$.

$$\mathbf{Y} = \begin{bmatrix} y_{t-1}^2 & y_{t-2}^2 & \cdots & y_{t-r}^2 & \hat{\nu}_{t-1} & \hat{\nu}_{t-2} & \cdots & \hat{\nu}_{t-q} & \hat{\xi}_t \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y_{T-1}^2 & y_{T-2}^2 & \cdots & y_{T-r}^2 & \hat{\nu}_{T-1} & \hat{\nu}_{T-2} & \cdots & \hat{\nu}_{T-q} & \hat{\xi}_T \end{bmatrix}$$

Let $\mathbf{Y}_t = (y_{t-1}^2, y_{t-2}^2, \ldots, y_{t-r}^2, \hat{\nu}_{t-1}, \hat{\nu}_{t-2}, \ldots, \hat{\nu}_{t-q}, \hat{\xi}_t)$, $t = 1, \ldots, T$, denotes the tth row of the design matrix $\mathbf{Y}$. Let also $\mathbf{B}^{(k)}$, for $k = 1, 2, 3$, respectively, represents the block vectors of NBB, MBB and CBB methods obtained from $\mathbf{Y}$ such that $\mathbf{B}_j^{(1)} = \{Y_{(j-1)\ell+1}, \ldots, Y_{j\ell}\}$ where $b = \lfloor T/\ell \rfloor$ and $j = 1, \ldots, b$, $\mathbf{B}_j^{(2)} = \{Y_j, \ldots, Y_{j+l-1}\}$ where $1 \leq j \leq N$ and $N = T - \ell + 1$ and $\mathbf{B}_j^{(3)} = \{Y_j, \ldots, Y_{j+l-1}\}$ where $1 \leq j \leq T$. Then obtain the block bootstrap observations $\{Y_1^*, \ldots, Y_T^*\}$, where $\mathbf{Y}_t^* = (y_{t-1}^{2*}, y_{t-2}^{2*}, \ldots, y_{t-r}^{2*}, \hat{\nu}_{t-1}^*, \hat{\nu}_{t-2}^*, \ldots, \hat{\nu}_{t-q}^*, \hat{\xi}_t^*)$, by sampling with replacement from $\mathbf{B}^{(k)}$. The ONBB and SB observations are obtained as follows.

- ONBB observations are obtained as ordering the bootstrapped non-overlapping blocks according to given labels to each original block. Suppose the data is divided into the four independent non-overlapping blocks. Then, the labels are determined as $B_1 = 1, B_2 = 2, B_3 = 3$ and $B_4 = 4$, and let the bootstrapped blocks are $B_1^* = B_4, B_2^* = B_2, B_3^* = B_3$ and $B_4^* = B_3$. As a consequence, the ONBB data is obtained as $\{B_2 \vdots B_3 \vdots B_3 \vdots B_4\}$.
- Let $B(i\ell) = (Y_i, \cdots, Y_{i+\ell-1})$, for $i \geq 1$, be the blocks of ℓ consecutive observations starting from Y_i. The observed time series data is wrapped around a circle in order to ensure that all starting points have equal probability of selection. Let $I_1, I_2, \cdots$ be the independently and identically distributed discrete uniform random variables

on $\{1,\cdots,T\}$ so that $P(I_1 = i) = 1/T$, for $i = 1,\cdots,T$. Let also $L_1, L_2, \cdots$ be the iid geometric random variables with parameter ρ such that $0 < \rho < 1$ and the probability mass function $P(L_1 = \ell) = \rho(1-\rho)^{\ell-1}$, for $\ell = 1,2,\cdots$. We assume that two sets $\{I_1, I_2, \cdots\}$ and $\{L_1, L_2, \cdots\}$ are independent and $\rho \to 0$ as $T\rho \to \infty$. Then, the SB data $\{Y_1^*,\cdots,Y_T^*\}$ are generated by sampling from $\{B_{I_1L_1}, B_{I_2L_2},\cdots\}$ where $B_{I_rL_r} = \{Y_{I_r},\cdots,Y_{I_r+L_r-1}\}$ for $r \geq 1$.

Step 4 Let $\mathbf{X}^*$ be the bootstrap analogue of X such that

$$\mathbf{X}^* = \begin{bmatrix} y_m^{*2} & y_{m-1}^{*2} & \cdots & y_{m-p+1}^{*2} & \hat{\nu}_m^* & \hat{\nu}_{m-1}^* & \cdots & \hat{\nu}_{m-q+1}^* \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_{T-1}^{*2} & y_{T-2}^{*2} & \cdots & y_{T-p}^{*2} & \hat{\nu}_{T-1}^* & \hat{\nu}_{T-2}^* & \cdots & \hat{\nu}_{T-q}^* \end{bmatrix}$$

Then calculate the block bootstrap estimators of ARMA coefficients as

$$\widehat{\phi}^* = (\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\mathbf{Z}_T^* = ((\widehat{\alpha_1+\beta_1})^*,\ldots,(\widehat{\alpha_r+\beta_r})^*,-\widehat{\beta}_1^*,\ldots,-\widehat{\beta}_q^*)'$$

where $\mathbf{Z}_T^* = \mathbf{X}^*\hat{\phi} + \hat{\xi}$. Also, calculate the corresponding $\widehat{\alpha}_i^*$'s as $\widehat{\alpha}_i^* = (\widehat{\alpha_i+\beta_i})^* - \widehat{\beta}_i^*$, for $i = 1,\ldots,p$, and $\widehat{\omega}^*$'s as in Step 1 but using bootstrap observations.

