A COUPLE OF NON REDUCED BIAS GENERALIZED MEANS IN EXTREME VALUE THEORY: AN ASYMPTOTIC COMPARISON

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Abstract:

- *Lehmer's mean-of-order p* (L_p) generalizes the arithmetic mean, and L_p *extreme value index* (EVI)-estimators can be easily built, as a generalization of the classical Hill EVI-estimators. Apart from a reference to the asymptotic behaviour of this class of estimators, an asymptotic comparison, at optimal levels, of the members of such a class reveals that for the optimal (p, k) in the sense of minimal mean square error, with k the number of top order statistics involved in the estimation, they are able to overall outperform a recent and promising generalization of the Hill EVI-estimator, related to the power mean, also known as Hölder's mean-of-order-p. A further comparison with other 'classical' non-reduced-bias estimators still reveals the competitiveness of this class of EVI-estimators.

Key-Words:

- *Heavy tails; Optimal tuning parameters; Semi-parametric estimation; Statistical extreme value theory.*

AMS Subject Classification:

- 62G32, 62E20.

1. GENERALIZED MEANS' ESTIMATORS AND SCOPE OF THE ARTICLE

Let us consider the notation $(X_{1:n}, \ldots, X_{n:n})$ for the ascending order statistics associated with a random sample of size n, $(X_1, \ldots, X_n)$, from a *cumulative distribution function* (CDF) F. Let us further assume that there exist sequences of real constants $\{a_n > 0\}$ and $\{b_n \in \mathbb{R}\}$ such that the maximum, linearly normalized, i.e. $(X_{n:n} - b_n)/a_n$, converges in distribution to a non-degenerate *random variable* (RV). Then (Gnedenko, 1943), the limit distribution is necessarily of the type of the general *extreme value* (EV) CDF, given by

$$\text{(1.1)} \qquad \mathrm{EV}_\xi(x) := \begin{cases} \exp(-(1+\xi x)^{-1/\xi}),\ 1+\xi x > 0, & \text{if } \xi \neq 0, \\[2ex] \exp(-\exp(-x)),\ x \in \mathbb{R}, & \text{if } \xi = 0. \end{cases}$$

The CDF F is then said to belong to the *max-domain of attraction* of EV_ξ, defined in (1.1), we use the notation $F \in \mathcal{D}_\mathcal{M}(\mathrm{EV}_\xi)$, and the parameter ξ is the *extreme value index* (EVI), the primary parameter of extreme events. It is well-known that the EVI measures the heaviness of the right-tail function $\overline{F}(x) := 1 - F(x)$, and the heavier the right-tail function, the larger ξ is. From a quantile point of view, with $F^{\leftarrow}(x) := \inf\{y : F(y) \geq x\}$ denoting the generalized inverse function of F, further consider $U(t) := F^{\leftarrow}(1 - 1/t),\ t \geq 1$, the *reciprocal tail quantile function* (RTQF). Then, with $\mathcal{R}_a$ denoting the class of regularly varying functions at infinity, with an index of regular variation equal to $a \in \mathbb{R}$, i.e. positive measurable functions $g(\cdot)$ such that for all $x > 0$, $g(tx)/g(t) \to x^a$, as $t \to \infty$, (see Bingham *et al.*, 1987, among others),

$$\text{(1.2)} \quad F \in \mathcal{D}_\mathcal{M}^+ := \mathcal{D}_\mathcal{M}(\mathrm{EV}_\xi)_{\xi>0} \iff \overline{F} \in \mathcal{R}_{-1/\xi} \text{ (Gnedenko, 1943)} \\ \iff U \in \mathcal{R}_\xi \text{ (de Haan, 1984).}$$

In this article we work with a Pareto-type underlying CDF, satisfying (1.2), i.e. with an associated positive EVI for maxima. These heavy-tailed models are quite common in a large variety of fields of application, like bibliometrics, biostatistics, computer science, insurance, finance, social sciences, statistical quality control and telecommunications, among others. For Pareto-type models, the classical EVI-estimators are the *Hill* (H) estimators (Hill, 1975), which are the averages of the log-excesses, i.e.

$$\text{(1.3)} \quad \hat{\xi}^{\mathrm{H}}(k) \equiv \mathrm{H}(k) := \frac{1}{k}\sum_{i=1}^{k} V_{ik}, \\ V_{ik} := \ln X_{n-i+1:n} - \ln X_{n-k:n}, \quad 1 \leq i \leq k < n.$$

One of the interesting facts concerning the H EVI-estimators is that various asymptotically equivalent versions of $\mathrm{H}(k)$ can be derived through essentially different methods, such as the maximum likelihood method or the mean excess function approach, showing that the Hill estimator is quite natural. Details can be found in Beirlant *et al.* (2004), among others. We merely note that from a quantile point of view, and with $U(\cdot)$ the RTQF, we can write the distributional identity $X \stackrel{d}{=} U(Y)$, with Y a unit Pareto RV, i.e. an RV with a CDF $F_Y(y) = 1 - 1/y,\ y \geq 1$. For the order statistics associated with a random unit Pareto sample $(Y_1, \ldots, Y_n)$, we have the distributional identity $Y_{n-i+1:n}/Y_{n-k:n} \stackrel{d}{=} Y_{k-i+1:k}$, $1 \leq i \leq k$. Moreover, $kY_{n-k:n}/n \xrightarrow[n\to\infty]{\mathbb{P}} 1$, i.e. $Y_{n-k:n} \stackrel{\mathbb{P}}{\sim} n/k$. Consequently, and provided that $k = k_n$, $1 \leq k < n$, is an intermediate sequence of integers, i.e. if

$$k = k_n \to \infty \qquad \text{and} \qquad k_n = o(n), \quad \text{as } n \to \infty, \tag{1.4}$$

we get

$$V_{ik} \stackrel{d}{=} \xi \ln Y_{k-i+1:k} + o_{\mathbb{P}}(1) \stackrel{d}{=} \xi E_{k-i+1:k} + o_{\mathbb{P}}(1), \tag{1.5}$$

with E denoting a standard exponential RV and the $o_{\mathbb{P}}(1)$-term uniform in i, $1 \leq i \leq k$ (see Caeiro *et al.*, 2016a, among others, for further details on this uniform behaviour). The log-excesses, V_{ik}, $1 \leq i \leq k$, in (1.3), are thus approximately the k order statistics of a sample of size k from an exponential parent with mean value ξ, motivating the H EVI-estimators in (1.3).

Beyond the average, the p-moments of log-excesses, i.e.

$$M_{k,n}^{(p)} := \frac{1}{k}\sum_{i=1}^{k} \left\{ \ln X_{n-i+1:n} - \ln X_{n-k:n} \right\}^p, \quad p \geq 1, \tag{1.6}$$

introduced in Dekkers *at al.* (1989) $\left[M_{k,n}^{(1)} \equiv \mathrm{H}(k)\right]$ have also played a relevant role in the EVI-estimation, and can more generally be parameterized in $p \in \mathbb{R} \setminus \{0\}$. Note next that a simple generalization of the mean is Lehmer's mean-of-order-p (see Havil, 2003, p. 121). Given a set of positive numbers $\mathbf{a} = (a_1, \ldots, a_k)$, such a mean generalizes both the arithmetic mean ($p = 1$) and the harmonic mean ($p = 0$), being defined as

$$L_p(\mathbf{a}) := \sum_{i=1}^{k} a_i^p \Big/ \sum_{i=1}^{k} a_i^{p-1}, \quad p \in \mathbb{R}.$$

Further note that $\lim_{p\to-\infty} L_p(\mathbf{a}) = \min_{1\leq i\leq k} a_i$ and $\lim_{p\to+\infty} L_p(\mathbf{a}) = \max_{1\leq i\leq k} a_i$.

