A STOCHASTIC STUDY FOR A GENERALIZED LOGISTIC MODEL

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Abstract:

- In this paper some properties of a generalized logistic discrete model are studied. Both autonomous and non-autonomous models are addressed, as well as the stochastic model, by varying the sequence of parameters that determine the sequence of mappings of the process. Some results on stability are established and the long-term behaviour of the orbits is studied.

Key-Words:

- *Periodic difference equations; stochastic difference equations; asymptotic stability; logistic equation; statistical dynamics.*

AMS Subject Classification:

- 37B55, 37C25, 37C75, 37H10, 37H20, 39A23; 39A30, 39A50.

1. INTRODUCTION

Dynamical systems occur in all branches of science. According to Martin Rasmussen [29], “the main goal of the study of a dynamical system is to understand the long behaviour of states in a system for which there is a deterministic rule for how a state evolves”. On the other hand, Christian Pötzsche [28] claims that “an understanding of the asymptotic behaviour of a dynamical system is probably one of the most relevant problems in sciences based on mathematical modeling”.

There are two approaches in the study of such mathematical models. The autonomous model where the system is governed by a single mapping and the non-autonomous model where the evolution in time is, in general, governed by a family of different mappings.

The non-autonomous systems arise naturally in the study of phenomena that evolve in time and cannot be ruled by the a single mapping by the simple fact that such phenomena do not repeat. For a general theory of non-autonomous (periodic) difference equations we refer a recent book by Luís [22] where the author presents the main concepts and results concerning periodic difference equations.

A generalization of discrete non-autonomous systems can be given by stochastic difference equations or random dynamical systems. The study of these systems are appropriate in the situation where the rules that govern the evolution of the system have a random nature.

Some works and authors in the field of random dynamical systems are worth-mention. The book of Arnold [5], where the author explores, separately, both random differential equations and random difference equations. The work of Kifer, [17] where the author studies basic connections between compositions of independent random transformations and corresponding Markov chains together with some applications. Liu in [21] reviews a selection of basic results in smooth ergodic theory and in the thermodynamic formalism of dynamical systems generated by compositions of random maps. An excellent tutorial on the asymptotic behaviours of random orbits of dynamical systems with random parameters may be found in the work of Ohno [27]. In 2009, Marie and Rousseau [25] presented a study of the recurrence behaviour in certain random dynamical systems and randomly perturbed dynamical systems. Baladi [6] uses transfer operators to construct invariant measures of chaotic dynamical systems. And to end this short list of references on random dynamical systems, we refer the excellent survey of Diaconis and Freedman [10] on iterated random functions, where the authors provide several examples under the unifying idea that the iterates of random Lipschitz functions converge if the functions are contracting on the average.

One of the well known models that have a discrete evolution is the quadratic model given by

$$x_{n+1} = \mu_n x_n(1 - x_n), \quad x \in [0, 1], \quad \mu_n \in (0, 4), n = 0, 1, 2, ... \tag{1.1}$$

When the sequence of parameters μ_n is constant, the model given by (1.1) is the well known logistic equation. The modern theory of discrete dynamical systems owns a great part of its development to the understanding of the dynamics of this equation, and may be found in many books on discrete dynamical systems, as the ones by Alligood, Sauer and Yorke [1, Chapter 1], by Devaney [9, Chapter 1], by Elaydi [11, Chapter 1] and by Zhang [30, Chapter 2], among others.

When the sequence of parameters is not constant, the dynamics of equation (1.1) is naturally more complex. Both, non-stochastic model, where the elements of the sequence of parameters are taken with a deterministic rule from the interval $(0, 4)$, and stochastic model, where the referred elements are taken randomly from the same interval, are far from being exhaustively studied. Some partial studies may be found in the literature. Grinfeld et al. [13] studied the bifurcation in $2-$periodic logistic equations. AlSharawi and Angelos [2] showed that when $\mu_{n+p} = \mu_n$, for all n, the $p-$periodic logistic equation (1.1) has cycles (periodic solutions) of minimal periods 1, p, $2p$, $3p$, $\ldots$. The same authors have also extended Singer's theorem to periodic difference equations, and used it to show that the $p-$periodic logistic equation has at most p stable cycles. Particular attention was given to the cases $p = 2$ and $p = 3$. AlSharawi et al. [3] and Alves [4] have, independently, presented an extension of Sharkovsky's theorem to periodic difference equations, where the main example is the periodic logistic equation.

In this paper some properties of a generalized logistic model given by

$$x_{n+1} = \mu_n x_n^k(1 - x_n), \tag{1.2}$$

where $x_n \in [0, 1]$, $k > 1$ and $\mu_n > 0$ for all $n = 0, 1, 2\ldots$, are studied. Some particular studies on the stability in both, non-autonomous (periodic) model (Section 2) and stochastic model (Section 3) are presented. In particular, the dynamical system defined by equation (1.2) when $k = 2$ and $\mu_n \in (0, 27/4]$ is deeply studied. The main focus of this study is the comprehension of the model's dynamics in the parameter space.