Step 5 Obtain block bootstrap volatilities as $\widehat{\sigma}_t^{2*} = \widehat{\omega}^* + \sum_{i=1}^p \widehat{\alpha}_i^* y_{t-i}^{2*} + \sum_{j=1}^q \widehat{\beta}_j^* \widehat{\sigma}_{t-j}^{2*}$ with $\widehat{\sigma}_0^{2*} = \widehat{\omega}/(1-\sum_{i=1}^r(\widehat{\alpha}_i + \widehat{\beta}_i))$.

Step 6 Calculate $h = 1,2,\ldots$ steps ahead block bootstrap future returns and volatilities with the following recursions:

$$\begin{aligned} \widehat{\sigma}_{T+h}^{2*} &= \widehat{\omega}^* + \sum_{i=1}^p \widehat{\alpha}_i^* y_{T+h-i}^{2*} + \sum_{j=1}^q \widehat{\beta}_j^* \widehat{\sigma}_{T+h-j}^{2*} \\ y_{T+h}^* &= \widehat{\sigma}_{T+h}^{2*}\widehat{\epsilon}_{T+h}^* \end{aligned}$$

where $y_{T+h}^* = y_{T+h}$ for $h \leq 0$ and $\widehat{\epsilon}_{T+h}^*$ is randomly drawn from $\widehat{F}_\epsilon$.

Step 7 Repeat Steps 3-6 B times to obtain bootstrap replicates of returns and volatilities $\{y_{T+h}^{*,1},\ldots,y_{T+h}^{*,B}\}$ and $\{\widehat{\sigma}_{T+h}^{2*,1},\ldots,\widehat{\sigma}_{T+h}^{2*,B}\}$ for each h. Note that B denotes the number of bootstrap replications.

As noted in [24], the one-step conditional variance is perfectly predictable if the model parameters are known, and the only uncertainty which is caused by the parameter estimation, is associated with the prediction of σ_{T+1}^2. On the other hand, there are further uncertainties about future errors when predicting two or more step ahead variances. Thus, it is more interesting issue to have prediction intervals for future volatilities. Now, let $G_y^*(k) = P(y_{T+h}^* \leq k)$ and $G_{\sigma^2}^*(k) =$

$P(\widehat{\sigma}^{*2}_{T+h} \leq k)$ be the block bootstrap distribution functions of unknown distribution functions of y_{T+h} and σ^2_{T+h}, respectively. Also let $G^*_{y,B}(k) = \#(y^{*,b}_{T+h} \leq k)/B$ and $G^*_{\sigma^2,B}(k) = \#(\widehat{\sigma}^{2*,b}_{T+h} \leq k)/B$, for $b = 1, \ldots, B$, be the corresponding Monte Carlo (MC) estimates. Then, the $100(1-\gamma)\%$ bootstrap prediction intervals for y_{T+h} and σ^2_{T+h}, respectively, are given by

$$\begin{aligned}\left[LB^*_{y,B}, UB^*_{y,B}\right] &= \left[Q^*_{y,B}(\gamma/2), Q^*_{y,B}(1-\gamma/2)\right], \\ \left[LB^*_{\sigma^2,B}, UB^*_{\sigma^2,B}\right] &= \left[Q^*_{\sigma^2,B}(\gamma/2), Q^*_{\sigma^2,B}(1-\gamma/2)\right]\end{aligned}$$

where $Q^*_{y,B} = G^{*-1}_{y}$ and $Q^*_{\sigma^2,B} = G^{*-1}_{\sigma^2}$.

3. NUMERICAL RESULTS

We performed a simulation study to investigate the performances of the block bootstrap prediction intervals constructed through the GARCH (1,1) model given in (3.1) below, and we compared our results with the method proposed by [24] (abbreviated as "PRR"). In brief, the PRR method uses quasi-maximum likelihood method to estimate the parameters and then, uses residual-based resampling to construct prediction intervals for future returns and volatilities. The comparison was made through the coverage probabilities and length of prediction intervals. It is worth the mention that we also checked the performances of the conventional block bootstrap methods. Roughly, we observed the coverage probabilities of other block bootstrap methods range in between 90%-94% for future returns while those range only in between 25%-60% for future volatilities. These results are not shown to save space, but are available from the authors upon request.

To discuss the numerical study we present here, let us start with the following GARCH(1,1) model.

$$\begin{aligned}y_t &= \sigma_t \epsilon_t \\ \sigma^2_t &= 0.05 + 0.1 y^2_{t-1} + 0.85 \sigma^2_{t-1},\end{aligned} \tag{3.1}$$

where ϵ_t follows a $N(0,1)$ distribution. The significance level γ is set to 0.05 to obtain 95% prediction intervals for future returns and volatilities. Since the block bootstrap methods are sensitive to the choice of the block length ℓ, we choose three different block lengths in our simulation study: $T^{1/3}, T^{1/4}, T^{1/5}$ as proposed by [17]. Let $h = 1, 2, \ldots, s,\ s \geq 1$, be defined as the lead time. We obtain the prediction intervals for next $s = 20$ observations. The experimental design is similar to those of [24] which is as follows:

Step 1 Simulate a GARCH(1,1) series with the parameters given in Equation 3.1 , for $h = 1, \ldots, s$, generate $R = 1000$ future values y_{T+h} and σ^2_{T+h} to calculate the average coverage probabilities and interval lengths (as well as their standard errors) for the prediction intervals.