The H EVI-estimators can thus be considered as the Lehmer mean-of-order-1 of the k log-excesses $\mathbf{V} := (V_{ik},\ 1 \leq i \leq k)$, in (1.3), $k < n$. We now more generally consider the Lehmer mean-of-order-p of those statistics. From (1.5), since $\mathbb{E}(E^p) = \Gamma(p+1)$ for any real $p > -1$, with $\Gamma(\cdot)$ denoting the complete Gamma function, the law of large numbers enables us to say that

$$\frac{1}{k}\sum_{i=1}^{k} V_{ik}^p \xrightarrow[n\to\infty]{\mathbb{P}} \Gamma(p+1)\xi^p.$$

Hence the reason for the class of *Lehmer mean-of-order-p* (L_p) EVI-estimators,

$$(1.7)\quad \hat\xi^{\mathrm{L}_p}(k) \equiv \mathrm{L}_p(k) := \frac{L_p(\mathbf{V})}{p} = \frac{1}{p}\,\frac{\sum_{i=1}^{k} V_{ik}^{p}}{\sum_{i=1}^{k} V_{ik}^{p-1}} = \frac{M_{k,n}^{(p)}}{pM_{k,n}^{(p-1)}} \qquad \big[\mathrm{L}_1(k) \equiv \mathrm{H}(k)\big],$$

consistent for all $\xi > 0$ and real $p > 0$, and where $M_{k,n}^{(p)}$ is given in (1.6).

As a possible competitive class of EVI-estimators, we further refer the one recently studied in Brilhante *et al.* (2013), Gomes and Caeiro (2014) and Caeiro *et al.* (2016a), among others, based on the power mean. Given a set of non-negative numbers $\mathbf{a} = (a_1, \ldots, a_k)$, such a mean generalizes the arithmetic mean ($p = 1$), the geometric mean ($p = 0$) and the harmonic mean ($p = -1$), being defined as

$$M_p(\mathbf{a}) := \left(\frac{1}{k}\sum_{i=1}^{k} a_i^p\right)^{1/p}, \quad p \in \mathbb{R}.$$

Further note that $\lim_{p\to 0} M_p(\mathbf{a}) \equiv M_0(\mathbf{a}) = \big(\prod_{i=1}^{k} a_i\big)^{1/k}$, $\lim\limits_{p\to-\infty} M_p(\mathbf{a}) = \min\limits_{1\le i\le k} a_i$ and $\lim\limits_{p\to+\infty} M_p(\mathbf{a}) = \max\limits_{1\le i\le k} a_i$. On the basis of the fact that the Hill EVI-estimator in (1.3) is the logarithm of the *geometric mean* of

$$(1.8)\qquad U_{ik} := X_{n-i+1:n}/X_{n-k:n}, \quad 1 \le i \le k < n,$$

the consideration of the power mean, also known as Hölder's *mean-of-order-p* (MO_p), of those same statistics leads to

$$(1.9)\qquad \hat\xi^{\mathrm{H}_p}(k) \equiv \mathrm{H}_p(k) := \begin{cases} \left(1 - \left(\frac{1}{k}\sum_{i=1}^{k} U_{ik}^p\right)^{-1}\right)/p, & \text{if } p < 1/\xi,\ p \ne 0,\\[2ex] \mathrm{H}(k), & \text{if } p = 0,\end{cases}$$

the so-called MO_p EVI-estimators, almost simultaneously considered, for $p \ge 0$, in Brilhante *et al.* (2013), Paulauskas and Vaičiulis (2013) and Beran *et al.* (2014). As a measure of comparison, and just as in Gomes and Henriques-Rodrigues (2016) (see also Gomes and Henriques-Rodrigues, 2017), the *Pareto probability weighted moments* (PPWM) EVI-estimators, introduced in Caeiro and Gomes (2011), and further studied in Caeiro *et al.* (2014, 2016b) will also be considered. The PPWM EVI-estimators, quite common in the areas of climatology and hydrology, are consistent only for $\xi < 1$, depend on the statistics $\hat a_j(k) := \frac{1}{k}\sum_{i=1}^{k} \big((i-1)/(k-1)\big)^j X_{n-i+1:n}$, $j = 0, 1$, and are defined by

$$(1.10)\qquad \hat\xi^{\mathrm{PPWM}}(k) \equiv \mathrm{PPWM}(k) := 1 - \frac{\hat a_1(k)}{\hat a_0(k) - \hat a_1(k)}, \quad 1 \le k < n.$$

We also mention the possibly *reduced-bias* (RB) class of EVI-estimators in

Caeiro and Gomes (2002b) (see also, Caeiro and Gomes, 2002a, 2014),

$$(1.11)\quad \hat{\xi}^{\mathrm{CG}_{p,\delta}}(k) \equiv \mathrm{CG}_{p,\delta}(k) := \frac{\Gamma(p)}{M_{k,n}^{(p-1)}} \left(\frac{M_{k,n}^{(\delta p)}}{\Gamma(\delta p+1)}\right)^{1/\delta}, \quad \delta > 0,\ p > 0$$

$$\big[\mathrm{CG}_{1,1}(k) \equiv \mathrm{H}(k)\big].$$

For $\delta = 2$ in (1.11), we obtain a class studied in Caeiro and Gomes (2002a), which generalizes the estimator $\mathrm{CG}_{1,2}(k) = \sqrt{M_{k,n}^{(2)}/2}$, studied in Gomes *et al.* (2000), where also $\mathrm{L}_2(k) = M_{k,n}^2/\big(2M_{k,n}^{(1)}\big)$ was introduced and studied both asymptotically and for finite samples. And we can also consider the class of L_p EVI-estimators in (1.7), as a non-RB particular case of (1.11). Indeed, $\mathrm{L}_p(k) \equiv \mathrm{CG}_{p,1}(k)$.

Remark 1.1. Note that all the aforementioned EVI-estimators are scale invariant, but not location-invariant. They can however become location-invariant if we apply the *peaks over random threshold* (PORT) methodology, basing them not on the original sample, but on the excesses over a central empirical quantile and even over the minimum of the available sample whenever possible, i.e. when the underlying parent F has a finite left endpoint. For details on the topic, see, among others, Araújo Santos *et al.* (2006), where the acronym PORT was introduced, Gomes *et al.* (2008), and more recently, Gomes and Henriques-Rodrigues (2016) and Gomes *et al.* (2016).

In Section 2, after the introduction of a few technical details in the field of *extreme value theory* (EVT), we deal with the asymptotic behaviour of the L_p EVI-estimators, in (1.7). In Section 3, it is shown that at optimal k-levels and for the optimal p, the members of such a class are able to overall outperform the optimal EVI-estimators in (1.9), which on its turn had been shown in Brilhante *et al.* (2013) to have a similar behaviour comparatively with the optimal Hill EVI-estimators, for an adequate optimal p ($\neq 0$). We next compare them, asymptotically and at optimal levels, with the optimal PPWM EVI-estimators, in (1.10). Finally, in Section 4, we advance with an overall comparison of a wide number of EVI-estimators, drawing some concluding remarks.

2. ASYMPTOTIC BEHAVIOUR OF THE EVI–ESTIMATORS

After a reference, in Section 2.1, to the most common second-order framework for heavy-tailed models, we briefly refer, in Section 2.2, the asymptotic behaviour of the EVI-estimators defined in Section 1. A recent review on the topic of statistical univariate EVT can be found in Gomes and Guillou (2015). See also Beirlant *et al.* (2012) and Scarrot and MacDonald (2012).

2.1. A few technical details in the field of EVT

In the area of statistical EVT and whenever working with large values, a model F is commonly said to be heavy-tailed whenever (1.2) holds. The second-order parameter ρ (≤ 0) rules the rate of convergence in any of the first-order conditions, in (1.2), and can be defined as the non-positive parameter appearing in the limiting relation

$$\lim_{t\to\infty} \frac{\ln U(tx) - \ln U(t) - \xi \ln x}{A(t)} = \begin{cases} \big(x^\rho - 1\big)/\rho, & \text{if } \rho < 0, \\ \ln x, & \text{if } \rho = 0, \end{cases} \tag{2.1}$$

which is assumed to hold for every $x > 0$, and where $|A|$ must then be of regular variation with index ρ. This condition has been widely accepted as an appropriate condition to specify the right-tail of a Pareto-type distribution in a semi-parametric way. For technical simplicity, we often assume that we are working in Hall-Welsh class of models (Hall and Welsh, 1985), with an RTQF,

$$U(t) = C\; t^\xi \Big(1 + \xi\; \beta\; t^\rho/\rho + o(t^\rho)\Big), \quad \text{as } t \to \infty, \tag{2.2}$$

$C > 0$, $\beta \ne 0$ and $\rho < 0$. Equivalently, we can say that, with (β, ρ) the vector of second-order parameters, the general second-order condition in (2.1) holds with $A(t) = \xi\beta t^\rho,\ \rho < 0$. Further details on second-order conditions can be found in Beirlant *et al.* (2004), de Haan and Ferreira (2006) and Fraga Alves *et al.* (2007), among others.