Finally, it should be mentioned that Marotto [26] studied the autonomous equation (1.2) when $k = 2$ and $\mu_n = \mu$, for all natural n. When $\mu_n = \mu$, for all n, the dynamical properties of the autonomous equation (1.2) have been addressed by several authors, like Levin and May [20], Hernández-Bermejo and Brenig [14], Briden and Zhang [7], among others.

2. Non-stochastic model

Let us consider the difference equation given by

$$x_{n+1} = \mu_n x_n^{k_n} (1 - x_n), \tag{2.1}$$

where $x_n \in [0, 1]$, $\mu_n > 0$ and $k_n = 2, 3, 4, \ldots$ for all non negative integer n.

Equation (2.1) may be represented by the map

$$f_n(x) = \mu_n x^{k_n} (1 - x).$$

In order to insure that $x_n \in I = [0, 1]$ for all n, we make the following assumption concerning the parameters

H: $\mu_n \leq \left(\dfrac{k_n + 1}{k_n}\right)^{k_n} (k_n + 1),\ n = 0, 1, 2 \ldots.$

Assumption **H** guarantees that all the orbits in (2.1) are bounded. Furthermore, it guarantees that f_n maps the interval I into the interval I for all $n = 0, 1, 2 \ldots$.

2.1. Autonomous equation

Let us first study the dynamics of the particular map $f(x) = \mu x^k (1 - x)$, with $x \in I$, $\mu > 0$ and $k = 2, 3, \ldots$. To find the fixed points of f we determine the solutions of the equation $\mu x^k (1 - x) = x$. After eliminating the trivial solution, $x = 0$, the positive fixed points are the solutions of

$$\mu x^{k-1} (1 - x) = 1, \tag{2.2}$$

or equivalently

$$\ln(\mu) = -(k - 1) \ln x - \ln(1 - x). \tag{2.3}$$

Letting $g(x) = -(k - 1) \ln x - \ln(1 - x)$, we see that $g(x) > 0$ for all $x \in (0, 1)$. Moreover, g is convex in the unit interval since $g'(x) > 0$, for all $x \in I$, and attains its minimum at $g(c_g)$ where $c_g = \frac{k-1}{k}$ is the unique critical point of g in the unit interval. Let $\mathbf{O}_\mu$ be the immediate basin of attraction of the origin.

1. If $g(c_g) > \ln(\mu)$, then Eq. (2.3) has no solution. Hence, $x^* = 0$ is the unique fixed point of the map f whenever $\mu < k \left(\frac{k}{k-1}\right)^{k-1}$. Under this scenario $x^* = 0$ is globally asymptotically stable, given that it is the unique fixed point in I. Notice that at the origin we have $f'(0) = 0$ and that $\mathbf{O}_\mu = [0, 1]$.

2. If $g(c_g) = \ln(\mu)$, then Eq. (2.3) has a unique solution, $x^* = \frac{k-1}{k} = c_g$. Hence, the map f has a unique positive fixed point when $\mu = k\left(\frac{k}{k-1}\right)^{k-1}$. In this case and using (2.2), we obtain $|f'(x^*)| = 1$ and $|f''(x^*)| = -k^2 < 0$, that allows us to conclude that x^* is an unstable fixed point, but semi-stable from the right. Moreover, its immediate basin of attraction is the set $\left[x^*, \max f^{-1}(\{x^*\})\right]$ where $f^{-1}(\{x^*\})$ is the pre-image of $\{x^*\}$. Notice that $\mathbf{O}_\mu = I \setminus \left[x^*, \max f^{-1}(\{x^*\})\right]$.

3. If $g(c_g) < \ln(\mu)$, then Eq. (2.3) has two positive solutions. Hence, the map f possesses two positive fixed points whenever $\mu > k\left(\frac{k}{k-1}\right)^{k-1}$. The smaller, denoted as $\mathbf{A}_\mu$, is known as a threshold point and the greater, denoted by $\mathbf{K}_\mu$, is known as a carrying capacity. Under this scenario, the fixed point $\mathbf{A}_\mu$ is always unstable and the fixed point $\mathbf{K}_\mu$ is locally asymptotically stable in the interval $\left(\mathbf{A}_\mu, \max f^{-1}(\{\mathbf{A}_\mu\}\right)$ if $\left|k - \mu\mathbf{K}_\mu^k\right| < 1$. Moreover, $\mathbf{O}_\mu = [0, \mathbf{A}_\mu) \cup \left(\max f^{-1}(\{\mathbf{A}_\mu\}), 1\right]$.

Notice that the sequence $a_k = \left(\dfrac{k+1}{k}\right)^k (k+1)$ that is used to define Assumption $\mathbf{H}$ is increasing for $k = 2, 3, \ldots$. We now resume the precedent ideas in the following result, for a general integer $k = 2, 3, \ldots$:

Theorem 2.1. *Let $f(x) = \mu x^k(1-x)$, $k = 2, 3, \ldots$. Then the following yields:*

1. *If $\mu < k\left(\frac{k}{k-1}\right)^{k-1}$, then $x^* = 0$ is a globally asymptotically stable fixed point of f and its basin of attraction is the unit interval.*