Step 2 Calculate bootstrap future values $y^{*,b}_{T+h}$ and $\sigma^{2*,b}_{T+h}$ for $h = 1, \ldots, s$ and $b = 1, \ldots, B$. Then estimate the coverage probabilities (C^*) of bootstrap prediction intervals for y^*_{T+h} and σ^{2*}_{T+h} as

$$\begin{aligned} C^*_{y_{T+h}} &= \frac{1}{R}\sum_{r=1}^{R} \mathbf{1}\{Q^*_{y_{T+h}}(\gamma/2) \le y^{*,r}_{T+h} \le Q^*_{y_{T+h}}(1-\gamma/2)\} \\ C^*_{\sigma^2_{T+h}} &= \frac{1}{R}\sum_{r=1}^{R} \mathbf{1}\{Q^*_{\sigma^2_{T+h}}(\gamma/2) \le \sigma^{2*,r}_{T+h} \le Q^*_{\sigma^2_{T+h}}(1-\gamma/2)\} \end{aligned}$$

where $\mathbf{1}$ represents the indicator function. The corresponding interval lengths (L^*) are calculated by

$$\begin{aligned} L^*_{y_{T+h}} &= Q^*_{y_{T+h}}(1-\gamma/2) - Q^*_{y_{T+h}}(\gamma/2) \\ L^*_{\sigma^2_{T+h}} &= Q^*_{\sigma^2_{T+h}}(1-\gamma/2) - Q^*_{\sigma^2_{T+h}}(\gamma/2) \end{aligned}$$

Step 3 Repeat Steps 1-2, $MC = 1000$ times to calculate the average values of $C^*_{y_{T+h}}$, $C^*_{\sigma^2_{T+h}}$, $L^*_{y_{T+h}}$ and $L^*_{\sigma^2_{T+h}}$.

Our results showed that the accuracy of the prediction intervals for volatilities are sensitive to the choice of block length parameter ℓ. The higher coverage probabilities are obtained for all the methods when $\ell = T^{1/5}$ is used, therefore to save space we present only the results obtained for the choices of block length parameter $\ell = T^{1/5}$. Table 1 summarizes the simulation results. More detailed results are presented in Figures 1 - 4. Our findings show that ONBB outperforms PRR and other block bootstrap methods in general. For coverage probabilities of future returns (see Figure 1), the performances of all the methods are almost the same. Also, all the proposed methods provide competitive interval lengths for returns (see Figure 3). For the prediction intervals of volatilities (please see Figure 4), the performance of ONBB is always better than PRR and other block bootstrap methods in small sample sizes especially for short-term forecasts, and it outperforms other methods also in large samples. PRR has better performances compared to non-ordered block bootstrap methods for short term forecasts, and all the methods have similar performances for long term forecasts. We note that the results obtained by MBB and CBB methods are quite similar, therefore to make the results more readable we present the results only for the CBB method.

Table 1: Prediction intervals for returns and volatilities of GARCH(1, 1) model.

Lead time	Sample size	Method	Average coverage for return (SE)	Average length for return (SE)	Average coverage for volatility (SE)	Average length for volatility (SE)
1	T	Empirical	0.95	3.814	0.95	-
		PRR	0.945(0.021)	3.748(0.874)	0.904(0.295)	0.649(0.520)
	300	ONBB	0.943(0.022)	3.690(0.704)	0.949(0.220)	0.720(0.592)
		NBB	0.941(0.041)	3.739(0.562)	0.847(0.360)	0.986(0.528)
		CBB	0.941(0.042)	3.737(0.558)	0.850(0.357)	0.991(0.536)
		SB	0.941(0.042)	3.731(0.564)	0.846(0.361)	1.001(0.544)
		PRR	0.946(0.011)	3.800(0.863)	0.952(0.214)	0.181(0.194)
		ONBB	0.948(0.015)	3.815(0.793)	0.995(0.070)	0.803(0.740)
	3000	NBB	0.948(0.045)	3.889(0.343)	0.892(0.310)	1.224(0.297)
		CBB	0.948(0.046)	3.886(0.340)	0.897(0.304)	1.230(0.297)
		SB	0.948(0.045)	3.888(0.347)	0.885(0.319)	1.232(0.300)
10	T	Empirical	0.95	3.946	0.95	1.389
		PRR	0.943(0.026)	3.846(0.712)	0.902(0.117)	1.564(1.387)
	300	ONBB	0.938(0.025)	3.723(0.530)	0.921(0.113)	1.541(1.181)
		NBB	0.937(0.032)	3.738(0.497)	0.898(0.141)	1.547(0.943)
		CBB	0.937(0.032)	3.736(0.503)	0.902(0.136)	1.549(0.944)
		SB	0.936(0.032)	3.721(0.499)	0.896(0.141)	1.516(0.923)
		PRR	0.946(0.012)	3.875(0.604)	0.941(0.036)	1.354(0.653)
		ONBB	0.947(0.014)	3.867(0.584)	0.955(0.059)	1.582(0.967)
	3000	NBB	0.947(0.029)	3.901(0.270)	0.939(0.097)	1.670(0.531)
		CBB	0.947(0.029)	3.907(0.275)	0.939(0.098)	1.669(0.533)
		SB	0.947(0.029)	3.897(0.278)	0.932(0.103)	1.647(0.541)
20	T	Empirical	0.95	3.948	0.95	1.661
		PRR	0.940(0.026)	3.876(0.647)	0.881(0.122)	1.771(1.515)
	300	ONBB	0.935(0.026)	3.741(0.507)	0.903(0.119)	1.646(0.990)
		NBB	0.934(0.029)	3.746(0.502)	0.895(0.128)	1.635(0.911)
		CBB	0.934(0.029)	3.740(0.498)	0.898(0.125)	1.640(0.900)
		SB	0.933(0.029)	3.727(0.499)	0.895(0.126)	1.623(0.919)
		PRR	0.946(0.012)	3.907(0.444)	0.940(0.033)	1.634(0.627)
		ONBB	0.946(0.014)	3.895(0.460)	0.949(0.063)	1.861(0.972)
	3000	NBB	0.946(0.020)	3.910(0.255)	0.948(0.073)	1.876(0.595)
		CBB	0.946(0.020)	3.913(0.255)	0.948(0.071)	1.872(0.583)
		SB	0.946(0.020)	3.900(0.259)	0.946(0.073)	1.859(0.598)

We also compared our proposed algorithm with the PRR in terms of their computing times. Let c_1 and c_2 be the obtained computing times for PRR and proposed algorithm, respectively. Figure 5 represents the ratio of computing times, c_1/c_2, for various sample sizes based on $B = 1000$ bootstrap replications and only one Monte Carlo simulation. As presented in Figure 5, the proposed algorithm has considerably less computational time such that PRR requires about 36 - 12 times more computing time (in small and large samples, respectively) than the proposed algorithm.