2.2. Asymptotic behaviour of the EVI-estimators under consideration

Trivial adaptations of the results in de Haan and Peng (1998), Caeiro and Gomes (2002b), Caeiro and Gomes (2011) and Brilhante *et al.* (2013), respectively for the H, $\mathrm{CG}_{p,\delta}$, PPWM and H_p classes of EVI-estimators, enable us to state:

Theorem 2.1. *Under the validity of the first-order condition, in* (1.2), *and for intermediate sequences* $k = k_n$, *i.e. if* (1.4) *holds, the classes of* H_p, PPWM *and* $\mathrm{CG}_{p,\delta}$ EVI-*estimators, respectively defined in* (1.9), (1.10), *and* (1.11), *generally denoted by* $\hat\xi^\bullet(k)$, *are consistent for the estimation of* $\xi > 0$, *provided that we work in* $\mathcal{S}_\bullet$, *where* $\mathcal{S}_{\mathrm{H}_p} = \{(\xi, p) : \xi > 0,\ p < 1/\xi\}$, $\mathcal{S}_{\mathrm{PPWM}} = \{\xi : 0 < \xi < 1\}$ *and* $\mathcal{S}_{\mathrm{CG}_{p,\delta}} = \{(\xi, p, \delta) : \xi > 0,\ p > 0,\ \delta > -1/p\}$.

Assume further that (2.1) *holds. Then, for* $\xi > 0$, *adequate regions of the spaces of parameters and with* $\mathcal{N}\left(\mu, \sigma^2\right)$ *standing for a normal* RV *with mean value* μ *and variance* σ^2,

$$\sqrt{k}\left(\hat\xi^\bullet(k) - \xi\right) \underset{n\to\infty}{\overset{d}{\longrightarrow}} \mathcal{N}\left(\lambda_A b_\bullet, \sigma_\bullet^2\right) \quad \textit{if} \quad \sqrt{k}\; A(n/k) \underset{n\to\infty}{\longrightarrow} \lambda_A, \ \textit{finite}. \tag{2.3}$$

Moreover

$$(2.4)\qquad b_{\mathrm{H}_p} = \frac{1-p\xi}{1-p\xi-\rho}, \quad \sigma^2_{\mathrm{H}_p} = \frac{\xi^2(1-p\xi)^2}{1-2p\xi} \quad \textit{if} \quad p<1/(2\xi),$$

$$b_{\mathrm{PPWM}} = \frac{(1-\xi)(2-\xi)}{(1-\xi-\rho)(2-\xi-\rho)}, \qquad \sigma^2_{\mathrm{PPWM}} = \frac{\xi^2(1-\xi)(2-\xi)^2}{(1-2\xi)(3-2\xi)}, \quad \textit{if} \quad \xi<1/2,$$

and

$$(2.5)\quad b_{\mathrm{CG}_{p,\delta}} = \frac{(1-\rho)^{-\delta p} - \delta(1-\rho)^{-p+1} + \delta - 1}{\delta\rho},$$

$$\sigma^2_{\mathrm{CG}_{p,\delta}} = \frac{\xi^2}{\delta^2}\left\{\frac{2\Gamma(2\delta p)}{\delta p\Gamma^2(\delta p)} + \frac{\delta^2\Gamma(2p-1)}{\Gamma^2(p)} - \frac{2\Gamma((\delta+1)p)}{p\Gamma(p)\Gamma(\delta p)} - (\delta-1)^2\right\}, \quad \textit{if} \quad p>1/2,\ \delta>0.$$

For the particular case $\delta = 1$, in (1.11), i.e. for the L_p EVI-estimators in (1.7), we can state:

Corollary 2.1. *Under the validity of the initial first-order conditions in* **Theorem 2.1**, *the class of* L_p *EVI-estimators, in* (1.7), *is consistent for the estimation of* ξ, *provided that we work in* $\mathcal{S}_{\mathrm{L}_p} = \{(\xi,p) : \xi>0, p>0\}$. *Under the second-order conditions of* **Theorem 2.1**, (2.3) *holds, with*

$$(2.6)\qquad b_{\mathrm{L}_p} = \frac{1}{(1-\rho)^p} \quad \textit{and} \quad \sigma^2_{\mathrm{L}_p} = \frac{\xi^2\ \Gamma(2p-1)}{\Gamma^2(p)} \quad \textit{if} \quad p>1/2.$$

More specifically, and for all $\rho \le 0$, *one can write the asymptotic distributional representation*

$$(2.7)\qquad \mathrm{L}_p(k) \overset{d}{=} \xi + \frac{\sigma_{\mathrm{L}_p}\, Z_k^{(p)}}{\sqrt{k}} + b_{\mathrm{L}_p}\, A(n/k) + o_{\mathbb{P}}(A(n/k)),$$

with $(b_{\mathrm{L}_p}, \sigma^2_{\mathrm{L}_p})$ *given in* (2.6), *and where* $Z_k^{(p)}$ *is an asymptotically standard normal* RV.

Remark 2.1. Note that regarding the L_p EVI-estimators, in (1.7), **Corollary 2.1** is a particular case of **Theorem 1** in Caeiro and Gomes (2002b), but generalizing now consistency for $p>0$ and asymptotic normality for $p>1/2$ rather than $p \ge 1$. Further note that for $\delta=1$ there is a full agreement between (2.6) and (2.5), the result provided in **Theorem 1** of Caeiro and Gomes (2002b). A detailed proof of **Corollary 2.1** can be found in Penalva *et al.* (2016).

Remark 2.2. Further note that for the MO_p EVI-estimators, denoted by H_p and defined in (1.9), a distributional representation of the type of the one in (2.7) holds for $p < 1/(2\xi)$, with $\big(b_{\mathrm{L}_p}, \sigma^2_{\mathrm{L}_p}\big)$ replaced by $\big(b_{\mathrm{H}_p}, \sigma^2_{\mathrm{H}_p}\big)$, given in (2.4).

For any $\xi > 0$, the asymptotic variance $\sigma^2_{\mathrm{L}_p}(\xi)$, in (2.6), has a minimum at $p = 1$. In **Figure 1** (*left*), we present the normalized standard deviation, $\sigma_{\mathrm{L}_p}(\xi)/\xi$, independent of ξ, as a function of p. On another side, the asymptotic bias ruler, $b_{\mathrm{L}_p}(\rho)$, also in (2.6), is independent of ξ and always decreasing in p. Such a performance is shown in **Figure 1** (*right*).

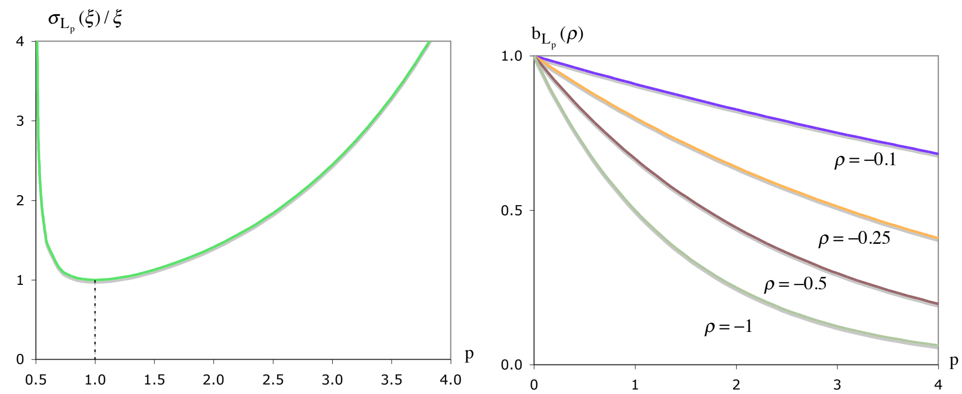

Figure 1: Graph of $\sigma_{\mathrm{L}_p}(\xi)/\xi$, as a function of $p > 1/2$ (*left*) and of the asymptotic bias ruler $b_{\mathrm{L}_p}(\rho)$, for $\rho = -0.1, -0.25, -0.5$ and -1, as a function of $p \geq 0$

The aforementioned results claim for an asymptotic study, at optimal (k, p), of the class of EVI-estimators in (1.7), a topic to be dealt with in Section 3.