2. *If $\mu = k\left(\frac{k}{k-1}\right)^{k-1}$, then the map has two fixed points, the origin and a positive fixed point $x^* = \frac{k-1}{k}$. This last one is locally asymptotically stable from the right and its immediate basin of attraction is the set $\left[x^*, \max f^{-1}(\{x^*\})\right]$. Moreover, $\mathbf{O}_\mu = I \setminus \left[x^*, \max f^{-1}(\{x^*\})\right]$.*

3. *If $\mu > k\left(\frac{k}{k-1}\right)^{k-1}$, then the map has three fixed points, the origin, a threshold fixed point $\mathbf{A}_\mu$ and a carrying capacity $\mathbf{K}_\mu$ such that $\mathbf{A}_\mu < \mathbf{K}_\mu$. The threshold fixed point is always unstable and if $|k - \mu\mathbf{K}_\mu^k| < 1$ the carrying capacity is locally asymptotically stable with a basin of attraction given by the set $\left(\mathbf{A}_\mu, \max f^{-1}(\{\mathbf{A}_\mu\})\right)$. Moreover, $\mathbf{O}_\mu = I \setminus \left[\mathbf{A}_\mu, \max f^{-1}(\{\mathbf{A}_\mu\})\right]$.*

Remark 2.1. Before ending this subsection and having in mind the next section, let us have a particular look in the dynamics of the autonomous equation when $k = 2$, i.e., the dynamics of the equation when the map is given by $f(x) = \mu x^2(1-x)$. We will be needing these results when studying the corresponding stochastic equation.

1. If $\mu < 4$, then the origin is a globally asymptotically stable fixed point provided that it is the unique fixed point in the unit interval.

2. If $\mu = 4$, then the map possesses two fixed points, the origin and $x^* = \frac{1}{2}$. The basin of attraction of the origin is

$$\mathbf{O}_4 = \left[0, \frac{1}{2}\right) \cup \left(\frac{1+\sqrt{5}}{4}, 1\right], \tag{2.4}$$

while the basin of attraction of the positive fixed point is $\left[\frac{1}{2}, \frac{1+\sqrt{5}}{4}\right]$. Notice that $x^* = \frac{1}{2}$ is a fixed point semi-stable from the right.

3. If $4 < \mu$, then the map has three fixed points, the origin, the threshold point $\mathbf{A}_\mu = \frac{1}{2}\left(1 - \sqrt{\frac{\mu-4}{\mu}}\right)$ and the carrying capacity $\mathbf{K}_\mu = \frac{1}{2}\left(1 + \sqrt{\frac{\mu-4}{\mu}}\right)$.
It is a straightforward computation to see that, when $\mu > 4$,

$$|f'(\mathbf{A}_\mu)| = 3 + \frac{\mu}{2}\left(-1 + \sqrt{\frac{\mu-4}{\mu}}\right) > 1.$$

Hence, the fixed point $\mathbf{A}_\mu$ is unstable.
Similarly, we see that

$$|f'(\mathbf{K}_\mu)| = \left|3 - \frac{\mu}{2}\left(1 + \sqrt{\frac{\mu-4}{\mu}}\right)\right| < 1 \text{ iff } 4 < \mu < \frac{16}{3}.$$

When $\mu = \frac{16}{3}$ we have $f'(\mathbf{K}_\mu) = -1$. Forward computations show that the Schwarzian derivative evaluated at the fixed point is negative, i.e., $Sf(\mathbf{K}_\mu) < 0$. Consequently, from Theorem 2 in [24] it follows that the fixed point $\mathbf{K}_\mu$ is asymptotically stable. Thus, the fixed point $x^* = \mathbf{K}_\mu$ is locally asymptotically stable whenever $4 < \mu \leq \frac{16}{3}$ and its basin of attraction is the set $\left(\mathbf{A}_\mu,\ \max f^{-1}(\{\mathbf{A}_\mu\})\right)$. Moreover,

$$\mathbf{O}_\mu = [0, \mathbf{A}_\mu) \cup \left(\max f^{-1}(\{\mathbf{A}_\mu\}), 1\right]. \tag{2.5}$$

2.2. Non-autonomous equation

We start this subsection presenting a result related to the non-autonomous equation (2.1) when $k = 2$ (although it may be extended for other values of the parameter k as well). It is not hard to prove the following:

Lemma 2.1. *Consider the non-autonomous difference equation given by*

$$x_{n+1} = \mu_n x_n^2 (1 - x_n), \tag{2.6}$$

where $x_n \in [0,1]$, $\mu_n \in \left(0, \frac{27}{4}\right]$, *for* $n = 0, 1, 2\ldots$, *and* $\mathbf{O}_\mu$ *the immediate basin of attraction of the origin. Then*

$$4 \leq \mu_1 \leq \mu_2 \leq \frac{27}{4} \Rightarrow \mathbf{O}_4 \supseteq \mathbf{O}_{\mu_1} \supseteq \mathbf{O}_{\mu_2} \supseteq \mathbf{O}_{\frac{27}{4}}, \tag{2.7}$$

where $\mathbf{O}_4$ *is given by (2.4) and*

$$\mathbf{O}_{\frac{27}{4}} = \left[0, \frac{9-\sqrt{33}}{18}\right) \cup \left(\max f^{-1}\left(\left\{\mathbf{A}_{\frac{27}{4}}\right\}\right), 1\right], \tag{2.8}$$

where $\max f^{-1}\left(\left\{\mathbf{A}_{\frac{27}{4}}\right\}\right) \approx 0.971\,62$.