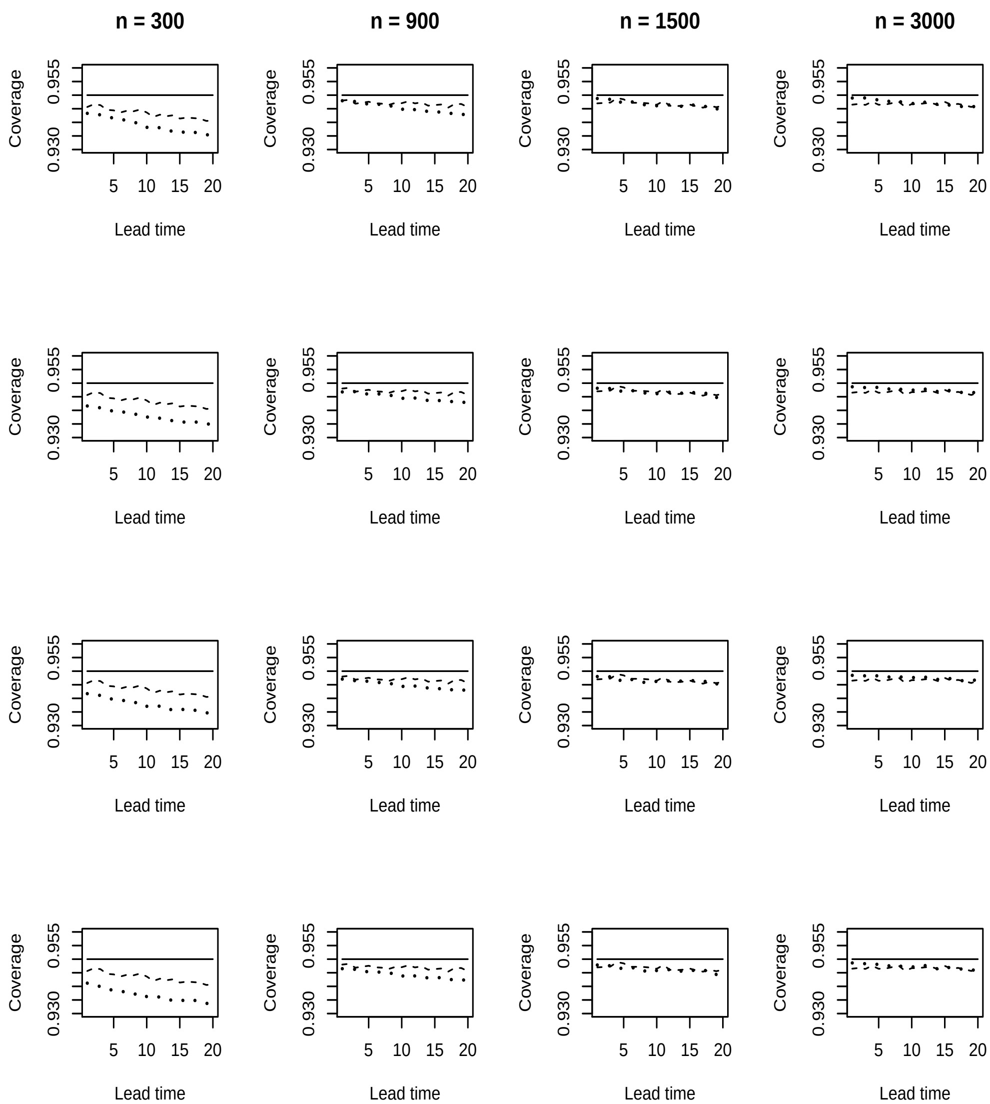

Figure 1: Estimated coverage probabilities of returns. First line: ONBB vs PRR, second line: NBB vs PRR, third line: CBB vs PRR, fourth line: SB vs PRR. Solid line represents the empirical coverage. Dashed line and dotted line represent the coverage probabilities obtained using PRR and proposed methods, respectively.

4. CASE STUDY

The JPY/USD daily exchange rate data were obtained starting from 3rd January, 2011 and ending on 30th April, 2015 (available at https://www.stlouisfed.org/). After excluding observations on weekends and inactive days, our final data consisted a total of 1071 observations. The daily logarithmic returns were obtained