3. ASYMPTOTIC COMPARISON AT OPTIMAL LEVELS

We next proceed to the comparison of the aforementioned non-RB EVI-estimators, generally denoted by $\hat{\xi}^{\bullet}(k)$, at their optimal levels. This is again done in a way similar to the one used in several articles, among which we refer Dekkers and de Haan (1993), de Haan and Peng (1998), Gomes and Martins (2001), Gomes *et al.* (2005, 2007, 2013, 2015), Gomes and Neves (2008), Gomes and Henriques-Rodrigues (2010, 2016), and Brilhante *et al.* (2013), among others. Let us assume that for any intermediate sequence of integers $k = k_n$, (2.3) holds. We write $\mathrm{Bias}_\infty\big(\hat{\xi}^{\bullet}(k)\big) := b_\bullet\, A(n/k)$ and $\mathrm{Var}_\infty\big(\hat{\xi}^{\bullet}(k)\big) := \sigma^2_\bullet/k$. The so-called *asymptotic mean square error* (AMSE) is then given by $\mathrm{AMSE}\big(\hat{\xi}^{\bullet}(k)\big) := \sigma^2_\bullet/k + b^2_\bullet\, A^2(n/k)$. Regular variation theory enabled Dekkers and de Haan (1993) to show that, whenever $b_\bullet \neq 0$, there exists a function $\varphi(n) = \varphi(n, \xi, \rho)$, such that

$$\lim_{n\to\infty} \varphi(n)\ \mathrm{AMSE}\big(\hat{\xi}^{\bullet}_0\big) = \big(\sigma^2_\bullet\big)^{-\frac{2\rho}{1-2\rho}}\, \big(b^2_\bullet\big)^{\frac{1}{1-2\rho}} =: \mathrm{LMSE}\left(\hat{\xi}^{\bullet}_0\right), \tag{3.1}$$

where $\hat{\xi}_0^\bullet := \hat{\xi}^\bullet(k_{0|\bullet}(n))$ and $k_{0|\bullet}(n) := \arg\min_k \mathrm{MSE}\big(\hat{\xi}^\bullet(k)\big)$. Moreover, if we slightly restrict the second-order condition in (2.1), assuming (2.2), we can write

$$k_{0|\bullet}(n) = \arg\min_k \mathrm{MSE}\big(\hat{\xi}^\bullet(k)\big) = \left(\frac{\sigma_\bullet^2\; n^{-2\rho}}{b_\bullet^2 \xi^2 \beta^2(-2\rho)}\right)^{1/(1-2\rho)} (1+o(1)).$$

We consider the following:

Definition 3.1. Given two biased estimators $\hat{\xi}^{(1)}(k)$ and $\hat{\xi}^{(2)}(k)$, for which (2.3) holds, with constants (σ_1, b_1) and (σ_2, b_2), $b_1, b_2 \neq 0$, respectively, both computed at their optimal levels, the *asymptotic root efficiency* (AREFF) of $\hat{\xi}_0^{(1)}$ relatively to $\hat{\xi}_0^{(2)}$ is

$$\text{(3.2)}\quad \mathrm{AREFF}_{1|2} \equiv \mathrm{AREFF}_{\hat{\xi}_0^{(1)}|\hat{\xi}_0^{(2)}} := \sqrt{\mathrm{LMSE}\big(\hat{\xi}_0^{(2)}\big)/\mathrm{LMSE}\big(\hat{\xi}_0^{(1)}\big)} = \left(\left(\frac{\sigma_2}{\sigma_1}\right)^{-2\rho}\left|\frac{b_2}{b_1}\right|\right)^{\frac{1}{1-2\rho}},$$

with LMSE defined in (3.1).

Remark 3.1. Note that the AREFF-indicator, in (3.2), has been conceived so that the highest the AREFF indicator is, the better is the estimator identified with the superscript (1).

The non-RB L_p, H_p, and PPWM EVI-estimators, respectively given in (1.7), (1.9) and (1.10), will be crucially included in the asymptotic comparison in Section 3.1.

3.1. Asymptotic comparison of EVI-estimators at optimal levels

Let us now turn back to the L_p EVI-estimators in (1.7), at optimal k-levels in the sense of minimum RMSE. We have

$$\mathrm{LMSE}(\mathrm{L}_{0|p}) = \Big(\xi^2\Gamma(2p-1)/\Gamma^2(p)\Big)^{-\frac{2\rho}{1-2\rho}}\Big((1-\rho)^{-2p}\Big)^{\frac{1}{1-2\rho}}$$

and

$$\text{(3.3)}\quad \mathrm{AREFF}_{\mathrm{L}}(p) \equiv \mathrm{AREFF}_{\mathrm{L}_{0|p}|\mathrm{L}_{0|1}} = \left(\Big(\Gamma(p)/\sqrt{\Gamma(2p-1)}\Big)^{-2\rho}(1-\rho)^{p-1}\right)^{\frac{1}{1-2\rho}}.$$

Remark 3.2. In Gomes *et al.* (2000) was shown that the AREFF of the optimal $\mathrm{L}_2(k)$ comparatively to the optimal $\mathrm{L}_1(k)$ is given by $[2^\rho(1-\rho)]^{1/(1-2\rho)}$, in agreement with (3.3). As noticed in the aforementioned article, $\mathrm{AREFF_L}(2) > 1 \iff -1 < \rho < 0$.

To measure the performance of $\mathrm{H}_{0|p}$, with H_p the MO_p EVI-estimator in (1.9), Brilhante *et al.* (2013) computed a similar AREFF-indicator, given by

$$\text{(3.4)}\quad \mathrm{AREFF_H}(p) \equiv \mathrm{AREFF}_{\mathrm{H}_{0|p}|\mathrm{H}_{0|0}} = \left(\left(\frac{\sqrt{1-2p\xi}}{1-p\xi} \right)^{-2\rho} \left| \frac{1-p\xi-\rho}{(1-\rho)(1-p\xi)} \right| \right)^{\frac{1}{1-2\rho}},$$

reparameterized in $(\rho,\ a = p\xi < 1/2)$, and denoted by $\mathrm{AREFF}^*_{a|0}$. In **Figure 2**, we picture $\mathrm{AREFF_L}(p)$ in (3.3) (*top*) and $\mathrm{AREFF}^*_{a|0}$ (*bottom*).

The gain in efficiency is not terribly high, but, at optimal levels, there is a wide region of the (p,ρ)-plane where the new class of L_p EVI-estimators performs better than the Hill EVI-estimators, with efficiencies slightly higher than the ones associated with the comparison of H_p and the Hill, in the (a,ρ)-plane. This result together with the fact that as far as we know, the EVI-estimators in (1.9) computed at the optimal (k,p) in the sense of maximal $\mathrm{AREFF_H}(p)$, with $\mathrm{AREFF_H}(p)$ given in (3.4), i.e. computed at $p_{\mathrm{M|H}} \equiv p_{\mathrm{M|H}}(\rho) := \arg\max_p \mathrm{AREFF_H}(p)$, explicitly given by

$$\text{(3.5)}\qquad p_{\mathrm{M|H}} = \varphi_\rho/\xi, \quad \text{with} \quad \varphi_\rho := 1-\rho/2-\sqrt{\rho^2-4\rho+2}\Big/2,$$

$b_{p_{\mathrm{M|H}}} \neq 0$, is, as expected, a non-RB EVI-estimator which is able to beat the Hill EVI-estimator in the whole (ξ,ρ)-plane, immediately leads us to think on what happens for the optimal value of p associated with the L_p EVI-estimation. Contrarily to the explicit expression for $p_{\mathrm{M|H}}$, in (3.5), the value of $p_{\mathrm{M|L}} = p_{\mathrm{M|L}}(\rho) := \arg\max_p \mathrm{AREFF_L}(p)$, with $\mathrm{AREFF_L}(p)$ given in (3.3), is an implicit function of ρ, easy to evaluate numerically. Some of those values are presented in Table 1.