Let us now turn our attention to the non-autonomous periodic equation (2.1). We will study the case where the sequence of maps is $p-$periodic, i.e., when $f_{n+p} = f_n$, for all $n = 0, 1, 2, \ldots$. Under this scenario, equation (2.1) is $p-$periodic.

The dynamics of the non-autonomous $p-$periodic equation (2.1) is completely determined by the following composition operator

$$\Phi_p = f_{p-1} \circ \ldots \circ f_1 \circ f_0.$$

From assumption $\mathbf{H}$ it follows that $\Phi_p(I) \subseteq I$ with $\Phi_p(0) = 0$ and $\Phi_p(1) = 0$. Hence, by the Brouwer's fixed point theorem [16], the composition operator Φ_p has a fixed point in the unit interval.

It is clear that $x^* = 0$ is a locally asymptotically stable fixed point of Φ_p provided that $|\Phi_p'(0)| = 0$. Now, if $\Phi_p(x) < x$, for all $x \in (0,1)$, then $x^* = 0$ is the unique fixed point of the composition operator Φ_p in the unit interval. In this case, $x^* = 0$ is a globally asymptotically stable fixed point and its basin of attraction is the entire unit interval. This is the case where local stability implies global stability in the sense that every orbit of $x_0 \in I$ converge to the origin.

Notice that, if C_{Φ_p} is the set of critical points of Φ_p, i.e., if C_{Φ_p} contains all the solutions in the unit interval of the p equations $\Phi_i(x) = c_i$, $i = 0, 1, \ldots, p-1$, where c_i is the critical point of the map f_i, then $\Phi_p(x) < x$, for all $x \in (0,1)$ if $\Phi_p(c_{\Phi_p}) < c_{\Phi_p}$, where $c_{\Phi_p} \in C_{\Phi_p}$.

Now, if $|\Phi_p(x)| > x$ for some $x \in (0,1)$,the composition operator Φ_p has more than one fixed point. We know from Coppel's Theorem [8] that every orbit converges to a fixed point if and only if the equation $\Phi_p \circ \Phi_p(x) = x$ has no solutions with the exception of the fixed points of Φ_p. It is not possible, in general, to say much concerning the number of fixed points of Φ_p since we have many scenarios. However, if all maps f_i have a threshold fixed point $\mathbf{A}_i$ and we let $\mathbf{A}_m = \min\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{p-1}\}$ and $\mathbf{A}_M = \max\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_{p-1}\}$, then one can show that the minimal positive fixed point of Φ_p, $\mathbf{A}_{\Phi_p}$, lies between $\mathbf{A}_m$ and

$\mathbf{A}_M$ and is, in fact, an unstable fixed point. Under this scenario, the immediate basin of attraction of the origin is $\cup_{i\geq 1} J_i$ where $J_i \subset I$ and

$$\Phi_p(J_i) \subset [0, \mathbf{A}_{\Phi_p}).$$

See Figure 1 for an example of this scenario.

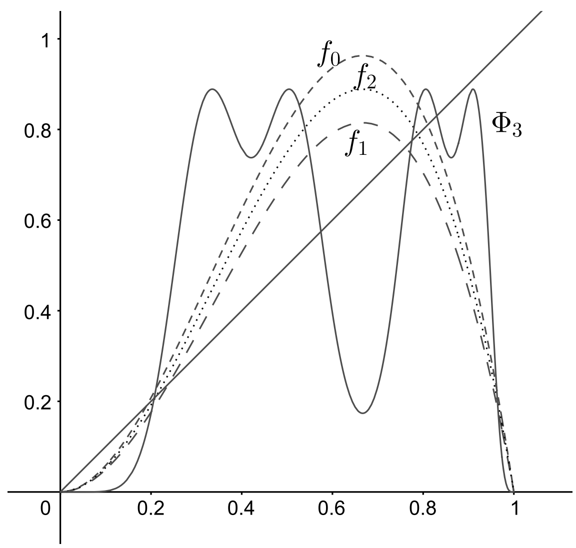

Figure 1: Composition of three generalized logistic maps. The composition map Φ_3 is represented by the solid curve and the individual maps are represented by the dashed curves. The values of parameters are $k = 2$, $\mu_0 = 6.5$ (f_0), $\mu_1 = 5.5$ (f_1) and $\mu_2 = 6$ (f_2).

We remark that each fixed point of the composition map Φ_p, with the exception of $x^* = 0$, generates a periodic orbit in equation (2.1). More precisely, if x^* is a non-trivial fixed point of Φ_p, then

$$\overline{C} = \{\overline{x}_0 = x^*, \overline{x}_1 = f_0(\overline{x}_0), \overline{x}_2 = f_1(\overline{x}_1), \ldots, \overline{x}_{p-1} = f_{p-2}(\overline{x}_{p-2})\}$$

is a periodic cycle of equation (2.1), which is locally asymptotically stable if

$$|\Phi_p'(x^*)| = \left|\prod_{i=0}^{p-1} f_i'(\overline{x}_i)\right| < 1.$$

Notice that, due the periodicity of the maps f_i, we have $\overline{x}_p = f_{p-1}(\overline{x}_{p-1}) = \overline{x}_0$, $\overline{x}_{p+1} = \overline{x}_1$, and so on.

