REVSTAT – Statistical Journal
Volume 13, Number 2, June 2015, 145–168

MODEL OF GENERAL SPLIT-BREAK PROCESS

Authors: Vladica Stojanović
– Faculty of Sciences and Mathematics,
University of Priština (in K. Mitrovica), Serbia
vladica.stojanovic@pr.ac.rs

Biljana Č. Popović
– Faculty of Sciences and Mathematics,
University of Niš, Serbia
bipop@pmf.ni.ac.rs

Predrag Popović
– Faculty of Civil Engineering and Architecture,
University of Niš, Serbia
popovicpredrag@yahoo.com

Received: May 2013 Revised: January 2014 Accepted: February 2014

Abstract:

- This paper presents a modification (and partly a generalization) of STOPBREAK process, which is the stochastic model of time series with permanent, emphatic fluctuations. The threshold regime of the process is obtained by using, so called, Noise indicator. We proceed to investigate the model which we named the General Split-BREAK (GSB) process. After brief recalling of its basic stochastic properties, we give some procedures of its parameters estimation. A Monte Carlo study of this process is also give, along with the application in the analysis of stock prices dynamics of several Serbian companies which were traded on Belgrade Stock Exchange.

Key-Words:

- *GSB process; STOPBREAK process; noise indicator; Split-MA process; stationarity; invertibility; parameters estimation.*

AMS Subject Classification:

- 62M10, 91B84.

1. INTRODUCTION

Starting from the fundamental results of Engle and Smith [4], who were the first who had introduced the stochastic process of permanent fluctuations, named STOPBREAK process, we had defined a new, modified version of the well known generalization of STOPBREAK process. In our model, we set the threshold noise indicator as we had recently done in the time series of ARCH type, described in Popović and Stojanović [12] and Stojanović and Popović [15]. Our model, named *the General Split-BREAK process* or, simply, *GSB process*, is at the same time the generalization of so called Split-BREAK model introduced in Stojanović *et al.* [16].

In the next section, Section 2, we shall briefly present the definition and the main stochastic properties of GSB model, described in detail in Stojanović *et al.* [17], and we will define the sequence of increments of the GSB process, called *Split-MA process*. Beside the standard investigation of the stochastic properties of Split-MA model, we will particularly give the conditions of its invertibility. The main result of this paper, procedures of the parameters estimation of GBS process, are described in Section 3. We will pay the special attention to the estimation of the threshold parameter, named *critical value of the reaction*. We shall prove the asymptotic properties of evaluated estimates. The following section, Section 4, is devoted to the Monte Carlo simulations of the innovations of GSB process. Section 5 describes an application of the estimation procedures on the real data of some trading volumes on Belgrade Stock Exchange. Finally, Section 6 is the conclusion.

2. THE GSB PROCESS. DEFINITION AND MAIN PROPERTIES

We shall suppose that (y_t) is the time series with the known values at time $t \in \{0, 1, ...T\}$ and $F = (\mathcal{F}_t)$ is a filtration defined on the probability space $(\Omega, \mathcal{F}, P)$. Following Engle and Smith [4], the sequence (y_t) will be a *General STOPBREAK process* if it satisfies the recurrent relation

$$A(L)\,B(L)\,y_t \;=\; q_{t-1}A(L)\,\varepsilon_t + (1-q_{t-1})\,B(L)\,\varepsilon_t\,, \qquad t=1,...,T\,, \tag{2.1}$$

where, $A(L) = 1 - \sum_{j=1}^{p} \alpha_j\, L^j$, $B(L) = 1 - \sum_{k=1}^{r} \beta_k\, L^k$, and L is the backshift operator. On the other hand, (ε_t) is a white noise, i.e. the i.i.d. sequence of random variables adapted to the filtration F, which satisfies

$$E\big(\varepsilon_t\,|\,\mathcal{F}_{t-1}\big) = 0\,, \quad \mathrm{Var}\big(\varepsilon_t\,|\,\mathcal{F}_{t-1}\big) = \sigma^2\,, \qquad t=1,...,T\,.$$

At last, (q_t) represents the sequence of random variables which depends on the white noise (ε_t), and in addition, $P\big\{0 \le q_t \le 1\big\} = 1$ for each $t = 0, 1, ..., T$. The sequence (q_t) displays the *(permanent) reaction* of the STOPBREAK process, because its values determine the amount of participation of previous elements of white noise process engaged in the definition of (y_t).