Table 1: Values of $p_{\mathrm{M|L}} = p_{\mathrm{M|L}}(\rho) := \arg\max_p \mathrm{AREFF_L}(p)$ for a few values of $|\rho|$

$\lvert\rho\rvert$	0^+	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.8	1.0	1.5	2	$+\infty$
$p_{\mathrm{M\vert L}}$	1	1.98	1.86	1.75	1.67	1.61	1.56	1.52	1.45	1.40	1.32	1.27	1

ρ \ p	1.00	1.05	1.10	1.15	1.20	1.25	1.30	1.35	1.40	1.45	1.50	1.55	1.60	1.65	1.70	1.75	1.80	1.85	1.90	1.95	2.00	2.05	2.10	2.15	2.20	2.25
0.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.10	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02
-0.20	1.00	1.01	1.01	1.02	1.02	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03
-0.30	1.00	1.01	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.03	1.03
-0.40	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02
-0.50	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.02
-0.60	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.02	1.02	1.02	1.02	1.01	1.01
-0.70	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00
-0.80	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99
-0.90	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98
-1.00	1.00	1.01	1.02	1.02	1.03	1.03	1.04	1.04	1.04	1.04	1.04	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.01	1.00	0.99	0.99	0.98	0.98	0.97
-1.10	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.04	1.04	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.98	0.97	0.96
-1.20	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.96
-1.30	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.95	0.95
-1.40	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.99	0.98	0.98	0.97	0.96	0.95	0.95	0.94
-1.50	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.98	0.98	0.97	0.96	0.96	0.95	0.94	0.93
-1.60	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.99	0.98	0.97	0.96	0.96	0.95	0.94	0.93	0.92
-1.70	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.99	0.98	0.97	0.97	0.96	0.95	0.94	0.93	0.93	0.92
-1.80	1.00	1.01	1.02	1.02	1.03	1.03	1.03	1.03	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.98	0.98	0.97	0.96	0.95	0.95	0.94	0.93	0.92	0.91
-1.90	1.00	1.01	1.02	1.02	1.02	1.03	1.03	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.99	0.98	0.97	0.96	0.96	0.95	0.94	0.93	0.92	0.91	0.91
-2.00	1.00	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.98	0.98	0.97	0.96	0.95	0.94	0.94	0.93	0.92	0.91	0.90

$AREFF_L(p) < 1$ $AREFF_L(p) = 1$ $1 < AREFF_L(p) \leq 1.025$ $AREFF_L(p) > 1.025$

ρ \ a	0.00	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	0.38	0.40	0.42	0.44	0.46	0.48	0.49
0.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
-0.10	1.00	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.01	1.01	1.01	1.01	1.00	0.99	0.98	0.96	0.91	0.87
-0.20	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.98	0.95	0.91	0.84	0.77
-0.30	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.00	0.99	0.98	0.96	0.93	0.88	0.79	0.70
-0.40	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.97	0.94	0.90	0.85	0.74	0.65
-0.50	1.00	1.00	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.00	0.99	0.98	0.95	0.93	0.88	0.82	0.71	0.61
-0.60	1.00	1.00	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.98	0.97	0.94	0.91	0.87	0.80	0.68	0.57
-0.70	1.00	1.00	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.98	0.96	0.93	0.90	0.85	0.78	0.65	0.54
-0.80	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.00	1.00	0.98	0.97	0.95	0.92	0.89	0.83	0.76	0.63	0.52
-0.90	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	0.99	0.98	0.96	0.94	0.91	0.87	0.82	0.74	0.62	0.50
-1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.00	1.00	0.99	0.97	0.96	0.93	0.90	0.87	0.81	0.73	0.60	0.48
-1.10	1.00	1.00	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.01	1.01	1.01	1.00	0.99	0.98	0.97	0.95	0.93	0.90	0.86	0.80	0.72	0.59	0.47
-1.20	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02	1.01	1.01	1.01	1.01	1.00	0.99	0.98	0.96	0.95	0.92	0.89	0.85	0.79	0.71	0.57	0.46
-1.30	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.96	0.94	0.92	0.88	0.84	0.78	0.70	0.56	0.45
-1.40	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	0.99	0.98	0.97	0.96	0.94	0.91	0.88	0.83	0.78	0.69	0.55	0.44
-1.50	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.95	0.93	0.91	0.87	0.83	0.77	0.68	0.55	0.43
-1.60	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.95	0.93	0.90	0.87	0.82	0.76	0.68	0.54	0.42
-1.70	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	0.99	0.99	0.98	0.96	0.95	0.92	0.90	0.86	0.82	0.76	0.67	0.53	0.41
-1.80	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.96	0.94	0.92	0.89	0.86	0.81	0.75	0.66	0.52	0.41
-1.90	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.96	0.94	0.92	0.89	0.85	0.81	0.75	0.66	0.52	0.40
-2.00	1.00	1.00	1.00	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.00	1.00	0.99	0.98	0.97	0.96	0.94	0.92	0.89	0.85	0.80	0.74	0.65	0.51	0.40

$AREFF^*_{a|0} < 1$ $AREFF^*_{a|0} = 1$ $1 < AREFF^*_{a|0} \leq 1.025$

Figure 2: $\mathrm{AREFF_L}(p)$, in (3.3) (*top*) and $\mathrm{AREFF}^*_{a|0}$ (*bottom*)

In **Figure 3**, we picture the indicator $\text{AREFF}_{\text{L}}(p)$, as a function of p for a few values of ρ.

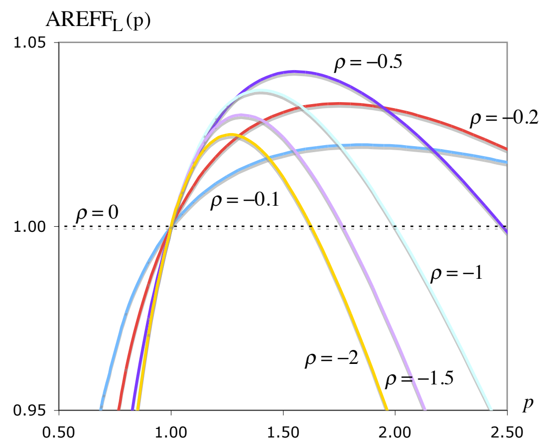

Figure 3: $\text{AREFF}_{\text{L}}(p)$, as a function of p, for $|\rho| = 0, 0.1, 0.2, 0.5(0.5)2$

Indeed, just as $\text{AREFF}_{\text{H}}(p_{_{\text{M}|\text{H}}}) > 1$, for any $\rho < 0$ and $\xi > 0$, also $\text{AREFF}_{\text{L}}(p_{_{\text{M}|\text{L}}}) > 1$, for any $\rho < 0$ and $\xi > 0$. Moreover,

$$\text{AREFF}_{\text{L}}(p_{_{\text{M}|\text{L}}}) > \text{AREFF}_{\text{H}}(p_{_{\text{M}|\text{H}}}),$$

as illustrated in **Figure 4**.

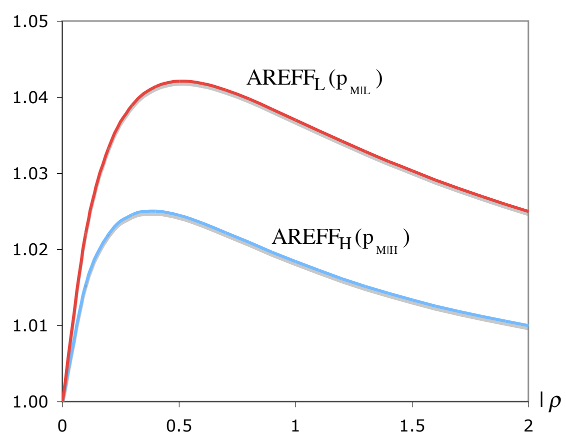

Figure 4: $\text{AREFF}_{\text{L}}(p_{_{\text{M}|\text{L}}})$ and $\text{AREFF}_{\text{H}}(p_{_{\text{M}|\text{H}}})$ as a function of $|\rho| = 0(0.1)2$

Just as done in Gomes and Henriques-Rodrigues (2016), and due to the competitive behaviour of the PPWM EVI-estimators, we still compare the L_p with the PPWM EVI-estimators, in (1.10), again at optimal levels. Whereas the gain in efficiency of the PPWM comparatively to the optimal H_p EVI-estimator happens in a wide region of the (ξ, ρ)-plane, $\text{L}^* := \text{L}_{p_{_{\text{M}|\text{L}}}}$ beats the optimal PPWM

EVI-estimator (now denoted P, for sake of simplicity) in a wider region of the (ξ, ρ)-plane, as can be seen in **Figure 5** (*bottom*). Indeed, in **Figure 5** (*top*), we reproduce the Figure in Gomes and Henriques-Rodrigues (2017), related to the comparative behaviour between $\mathrm{H}^* := \mathrm{H}_{p_{\mathrm{M|H}}}$ and the optimal PPWM EVI-estimator.