From the dynamical point of view, it is interesting to know the region where the stability of the fixed points occurs. Since we are not able to find explicitly the fixed points of the composition map Φ_p for general values of the parameters k_i and μ_i, $i = 0, 1, \ldots, p-1$, we will particularize and study the cases where this is possible as are the cases when $p = 2, 3, 4$ and $k = 2$, i.e., we will study the dynamics of the system when the sequence of maps is 2−periodic and given by

$$f_{n\ mod(2)}(x) = \mu_{n\ mod(2)}\ x^k(1-x), \quad k = 2, 3, 4.$$

Let us start with the case $k = 2$. Following the techniques employed in [23], one can find the region of local stability of the fixed points of the composition map $\Phi_2 = f_1 \circ f_0$ by calculating the boundary where the absolute value of $\Phi_2'(x^*)$ is equal to one. Since the computations are long we will omit them here. The stability regions are depicted in Figure 2, in the parameter space $\mu_0 O \mu_1$.

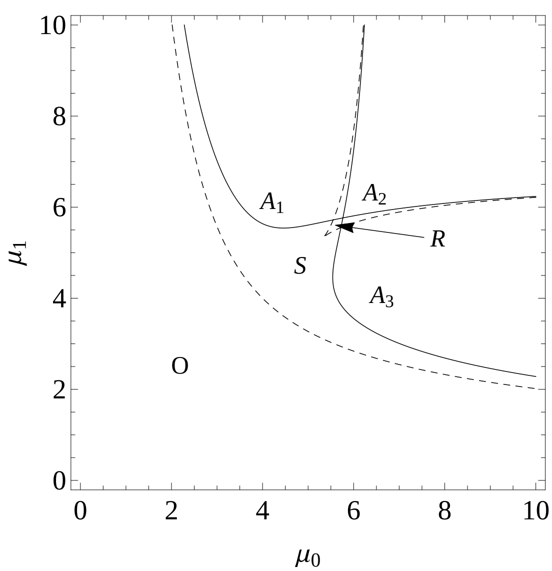

Figure 2: Region of local stability, in the parameter space $\mu_0 O \mu_1$ where the fixed points of $f_1 \circ f_0$ are locally asymptotically stable and the maps are given by $f_i(x) = \mu_i x^2(1-x)$, $i = 0, 2$.

If the parameters μ_0 and μ_1 belong to the region **O**, then the origin is a fixed point globally asymptotically stable. Once the parameters cross the dashed curve, from Region **O** to Region **S**, a bifurcation occurs, known as saddle-node bifurcation. The fixed point $x^* = 0$ becomes unstable and a new locally stable fixed point of Φ_2 is born. This fixed point is, in fact, a 2−periodic cycle of the 2−periodic equation (2.1). Now if the parameters μ_0 and μ_1 cross the dashed curve from Region **S** to Region **R,** a saddle-node bifurcation occurs. The 2−periodic cycle becomes unstable and a new locally asymptotically stable 2−periodic cycle is born.

At the solid curve a new type of bifurcation occurs known as a period-doubling bifurcation. Hence, when the parameters cross the solid curve from Region S to Region A_i, $i = 1, 2, 3$, the 2−periodic cycle of equation (2.1) becomes unstable and a new locally asymptotically stable 4−periodic cycle is born.

Following a similar idea as before, we are able to find (numerically) the regions of local stability of the 4−periodic cycle identified before. We notice that this scenario of period-doubling bifurcation continues route to chaos.

For a general framework of bifurcation in one-dimensional periodic difference equations, we refer the work of Elaydi, Luís, and Oliveira in [12].

Now, following the same techniques as before, we are able to find the regions of local stability of fixed points when $k = 3$ and $k = 4$. These regions are represented in Figure 3. As we can observe, they are similar to the case $k = 2$ and the conclusions follow in the same fashion.

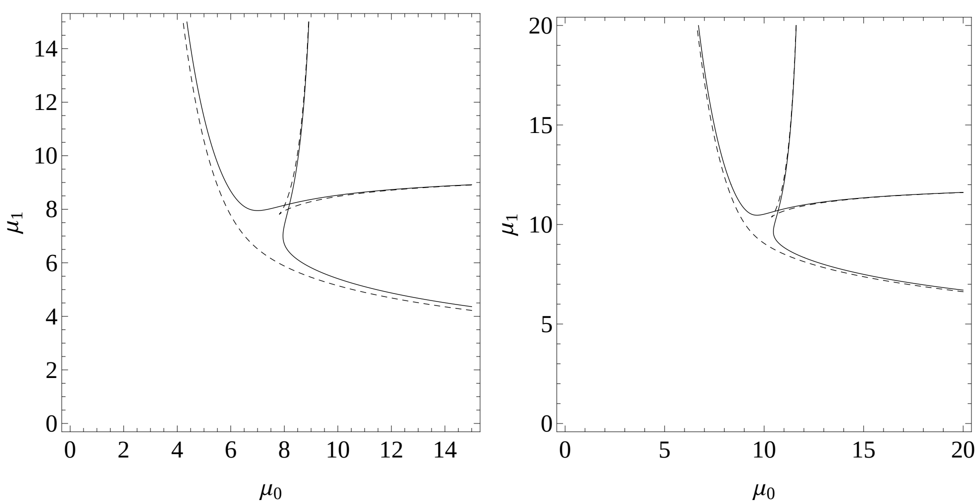

Figure 3: Regions of local stability, in the parameter space, of the 2−periodic equation when $k = 3$ (left) and $k = 4$ (right).