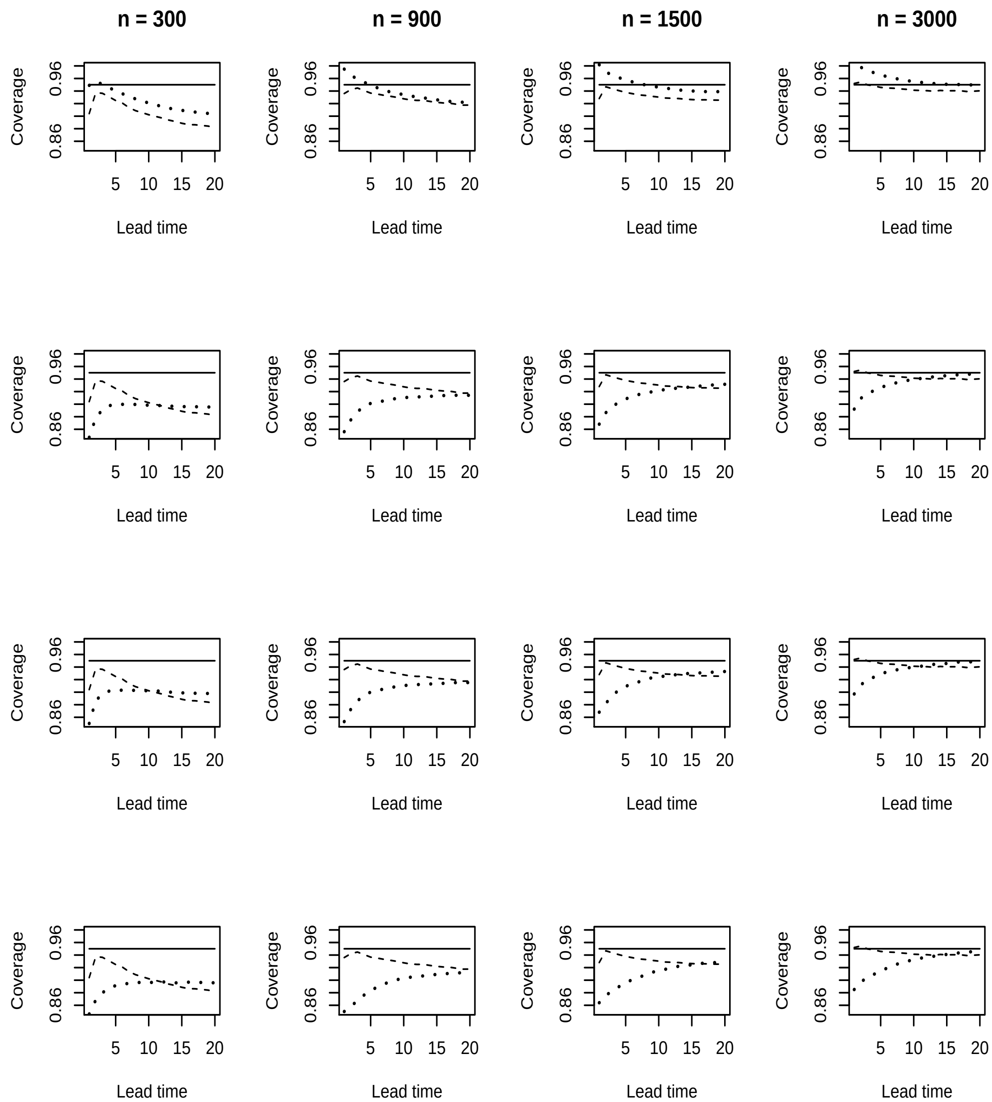

Figure 2: Estimated coverage probabilities of volatilities. First line: ONBB vs PRR, second line: NBB vs PRR, third line: CBB vs PRR, fourth line: SB vs PRR. Solid line represents the empirical coverage. Dashed line and dotted line represent the coverage probabilities obtained using PRR and proposed methods, respectively.

as $y_t = 100{*}\log(P_t/P_{t-1})$, where P_t was the closing price on t-th day. The time series plots of the exchange rates and returns are presented in Figure 6. We checked the stationary status of the return series by applying the Ljung-Box and Augmented Dickey - Fuller t-statistic tests and small p-values reject the null hypothesis against stationary alternative and suggest that the return series is a mean-zero stationary process. Table 2 reports the sample statistics of y_t series, and it shows that the estimated kurtosis is higher than 3 which indicates that the distribution