ρ \ ξ	0.05	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.21	0.23	0.25	0.27	0.29	0.31	0.33	0.35	0.37	0.39	0.41	0.43	0.45
0.00	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
-0.05	P	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*
-0.07	P	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*
-0.09	P	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*
-0.11	P	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*
-0.13	P	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*
-0.15	P	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*
-0.17	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*
-0.19	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*
-0.21	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*
-0.23	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*
-0.25	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*
-0.27	P	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*
-0.29	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.31	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.33	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.35	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.37	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.39	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.41	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.43	P	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.45	P	P	P	P	P	P	P	P	P	P	P	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*

ρ \ ξ	0.05	0.07	0.09	0.11	0.13	0.15	0.17	0.19	0.21	0.23	0.25	0.27	0.29	0.31	0.33	0.35	0.37	0.39	0.41	0.43	0.45
0.00	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
-0.05	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.07	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.09	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.11	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.13	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.15	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.17	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.19	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.21	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.23	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.25	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.27	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.29	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.31	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.33	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.35	L*	L*	L*	P	P	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.37	L*	L*	L*	P	P	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.39	L*	L*	P	P	P	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.41	L*	L*	P	P	P	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.43	L*	P	P	P	P	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.45	L*	P	P	P	P	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*

Figure 5: Best EVI-estimator asymptotically and at optimal levels for a choice between H* and PPWM (*top*) and between L* and PPWM (*bottom*)

So far, asymptotically and for a heavy right-tail, the class of Lehmer's EVI-estimators, in (1.7), seems indeed to be the most competitive class of non-RB EVI-estimators in the literature. Note however that further classes of generalized means, among which we mention the ones studied in Paulauskas and Vaičiulis (2017), may possibly provide even more astonishing results.

4. An asymptotic comparison with other EVI-estimators at optimal levels

As mentioned above, the optimal MO_p EVI-estimator (H^*), associated with a value $p_{_{\mathrm{M|H}}} \neq 0$, can beat the optimal Hill EVI-estimator in the whole (ξ,ρ)-plane. But it is now beaten by the optimal Lehmer EVI-estimator (L^*), also in the whole (ξ,ρ), an atypical behaviour among other classical EVI-estimators. We thus consider now sensible to compare H^* and L^* with the most common EVI-estimators in the literature, non generally RB, but possibly RB in some regions of the (ξ,ρ)-plane.

We shall take into account the *moment* (M) EVI-estimators, studied in Dekkers *et al.* (1989), based on $\big(M_{k,n}^{(1)}, M_{k,n}^{(2)}\big)$, with $M_{k,n}^{(p)}$ defined in (1.6). They are consistent for all $\xi \in \mathbb{R}$, being given by

$$\hat\xi^{\mathrm{M}}(k) \equiv \mathrm{M}(k) := M_{k,n}^{(1)} + \tfrac{1}{2}\Big\{1 - \Big(M_{k,n}^{(2)}/\big(M_{k,n}^{(1)}\big)^2 - 1\Big)^{-1}\Big\}. \tag{4.1}$$

We additionally consider the *generalized Hill* (GH) EVI-estimators (Beirlant *et al.*, 1996), based on the Hill estimators in (1.3) and with the functional form

$$\hat\xi^{\mathrm{GH}}(k) \equiv \mathrm{GH}(k) := \hat\xi^{\mathrm{H}}(k) + \frac{1}{k}\sum_{i=1}^{k}\Big\{\ln \hat\xi^{\mathrm{H}}(i) - \ln \hat\xi^{\mathrm{H}}(k)\Big\}, \tag{4.2}$$

further studied in Beirlant *et al.* (2005). Just as in de Haan and Ferreira (2006), we also consider, for $\xi < 1$, the *generalized Pareto* (GP) PWM (GPPWM) EVI-estimators, based on the sample of exceedances over the high random level $X_{n-k:n}$ and defined by

$$\hat\xi^{\mathrm{GPPWM}}(k) \equiv \mathrm{GPPWM}(k) := 1 - \frac{2\hat a_1^\star(k)}{\hat a_0^\star(k) - 2\hat a_1^\star(k)}, \tag{4.3}$$

with $k = 1, \ldots, n-1$, and

$$\hat a_s^\star(k) := \frac{1}{k}\sum_{i=1}^{k}\left(\frac{i-1}{k-1}\right)^s (X_{n-i+1:n} - X_{n-k:n}), \quad s = 0, 1.$$

Finally, with $U_{ik},\ 1 \le i \le k$, given in (1.8), and the notation

$$L_{k,n}^{(j)} := \frac{1}{k}\sum_{i=1}^{k}\Big(1 - U_{ik}^{-1}\Big)^j, \quad j \ge 1,$$

we further consider the *mixed moment* (MM) EVI-estimators (Fraga Alves *et al.*, 2009), defined by

$$(4.4)\quad \hat{\xi}^{\mathrm{MM}}(k) \equiv \mathrm{MM}(k) := \frac{\hat{\varphi}_{k,n}-1}{1+2\min\left(\hat{\varphi}_{k,n}-1,0\right)}, \qquad \text{with} \quad \hat{\varphi}_{k,n} := \frac{M_{k,n}^{(1)}-L_{k,n}^{(1)}}{\left(L_{k,n}^{(1)}\right)^2}.$$

The estimators in (4.3) are consistent only for $0<\xi<1$. The estimators in (4.1), (4.2) and (4.4) are consistent for any $\xi \in \mathbb{R}$, but will be here considered only for $\xi>0$.

Remark 4.1. Note that the MM EVI-estimators, in (4.4), are, for a wide class of models with $\xi>0$, very close to the implicit ML EVI-estimators, based on the excesses $W_{ik} := X_{n-i+1:n} - X_{n-k:n}$, $1 \le i \le k < n$ (see Fraga Alves *et al.*, 2009, for details on the topic). A comprehensive study of the asymptotic properties of the aforementioned ML EVI-estimators has been undertaken in Drees *et al.* (2004).

Remark 4.2. Further note that all the aforementioned EVI-estimators in this section are scale invariant. The GPPWM and the ML EVI-estimators are also location invariant, and can be regarded as classes of PORT EVI-estimators. We can further consider PORT-M, GH and MM EVI-estimators.

Under the validity of the second-order condition in (2.1), and for intermediate $k=k_n$, (2.3) holds, with

$$b_{\mathrm{M}} = b_{\mathrm{GH}} = \frac{\xi-\xi\rho+\rho}{\xi(1-\rho)^2}, \quad \sigma^2_{\mathrm{M}} = \sigma^2_{\mathrm{GH}} = 1+\xi^2,$$

$$b_{\mathrm{MM}} = b_{\mathrm{ML}} = \frac{(1+\xi)(\xi+\rho)}{\xi(1-\rho)(1+\xi-\rho)}, \quad \sigma^2_{\mathrm{MM}} = \sigma^2_{\mathrm{ML}} = (1+\xi)^2,$$

and for $\xi<1/2$,

$$b_{\mathrm{GPPWM}} = \frac{(\xi+\rho)\, b_{\mathrm{PPWM}}}{\xi} \qquad \text{and} \qquad \sigma^2_{\mathrm{GPPWM}} = \frac{(1-\xi+2\xi^2)(1-\xi)(2-\xi)^2}{(1-2\xi)(3-2\xi)}.$$

As happened before with the optimal MO_p EVI-estimator, the optimal Lehmer EVI-estimator can be beaten by the optimal M (and GH) EVI-estimator in a region close to $\xi=-\rho/(1-\rho)$, where $b_{\mathrm{M}}=b_{\mathrm{GH}}=0$. The optimal MM EVI-estimator in (4.4), asymptotically equivalent to the optimal ML-estimator, unless $\xi+\rho=0$ and $(\xi,\rho)\neq(0,0)$, outperforms the M EVI-estimator at optimal levels, in a region around $\xi+\rho=0$, and can even outperform the optimal Lehmer EVI-estimator. The GPPWM EVI-estimator, in (4.3), is RB for $\xi+\rho=0$, and can beat the MM EVI-estimator in a short region of the (ξ,ρ)-plane, as can be seen in **Figure 6**, where we exhibit the comparative behaviour of all 'classical' EVI-estimators under consideration, including both the L* and the H* classes (**Figure 6**, *bottom*), after including only the H* class (**Figure 6**, *top*), as done in Brilhante *et al.* (2013). The GPPWM and PPWM EVI-estimators are respectively denoted by GP and P. The PPWM, despite of non-RB, can beat even the optimal Lehmer for a few values of ξ around 0.1, as detected before (see also **Figure 5**, *bottom*).