3. Stochastic model

In this section, we will consider the stochastic version of the difference equation (2.1) when $k_n = 2$, for all n, defined by the equation

$$x_{n+1} = f_n(x_n) = b(\mu_n, x_n) = \mu_n x_n^2 (1 - x_n), \tag{3.1}$$

with $x_0 \in I = [0, 1]$, $\{\mu_n, n \in \mathbb{N}_0\}$ a sequence of independent and identically distributed random variables with support contained in $S = \left(0, \frac{27}{4}\right]$ and common probability density function ϕ.

3.1. Stochastic kernel and asymptotic behaviour

Notice that x_n, for $n \in \mathbb{N}$, defined by (3.1) is an absolutely continuous random variable (with respect to Lebesgue measure). Let f_n be the probability density function of x_n. For each $n \in \mathbb{N}$, the random variables μ_n and x_n are independent and hence their joint probability density function is the product of the individual probability density functions ϕf_n. Let h be an arbitrary bounded function defined in I ($h \in L^\infty(I)$). We have

$$E\left[h\left(x_{n+1}\right)\right] = \int_I h(x)\, f_{n+1}(x)\, dx, \tag{3.2}$$

and, on the other hand,

$$E\left[h\left(x_{n+1}\right)\right] = E\left[h\left(b\left(\mu_n, x_n\right)\right)\right] = \int_I \int_S h\left(b\left(u,x\right)\right)\phi\left(u\right) f_n\left(x\right) du dx.$$

Letting $y = b(u,x) = ux^2(1-x)$ in the inner integral, we obtain

$$E\left[h\left(x_{n+1}\right)\right] = \int_I \left[\int_0^{\frac{27}{4}x^2(1-x)} h(y)\,\phi\left(\frac{y}{x^2(1-x)}\right) f_n(x)\,\frac{1}{x^2(1-x)}dy\right] dx. \tag{3.3}$$

Let $\gamma_1 : [0,1] \to \left[0, \frac{2}{3}\right]$ be the inverse function of $\gamma : \left[0, \frac{2}{3}\right] \to [0,1]$ and $\gamma_2 : [0,1] \to \left[\frac{2}{3}, 1\right]$ the inverse function of $\gamma : \left[\frac{2}{3}, 1\right] \to [0,1]$, i.e.,

$$\gamma_1(y) = \frac{1}{3}\left(\sqrt[3]{2\sqrt{y^2-y}-2y+1} + \frac{1}{\sqrt[3]{2\sqrt{y^2-y}-2y+1}} + 1\right)$$

and

$$\gamma_2(y) = -\frac{1}{6}\left(1+i\sqrt{3}\right)\sqrt[3]{2\sqrt{y^2-y}-2y+1} - \frac{1-i\sqrt{3}}{6\sqrt[3]{2\sqrt{y^2-y}-2y+1}} + \frac{1}{3}.$$

The functions γ, γ_1 and γ_2 are represented in Figure 4.

Inverting the integration order in (3.3) we obtain

$$E\left[h\left(x_{n+1}\right)\right] = \int_I h(y)\left[\int_{\gamma_1(y)}^{\gamma_2(y)} \phi\left(\frac{y}{x^2(1-x)}\right) f_n(x)\,\frac{1}{x^2(1-x)}dx\right] dy. \tag{3.4}$$

Comparing (3.2) and (3.4), since h is arbitrary, it follows that

$$f_{n+1}(y) = \int_I \phi\left(\frac{27}{4}\frac{y}{\gamma(x)}\right) f_n(x)\,\frac{27}{4}\frac{1}{\gamma(x)} I_{[\gamma_1(y),\gamma_2(y)]}(x)\, dx$$

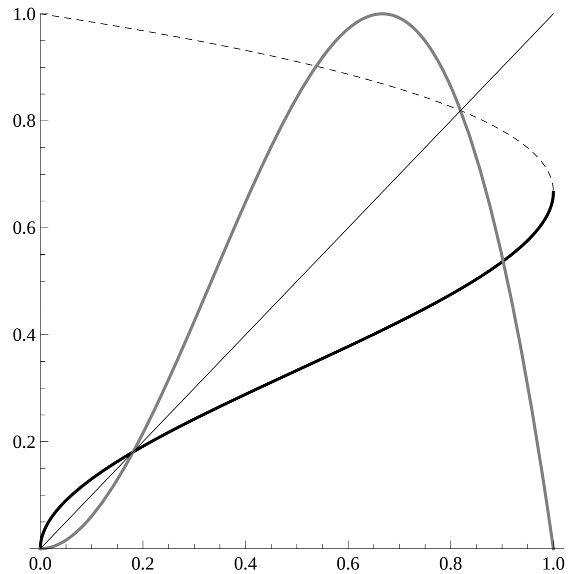

Figure 4: Graphs of γ (grey solid line), γ_1 (black solid line) and γ_2 (dashed line) in the unit interval.