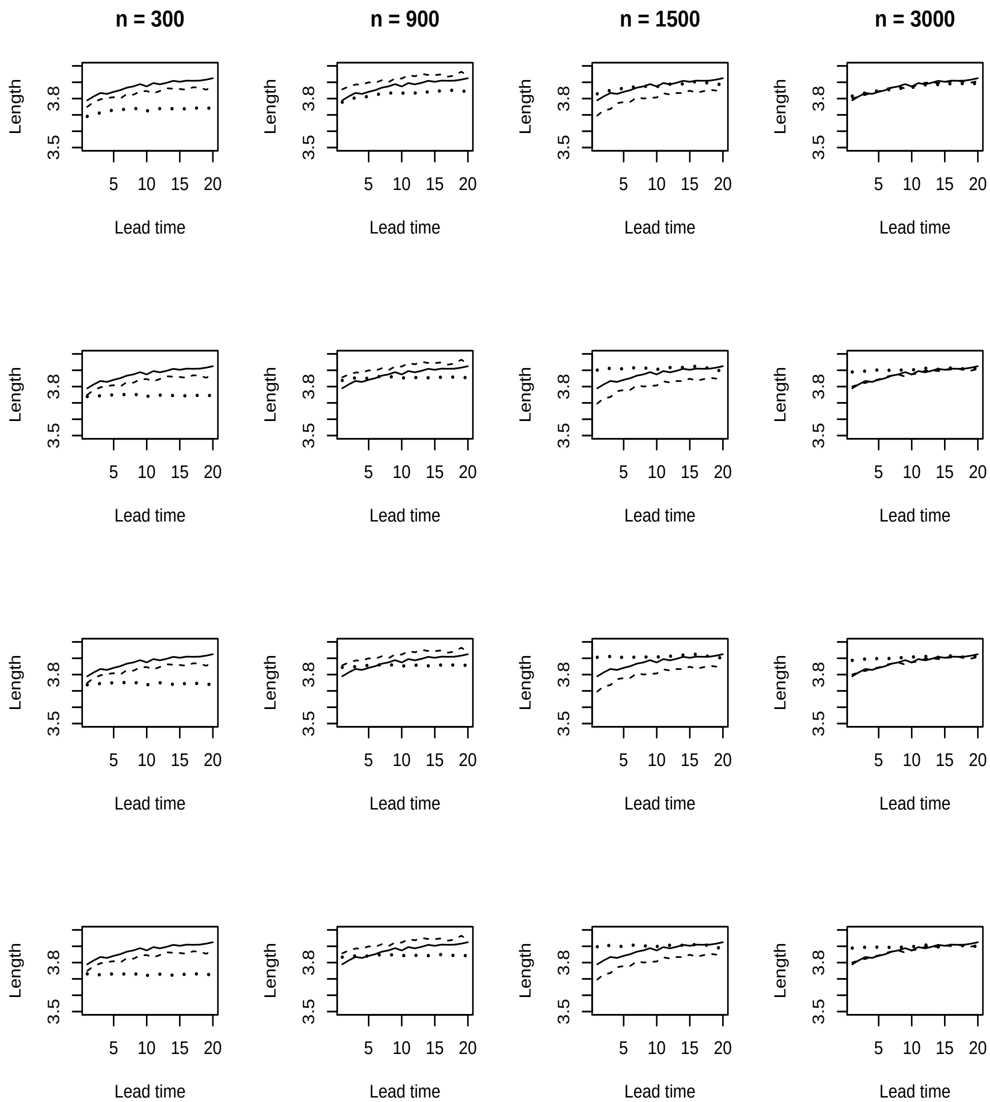

Figure 3: Estimated lengths of prediction intervals of returns. First line: ONBB vs PRR, second line: NBB vs PRR, third line: CBB vs PRR, fourth line: SB vs PRR. Solid line represents the empirical interval lengths. Dashed line and dotted line represent the interval lengths obtained using PRR and proposed methods, respectively.

of the returns was leptokurtic. Next, we checked for the Gaussianity of the return series and the p-value $= 0.000$ of Jarque-Bera test indicated that y_t was not Gaussian. Further, we performed the Box-Pierce test to test for auto-correlations in the absolute and squared returns and smaller p-values indicated that the absolute and squared returns are highly auto-correlated. The auto-correlations of returns, absolute and squared returns are presented in Table 3. All of our preliminary exploratory analyses suggested the presence of conditional heteroscedasticity in

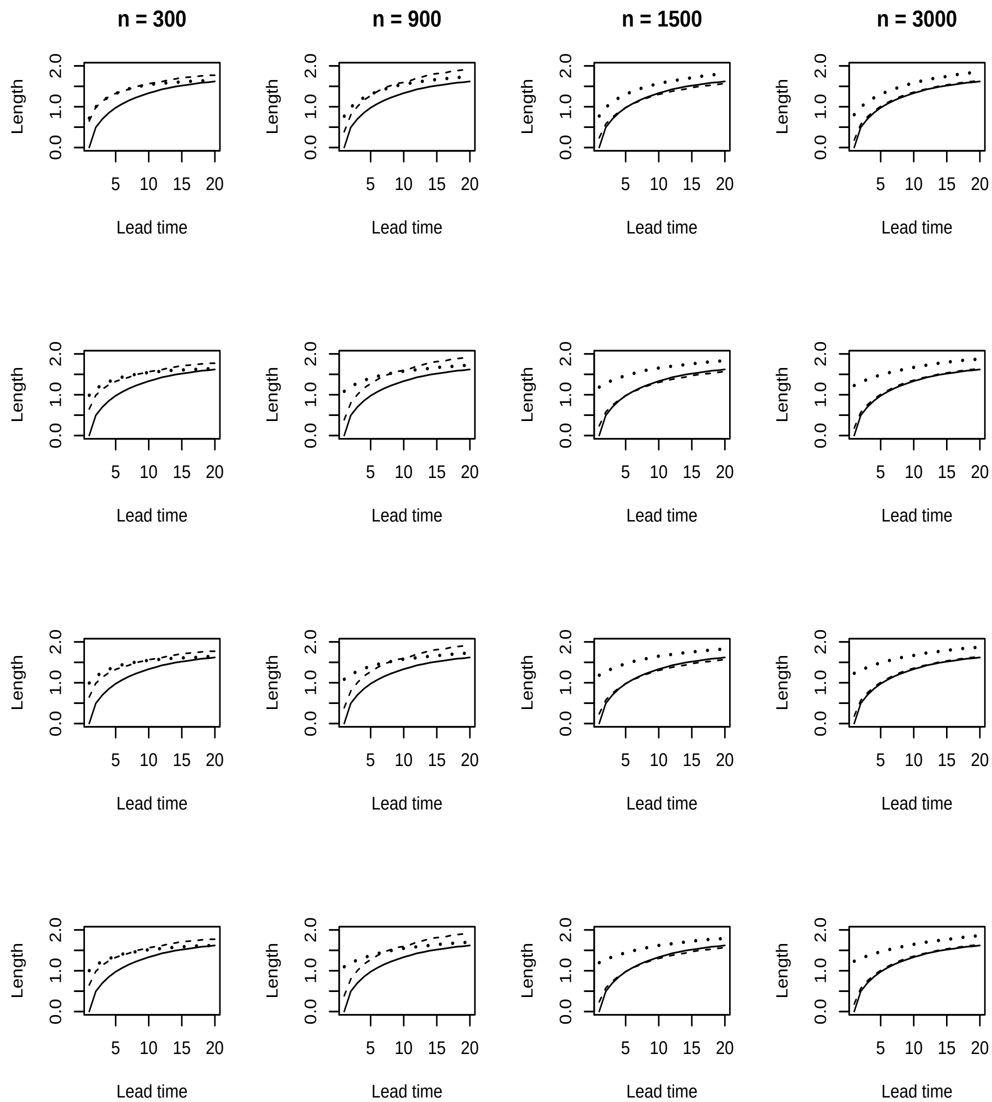

Figure 4: Estimated lengths of prediction intervals of volatilities. First line: ONBB vs PRR, second line: NBB vs PRR, third line: CBB vs PRR, fourth line: SB vs PRR. Solid line represents the empirical interval lengths. Dashed line and dotted line represent the interval lengths obtained using PRR and proposed methods, respectively.

the series. To find the optimal lag for the GARCH model to model the return series we defined many possible subsets of the GARCH(p,q)models with different p and q values. To choose the best model we used Akaike information (AIC) criterion (since it is proposed to determine the best model for forecasting) and the results show that GARCH(1,1) model is optimal according to AIC.