ρ \ ξ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
0.00	MM	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*	H*
-0.10	H*	ML	MM	GP	GP	MM	MM	MM	MM	M	M	M	M	M	M	M	M	M	M	M	M
-0.20	H*	P	ML	GP	MM	MM	MM	MM	MM	MM	MM	M	M	M	M	M	M	M	M	M	M
-0.30	H*	P	P	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	M	M	M	M	M	M	M	M
-0.40	H*	P	P	M	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	M	M	M	M	M	M
-0.50	H*	P	P	M	M	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	M	M	M	M
-0.60	H*	P	P	H*	M	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	M	M
-0.70	H*	P	P	H*	M	M	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM
-0.80	H*	P	P	H*	M	M	M	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM
-0.90	H*	P	P	H*	H*	M	M	M	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM
-1.00	H*	P	P	H*	H*	M	M	M	MM	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM
-1.10	H*	P	H*	H*	H*	M	M	M	H*	MM	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM
-1.20	H*	P	H*	H*	H*	M	M	M	H*	H*	MM	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM
-1.30	H*	P	H*	H*	H*	H*	M	M	H*	H*	H*	MM	MM	ML	MM	MM	MM	MM	MM	MM	MM
-1.40	H*	P	H*	H*	H*	H*	M	M	H*	H*	H*	H*	MM	MM	ML	MM	MM	MM	MM	MM	MM
-1.50	H*	P	H*	H*	H*	H*	M	M	H*	H*	H*	H*	MM	MM	MM	ML	MM	MM	MM	MM	MM
-1.60	H*	P	H*	H*	H*	H*	M	M	H*	H*	H*	H*	H*	MM	MM	MM	ML	MM	MM	MM	MM
-1.70	H*	P	H*	H*	H*	H*	M	M	H*	H*	H*	H*	H*	H*	MM	MM	MM	ML	MM	MM	MM
-1.80	H*	P	H*	H*	H*	H*	M	M	H*	H*	H*	H*	H*	H*	H*	MM	MM	MM	ML	MM	MM
-1.90	H*	P	H*	H*	H*	H*	H*	M	H*	H*	H*	H*	H*	H*	H*	H*	MM	MM	MM	ML	MM
-2.00	H*	P	H*	H*	H*	H*	H*	M	H*	H*	H*	H*	H*	H*	H*	H*	MM	MM	MM	MM	ML

ρ \ ξ	0.00	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90	1.00	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.80	1.90	2.00
0.00	MM	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.10	L*	ML	MM	GP	GP	MM	MM	MM	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.20	L*	L*	ML	GP	MM	MM	MM	MM	MM	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.30	L*	L*	L*	ML	MM	MM	MM	MM	MM	MM	MM	L*	L*	L*	L*	L*	L*	L*	L*	L*	L*
-0.40	L*	P	L*	M	ML	MM	MM	MM	MM	MM	MM	MM	MM	L*	L*	L*	L*	L*	L*	L*	L*
-0.50	L*	P	L*	M	M	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	L*	L*	L*	L*	L*	L*
-0.60	L*	P	L*	L*	M	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	L*	L*	L*	L*
-0.70	L*	P	L*	L*	M	M	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	L*	L*	L*
-0.80	L*	P	L*	L*	M	M	M	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	L*	L*
-0.90	L*	L*	L*	L*	L*	M	M	M	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM
-1.00	L*	L*	L*	L*	L*	M	M	M	MM	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM	MM
-1.10	L*	L*	L*	L*	L*	M	M	M	L*	MM	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM	MM
-1.20	L*	L*	L*	L*	L*	M	M	M	L*	L*	MM	MM	ML	MM	MM	MM	MM	MM	MM	MM	MM
-1.30	L*	L*	L*	L*	L*	L*	M	M	L*	L*	L*	MM	MM	ML	MM	MM	MM	MM	MM	MM	MM
-1.40	L*	L*	L*	L*	L*	L*	M	M	L*	L*	L*	L*	MM	MM	ML	MM	MM	MM	MM	MM	MM
-1.50	L*	L*	L*	L*	L*	L*	M	M	L*	L*	L*	L*	MM	MM	MM	ML	MM	MM	MM	MM	MM
-1.60	L*	L*	L*	L*	L*	L*	M	M	L*	L*	L*	L*	L*	MM	MM	MM	ML	MM	MM	MM	MM
-1.70	L*	L*	L*	L*	L*	L*	M	M	L*	L*	L*	L*	L*	L*	MM	MM	MM	ML	MM	MM	MM
-1.80	L*	L*	L*	L*	L*	L*	M	M	L*	L*	L*	L*	L*	L*	L*	MM	MM	MM	ML	MM	MM
-1.90	L*	L*	L*	L*	L*	L*	L*	M	L*	L*	L*	L*	L*	L*	L*	L*	MM	MM	MM	ML	MM
-2.00	L*	L*	L*	L*	L*	L*	L*	M	L*	L*	L*	L*	L*	L*	L*	L*	MM	MM	MM	MM	ML

Figure 6: Comparative overall behaviour of the EVI-estimators under study, considering only the optimal H_p, denoted H* (*top*) and including both H* and the optimal L_p, denoted L* (*bottom*)

Remark 4.3. As already mentioned in Brilhante *et al.* (2013), note that in the region $\xi + \rho \neq 0$ and $\xi \neq -\rho/(1-\rho)$, where a further study under the third-order framework is needed, all RB EVI-estimators, like the corrected-Hill EVI-estimators in Caeiro *et al.* (2005), overpass at optimal levels all classical and non-RB EVI-estimators available in the literature. They were thus not included in **Figure 6**, so that we can see the comparative behaviour of the non-RB EVI-estimators. A similar comment applies to the optimal $\mathrm{CG}_{p,\delta}$ EVI-estimators, in (1.11).

Remark 4.4. As expected, none of the estimators can always dominate the alternatives, but the L_p EVI-estimators have a quite interesting performance, being unexpectedly able to beat the $\mathrm{MO}_p \equiv \mathrm{H}_p$ EVI-estimators at optimal levels in the whole (ξ, ρ)-plane.

Remark 4.5. For a final adaptive EVI-estimation, i.e. for the choice of (k, p) in (1.7), a double-bootstrap algorithm, of the type of **Algorithm 4.1** in Brilhante *et al.* (2013), now based on the asymptotic behaviour in (2.7), can be used. Such an algorithm relies on the minimization of a bootstrap estimate of the AMSE. Also, the slight modification of the semi-parametric bootstrap method in Caers *et al.* (1999), provided in the **Algorithm 4.3** of Caeiro and Gomes (2015) is expected to provide an adequate estimation of the bootstrap MSE. Alternatively, one can use any of the available methods based on sample-path stability (see also Caeiro and Gomes, 2015, among others).

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REFERENCES

[1] Araújo Santos, P., Fraga Alves, M.I. and Gomes, M.I. (2006). Peaks over random threshold methodology for tail index and quantile estimation. *Revstat* **4**, 3, 227–247.

[2] Beirlant, J., Vynckier, P. and Teugels, J. (1996). Excess functions and estimation of the extreme-value index. *Bernoulli* **2**, 293–318.

[3] Beirlant, J., Goegebeur, Y., Segers, J. and Teugels, J. (2004). *Statistics of Extremes. Theory and Applications.* Wiley.

[4] Beirlant, J., Dierckx, G. and Guillou, A. (2005). Estimation of the extreme-value index and generalized quantile plots. *Bernoulli* **11**, 6, 949–970.

[5] Beirlant, J., Caeiro, F. and Gomes, M.I. (2012). An overview and open researh topics in statistics of univariate extremes. *Revstat* **10**, 1, 1–31.