(where $I_A(v) = 1$ if $v \in A$, $I_A(v) = 0$, otherwise).

It is not difficult to prove that if f_n is supported on $S_n \subseteq I$, then f_{n+1} is supported on $S_{n+1} \subseteq I$.

Let $f \in L^1(I)$, i.e., such that $\int_I |f(x)|\, dx < +\infty$ and $P : L^1(I) \to L^1(I)$ the operator defined by

$$Pf(u) = \int_I L(u,v)\, f(v)\, dv, \tag{3.5}$$

were L is defined for $(u,v) \in$ on $I \times I$ by

$$L(u,v) = \phi\left(\frac{u}{v^2(1-v)}\right)\frac{1}{v^2(1-v)} I_{[\gamma_1(u),\gamma_2(u)]}(v). \tag{3.6}$$

Notice that

$$\int_I L(u,v)\, du = \int_0^{\frac{27}{4}} \phi(y)\, dy = 1,$$

i.e., L is a stochastic kernel on $I \times I$, since, in addition, $L \geq 0$, and also that

$$P^{n+1} f(u) \;=\; \int_I f(v)\, L_{n+1}(u,v)\, dv$$

with

$$L_{n+1}(v_0, v_{n+1}) \;=\; \int_{I^n} \prod_{i=1}^{n+1} L(v_{i-1}, v_i)\, dv_n ... dv_2 dv_1.$$

In the sequel will study the asymptotically behaviour of the sequence $\{P^n, n \in \mathbb{N}\}$. Suppose ϕ is a bounded probability density function with support $[a,b] \subset \left(0, \frac{27}{4}\right]$ and consider the function

$$h_u(v) = \frac{u}{v^2(1-v)},$$

defined for $v \in (0,1)$ and $u \in I$ (cf. Figure 5 for some graphical examples). The minimum of $h_u(v)$ is obtained when $v = \frac{2}{3}$ and is given by $h_u\left(\frac{2}{3}\right) = u\frac{27}{4}$.

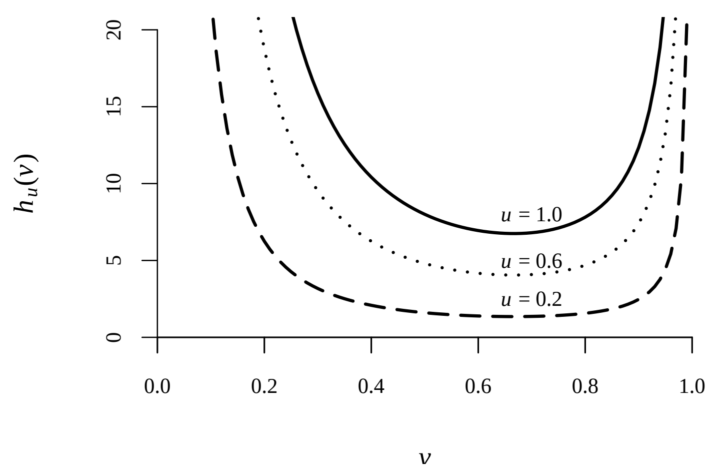

Figure 5: Graphs of h_u when $u = 1$ (solid line), $u = 0.6$ (dotted line) and $u = 0.2$ (dashed line).

Notice that (cf. (3.6))

$$L(u,v) = \phi(h_u(v)) \frac{1}{v^2(1-v)} I_{[\gamma_1(u), \gamma_2(u)]}(v).$$

There are three possibilities, for a given u:

1. If u is such that $h_u\left(\frac{2}{3}\right) > b$, i.e., if $u > \frac{4}{27}b$, then $L(u,v) = 0$, for all $v \in I$.

2. If u is such that $a \le h_u\left(\frac{2}{3}\right) \le b$, i.e., if $\frac{4}{27}a \le u \le \frac{4}{27}b$, then

$$L(u,v) = \phi(h_u(v)) \frac{1}{v^2(1-v)} I_{[\gamma_1(u), \gamma_2(u)] \cap V(u)}(v) \le \frac{27}{4}\frac{b}{a}M, \tag{3.7}$$

where $V(u) = \left[\min h_u^{-1}(\{b\}), \max h_u^{-1}(\{b\})\right]$ and $M = \sup\limits_{u,v \in [0,1]} \phi(h_u(v))$.

3. Finally, for u such that $h_u\left(\frac{2}{3}\right) < a$, L is null if $v \notin \{v : a \le h_u(v) \le b\}$, and the same condition (3.7) is obtained.