To obtain out-of sample prediction intervals for the real data, we divide

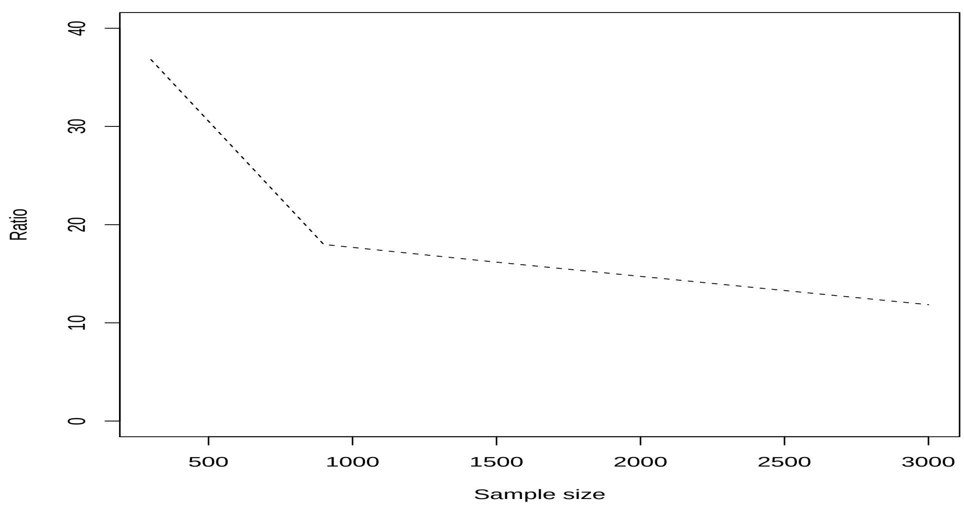

Figure 5: Ratio of the estimated computing times for PRR and proposed algorithm.

Table 2: Sample statistics for y_t

T	Mean	Median	SD	Skewness	Kurtosis	Min.	Max.
1070	0.04	0.01	0.58	0.64	7.21	-2.13	3.34

Table 3: Autocorrelations of y_t at lag k, $k = 1, 2, 5, 10, 16, \cdots, 20$.

Autocorrelations	r(1)	r(2)	r(5)	r(10)	r(16)	r(17)	r(18)	r(19)	r(20)
y_t	0.008	-0.006	-0.009	-0.027	-0.085	0.015	-0.016	0.011	0.097
$\|y_t\|$	0.117	0.107	0.111	0.125	0.119	0.070	0.084	0.132	0.096
y_t^2	0.094	0.091	0.066	0.085	0.070	0.027	0.021	0.097	0.083

the full data into the following two parts: The model is constructed based on the observations from 3rd January, 2011 to 19th March, 2015 (1041 observations in total) to calculate 30 steps ahead predictions from 20th March to 30th April, 2015 and compare with the actual values. The fitted models for the PRR and proposed block bootstrap methods are obtained as in Equations 4.1 and 4.2, respectively.

$$y_t^2 = 0.0054 + 0.0569 y_{t-1}^2 + 0.9283 \hat{\sigma}_{t-1}^2, \tag{4.1}$$

$$y_t^2 = 0.0150 + 0.9556 y_{t-1}^2 + \nu_t - 0.8805 \nu_{t-1}, \tag{4.2}$$

where $\widehat{\omega} = 0.0150$, $\widehat{\alpha}_1 = 0.0750$ and $\widehat{\beta}_1 = 0.8805$ for the model estimated by 4.2. The 30 steps ahead prediction intervals for returns y_{T+h} based on the models given in Equations 4.1 and 4.2, together with the true returns are presented in Figure 7. The intervals obtained using all the methods are similar and they include all of the true values of returns (only PRR fails to cover the 13th point).

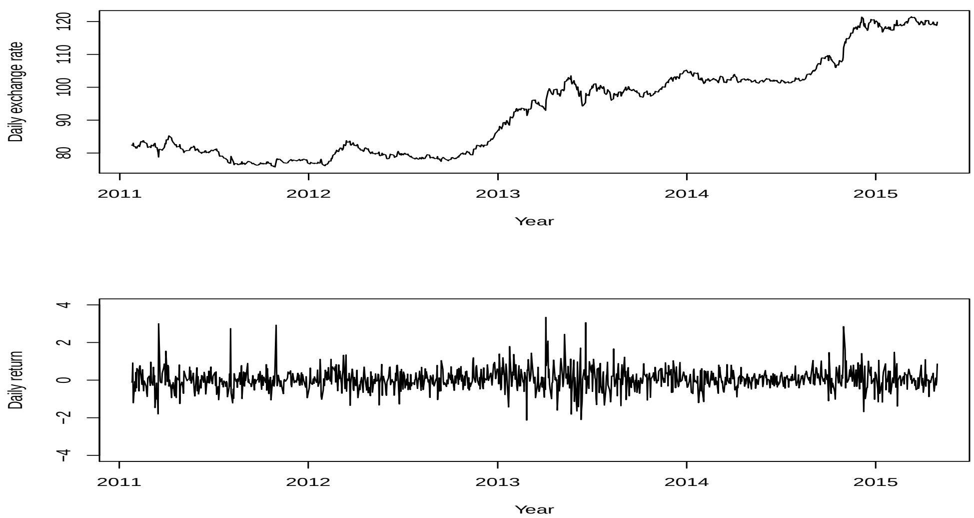

Figure 6: Time series plots of JPY/USD daily exchange rates and returns from 3rd January, 2011 to 30th April, 2015.