[6] Beran, J., Schell, D. and Stehlík, M. (2014). The harmonic moment tail index estimator: asymptotic distribution and robustness. *Ann. Inst. Statist. Math.* **66**, 193–220.

[7] Bingham, N., Goldie, C.M. and Teugels, J.L. (1987). *Regular Variation.* Cambridge Univ. Press, Cambridge.

[8] Brilhante, M.F., Gomes, M.I. and Pestana, D. (2013). A simple generalization of the Hill estimator. *Computat. Statistics and Data Analysis* **57**, 1, 518–535.

[9] Caeiro, F. and Gomes, M.I. (2002a). A class of 'asymptotically unbiased' semi-parametric estimators of the tail index. *Test* **11**, 2, 345–364.

[10] Caeiro, F. and Gomes, M.I. (2002b). Bias reduction in the estimation of parameters of rare events. *Theory of Stochastic Processes* **8**, 24, 67–76.

[11] Caeiro, F. and Gomes, M.I. (2011). Semi-parametric tail inference through probability-weighted moments. *J. Statistical Planning and Inference* **141**, 2, 937–950.

[12] Caeiro, F. and Gomes, M.I. (2014). Comparison of asymptotically unbiased extreme value index estimators: a Monte Carlo simulation study. *AIP Conference Proceedings* **1618**, 551–554.

[13] Caeiro, F. and Gomes, M.I. (2015). Threshold Selection in Extreme Value Analysis. Chapter in: Dipak Dey and Jun Yan, *Extreme Value Modeling and Risk Analysis*, Chapman-Hall/CRC, ISBN 9781498701297, 69–87.

[14] Caeiro, F., Gomes, M.I. and Pestana, D.D. (2005). Direct reduction of bias of the classical Hill estimator. *Revstat* **3**, 2, 111–136.

[15] Caeiro, F., Gomes, M.I. and Vandewalle, B. (2014). Semi-parametric probability-weighted moments estimation revisited. *Methodology and Computing in Applied Probability* **16**, 1, 1–29.

[16] Caeiro, F., Gomes, M.I., Beirlant, J. and de Wet, T. (2016a). Mean-of-order p reduced-bias extreme value index estimation under a third-order framework. *Extremes* **19**, 4, 561–589.

[17] Caeiro, F., Gomes, M.I. and Henriques-Rodrigues, L. (2016b). A location invariant probability weighted moment EVI estimator. *International J. of Computer Mathematics* **93**, 4, 676-695.

[18] Caers, J., Beirlant, J. and Maes, M.A. (1999). Statistics for modeling heavy tailed distributions in geology: Part I. Methodology. *Mathematical Geology* **4**, 391–410.

[19] Dekkers, A.L.M. and de Haan, L. (1993). Optimal choice of sample fraction in extreme-value estimation. *J. Multivariate Analysis* **47**, 173–195.

[20] Dekkers, A., Einmahl, J. and de Haan, L. (1989). A moment estimator for the index of an extreme-value distribution. *Annals of Statistics* **17**, 1833–1855.

[21] Drees, H., Ferreira, A. and de Haan, L. (2004). On maximum likelihood estimation of the extreme value index. *Ann. Appl. Probab.* **14**, 1179–1201.

[22] Fraga Alves, M.I., Gomes, M.I., de Haan, L. and Neves, C. (2007). A note on second order conditions in extreme value theory: linking general and heavy tails conditions. *Revstat* **5**, 3, 285–305.

[23] Fraga Alves, M.I., Gomes, M.I., de Haan, L. and Neves, C. (2009). The mixed moment estimator and location invariant alternatives. *Extremes* **12**, 149–185.

[24] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. *Annals of Mathematics* **44**, 6, 423–453.

[25] Gomes, M.I. and Caeiro, F. (2014). Efficiency of partially reduced-bias mean-of-order-p versus minimum-variance reduced-bias extreme value index estimation. In M. Gilli *et al.* (eds.), *Proceedings of COMPSTAT* 2014, The International Statistical Institute/International Association for Statistical Computing, 289–298.

[26] Gomes, M.I. and Guillou, A. (2015). Extreme value theory and statistics of univariate extremes: a review. *International Statistical Review* **83**, 2, 263–292.

[27] Gomes, M.I. and Henriques-Rodrigues, L. (2010). Comparison at optimal levels of classical tail index estimators: a challenge for reduced-bias estimation? *Discussiones Mathematica: Probability and Statistics* **30**, 1, 35–51.

[28] Gomes, M.I. and Henriques Rodrigues, L. (2016). Competitive estimation of the extreme value index. *Statist. Probab. Letters* **117**, 128–135.

[29] Gomes, M.I. and Henriques-Rodrigues, L. (2017). Erratum to: Competitive estimation of the extreme value index [*Statist. Probab. Letters* **117** (2016) 128–135]. *Statist. Probab. Letters* **130**, 40–41.

[30] Gomes, M.I. and Martins, M.J. (2001). Generalizations of the Hill estimator — asymptotic versus finite sample behaviour. *J. Statist. Planning and Inference* **93**, 161–180.

[31] Gomes, M.I. and Neves, C. (2008). Asymptotic comparison of the mixed moment and classical extreme value index estimators. *Statist. Probab. Letters* **78**, 6, 643–653.

[32] Gomes, M.I., Martins, M.J. and Neves, M.M. (2000). Alternatives to a semi-parametric estimator of parameters of rare events—the Jackknife methodology. *Extremes* **3**, 3, 207–229.

[33] Gomes, M.I., Miranda, C. and Pereira, H. (2005). Revisiting the role of the Jackknife methodology in the estimation of a positive extreme value index. *Comm. in Statistics—Theory and Methods* **34**, 1–20.

[34] Gomes, M.I., Miranda, C. aand Viseu, C. (2007). Reduced-bias tail index estimation and the Jackknife methodology. *Statistica Neerlandica* **61**, 2, 243–270.

[35] Gomes, M.I., Fraga Alves, M.I. and Araújo-Santos, P. (2008). PORT Hill and moment estimators for heavy-tailed models. *Commun. in Statist.—Simul. and Comput.* **37**, 1281–1306.

[36] Gomes, M.I., Martins, M.J. and Neves, M.M. (2013). Generalized jakknife-based estimators for univariate extreme-value modelling. *Comm. in Statistics—Theory and Methods* **42**, 7, 1227–1245.

[37] Gomes, M.I., Figueiredo, F., Martins, M.J. and Neves, M.M. (2015). Resampling methodologies and reliable tail estimation. *South African Statistical J.* **49**, 1–20.

[38] Gomes, M.I., Henriques-Rodrigues, L. and Manjunath, B.G. (2016). Mean-of-order-p location-invariant extreme value index estimation. *Revstat* **14**, 3, 273–296.

[39] de Haan, L. (1984). Slow variation and characterization of domains of attraction. In J. Tiago de Oliveira, ed., *Statistical Extremes and Applications*. D. Reidel, Dordrecht, 31–48.

[40] de Haan, L. and Ferreira, A. (2006). *Extreme Value Theory: An Introduction*. Springer Science+Business Media, LLC, New York.

[41] de Haan, L. and Peng, L. (1998). Comparison of extreme value index estimators. *Statistica Neerlandica* **52**, 60–70.

[42] Hall, P. and Welsh, A.W. (1985). Adaptive estimates of parameters of regular variation. *Ann. Statist.* **13**, 331–341.

[43] Havil, J. (2003). *Gamma: Exploring Euler's Constant.* Princeton, NJ: Princeton Univ. Press.

[44] Hill, B.M. (1975). A simple general approach to inference about the tail of a distribution. *Ann. Statist.* **3**, 1163–1174.

[45] Paulauskas, V. and Vaičiulis, M. (2013). On the improvement of Hill and some others estimators. *Lith. Math. J.* **53**, 336–355.

[46] Paulauskas, V. and Vaičiulis, M. (2017). A class of new tail index estimators. *Annals of the Institute of Statistical Mathematics* **69**, 661–487.

[47] Penalva, H., Caeiro, F., Gomes, M.I. and Neves, M. (2016). *An Efficient Naive Generalization of the Hill Estimator—Discrepancy between Asymptotic and Finite Sample Behaviour*. Notas e Comunicações CEAUL 02/2016.

[48] Scarrot, C. and MacDonald, A. (2012). A review of extreme value threshold estimation and uncertainty quantification. *Revstat* **10**, 1, 33–60.