We can then conclude that $\forall u, v \in I$ we have

$$L(u,v) \leq \frac{27}{4}\frac{b}{a}M.$$

Since $\int_I \frac{b}{a}\frac{27}{4} M dx < +\infty$, we have proven the following result (cf. [18], p. 99 and Theorem 5.7.3 in p. 118):

Theorem 3.1. *The sequence $\{P^n, n \in \mathbb{N}\}$, where P is defined by (3.5), is asymptotically periodic.*

This means that there exists a finite sequence of densities $g_1, ...g_r$, a sequence of linear functionals $\lambda_1, ..., \lambda_r$, and a permutation ω of the integers $1, ..., r$ such that

$$Pg_i = g_{\omega(i)}, \quad g_i g_j = 0 \text{ for } i \neq j$$

and

$$\lim_{n\to\infty} \left\| P^n f - \sum_{i=1}^{r} \lambda_i(f) g_{\omega^n(i)} \right\| = 0 \text{ for} f \in L^1.$$

For better understanding the behaviour of the sequence $\{P^n, n \in \mathbb{N}\}$, where P is defined by (3.5), let the parameters μ_n, for $n \in \mathbb{N}$ from the stochastic difference equation (3.1) be uniform in an interval $C \subseteq S = (0, 27/4]$, i.e., let $\phi(x) = \frac{1}{|C|} I_C(x)$. The asymptotic behaviour of the process depends on the set C. For example, if $C = S$, i.e., if $\phi(x) = \frac{4}{27} I_S(x)$, then at the instant n the system can be in one of the following intervals:

$$E_1 = \left[0, \mathbf{A}_{\frac{27}{4}}\right), \quad E_2 = \left(\mathbf{A}_{\frac{27}{4}}, \frac{1}{2}\right), \quad E_3 = \left[\frac{1}{2}, \frac{1+\sqrt{5}}{4}\right],$$

$$E_4 = \left(\frac{1+\sqrt{5}}{4}, \max f^{-1}\left(\left\{\mathbf{A}_{\frac{27}{4}}\right\}\right)\right), \quad E_5 = \left(\max f^{-1}\left(\left\{\mathbf{A}_{\frac{27}{4}}\right\}\right), 1\right],$$

where, recall, $\mathbf{A}_{\frac{27}{4}} = \frac{9-\sqrt{33}}{18}$. Consider $P_n = [p_{i,j,n}]_{i,j\in\{1,...,5\}}$ where $p_{i,j,n} = P\left(x_{n+1} \in E_j | x_n \in E_i\right)$. We have

$$P_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ p_{2,1,n} & p_{2,2,n} & p_{2,3,n} & p_{2,4,n} & p_{2,5,n} \\ p_{3,1,n} & p_{3,2,n} & p_{3,3,n} & p_{3,4,n} & p_{3,5,n} \\ p_{4,1,n} & p_{4,2,n} & p_{4,3,n} & p_{4,4,n} & p_{4,5,n} \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $p_{i,j,n} \neq 0$ for $i \in \{2,3,4\}$ and $j \in \{1,..,5\}$, the fixed point zero will attract all points with probability one. Also, if there exists a natural number n_0 such that $p_{i,j,n_0} = 0$, then $p_{i,j,n} = 0$, for all $n \geq n_0$.

On the other hand, if, e.g., $C = (4, 16/3)$ and $x_0 \in E_3$, the system will remain in E_3 (Figure 6 represents two samples of the position of the system after 20000 steps). Hence, in this case, there exists a set of positive Lebesgue measure where the inequality $P^n f > 0$ holds for $n \geq n_0(f)$, for every probability density function, f, with support on the positive real numbers set. Using, e.g., Lemma 1 from [19], we can then conclude the following result:

Corollary 3.1. *If ϕ is the uniform distribution based on a non null subset of $(4, \frac{16}{3})$, the sequence $\{P^n, n \in \mathbb{N}\}$, where P is defined by (3.5) and (3.6), is asymptotically stable, i.e., there exists a probability density function f^* on R^+ such that $Pf^* = f^*$ and*

$$lim_{x\to\infty} \|P^n f - f^*\| = 0,$$

for any probability density function f on R^+, where $\|.\|$ denotes the norm in L^1.

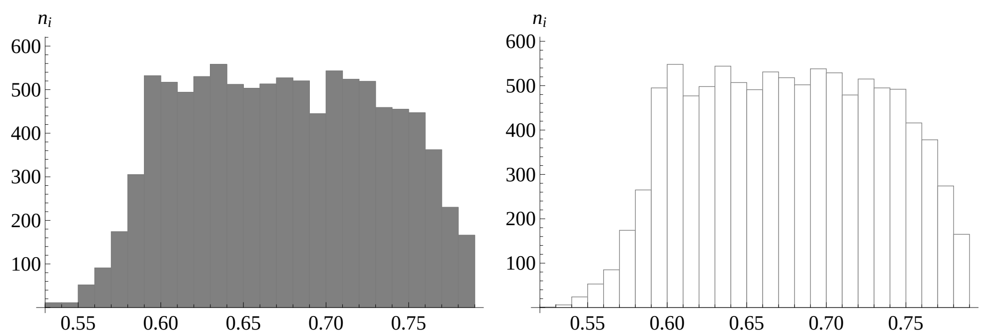

Figure 6: Two samples of size 10000 of the random variable x_{20000} when the sequence μ_n is uniformly distributed in $(4, 10/3)$ and $x_0 = 0.6$.

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