Figure 8 shows the predicted intervals for 30 steps ahead volatilities σ^2_{T+h}. The true values of the volatilities can not be observed directly. We calculate the realized volatility by summing squared returns at day t, $\sigma_t^2 = y_{t,1}^2 + \ldots + y_{t,n}^2$, where n is the number of observations recorded during day t as proposed by [1]. Since our data is from 24 hour open trading market, the realized volatilities are computed by using one-minute returns based on tick-by-tick prices such that $n = 1440$ approximately. Figure 8 indicates that the PRR and ONBB methods produce narrower prediction intervals than the one obtained by other block bootstrap methods.

5. CONCLUSION

In this paper, we propose a novel resampling algorithm to obtain prediction intervals for returns and volatilities under GARCH models, and we compare the performances of the methods by both simulations and a case study. Our idea is based on using the ARMA representation of the GARCH models. Under ARMA representation, estimation of parameters becomes linear, which allows us to have a valid prediction intervals for the block bootstrapping procedure.

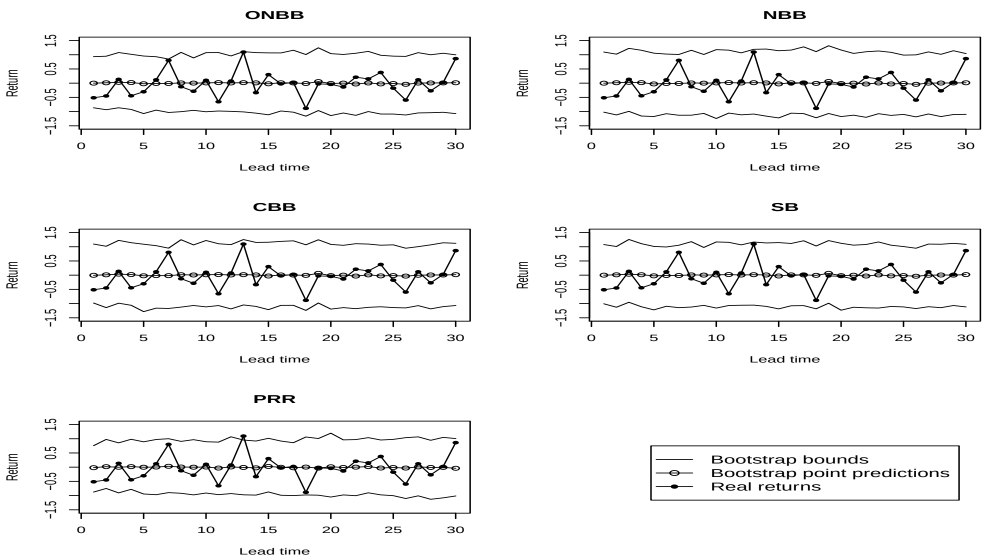

Figure 7: 95% prediction intervals of returns from 20th March, 2015 to 30th April, 2015.

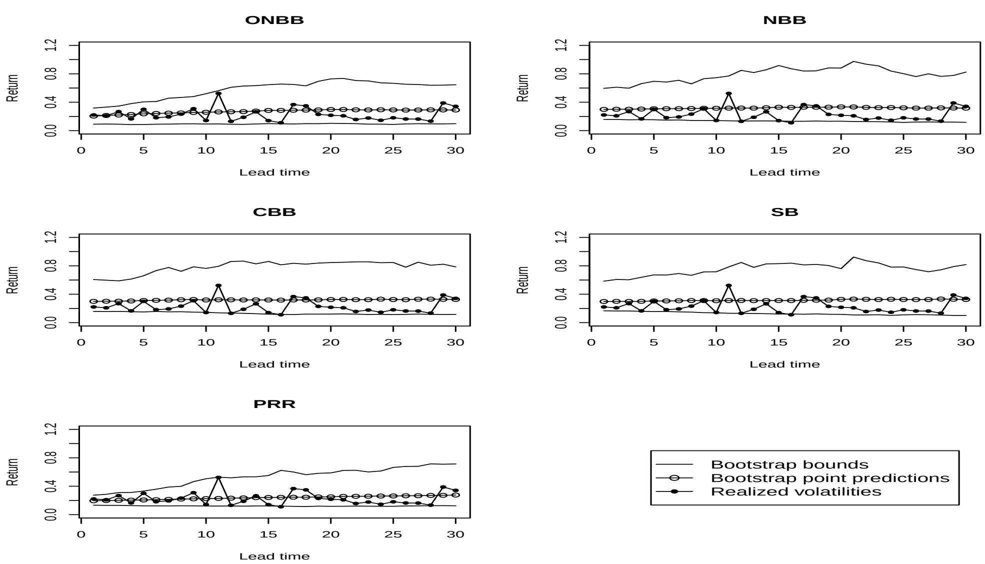

Figure 8: 95% prediction intervals of volatilities from 20th March, 2015 to 30th April, 2015.

Our findings show that our proposed ONBB method: (i) is a good competitor or even better, (ii) is computationally more efficient than traditional method(s). Also, the proposed algorithm improves the performances of the non-ordered block bootstrap methods significantly compared to their conventional counterparts.

As a future research, the performances of the proposed methods can also be studied for forecasting time series with BOOT.EXPOS procedure as studied by

[12] or they can also be used in other statistical inference problems for dependent data.

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