In that way, the structure of the sequence (q_t) determines the character and the properties of the STOPBREAK process, which vary among the well known linear stochastic models. This model was investigated later by several authors, for instance Gonzáles [6], or Diebold [3], whose works were based on certain variations of the reaction (q_t). On the other hand, many authors, for instance Huang and Fok [9] or Kapetanios and Tzavalis [10], have studied mainly the practical application of the STOPBREAK (and some similar) processes.

Similarly as in the definition of Split-ARCH model [12, 15], we shall suppose in the following that

$$(2.2)\qquad q_t \,=\, I(\varepsilon^2_{t-1} > c) \,=\, \begin{cases} 1\,, & \varepsilon^2_{t-1} > c \\ 0\,, & \varepsilon^2_{t-1} \le c\,, \end{cases} \qquad t=1,\dots,T\,,$$

i.e., that the permanent reaction (2.2) represents, so-called *a Noise indicator*. Remark that, according to (2.2), it follows that

$$E\big(q_t\,\varepsilon_t\,|\,\mathcal{F}_{t-1}\big) \,=\, q_t E\big(\varepsilon_t\,|\,\mathcal{F}_{t-1}\big) \,=\, 0\;,$$

and it can be seen that the sequence $(q_t\,\varepsilon_t)$ is a martingale difference, as in the definition of basic STOPBREAK model [4].

However, it seems that in the case of general STOPBREAK process this formulation of reaction (q_t) is inadequate. The primary reason for such opinion is the fact that the model (2.1) includes only "directly previous" realizations of (q_t), which are obtained at the moment $t-1$. Therefore, the general STOPBREAK process (2.1) with the reaction (2.2) operates in (only) two different regimes

$$(2.3)\qquad \varepsilon_t \,=\, \begin{cases} A(L)\,y_t\,, & q_{t-1}=0 \;\;(\text{w.p. } b_c) \\ B(L)\,y_t\,, & q_{t-1}=1 \;\;(\text{w.p. } a_c)\,, \end{cases}$$

where $a_c = E(q_t) = P\{\varepsilon^2_{t-1} > c\}$, $b_c = 1 - a_c$ and "w.p." stands for "with probability". Therefore, the equality (2.3) defines the well known Thresholds Autoregressive (TAR) model introduced by Tong [18] and discussed in details, for instance by Chan [2], Hansen [8] and, in the newest time, Scarrott and MacDonald [13] and some other authors. Based on this, here we discuss the different generalization of Split-BREAK process, more general than that of Engle and Smith [4].

Definition 2.1. Let L be a backshift operator, $A(L) = 1 - \sum_{i=1}^{m} \alpha_i L^i$, $B(L) = 1 - \sum_{j=1}^{n} \beta_j L^j$, $C(L) = 1 - \sum_{k=1}^{p} \gamma_k L^k$, and (q_t) the noise indicator defined with (2.2). Then, the sequence (y_t) represents *the General Split-BREAK* (*GSB*) *process* if it satisfies

$$(2.4)\qquad A(L)\,y_t \,=\, B(L)\,q_t\,\varepsilon_t + C(L)\,(1-q_t)\,\varepsilon_t\,, \qquad t \in \mathbf{Z}\;.$$

Note that the definition above represents the general stochastic model which as its specific forms, contains the most of other, well known models. In dependence of $A(L)$, $B(L)$ and $C(L)$ we have, for example, the following processes:

$$\begin{array}{lll} A(L)=B(L)=C(L)=1: & y_t=\varepsilon_t & \text{(White Noise)} \\ A(L)=1,\ \ B(L)=C(L)\neq 1: & y_t=B(L)\varepsilon_t & \text{(MA model)} \\ A(L)\neq 1,\ \ B(L)=C(L)=1: & A(L)y_t=\varepsilon_t & \text{(AR model)}\ . \end{array}$$

Finally, in the case when $A(L)=C(L)=1-L$ and $B(L)=1$ we get the Split-BREAK process introduced by Stojanović *et al.* [16]. In the following, we shall analyze some specificity of the model (2.4) and suppose $A(L)=C(L)\neq 1$ and $B(L)=1$. Thus, the defined model can be written in the form

$$y_t-\sum_{j=1}^{p}\alpha_j\, y_{t-j}\ =\ \varepsilon_t-\sum_{j=1}^{p}\alpha_j\,\theta_{t-j}\,\varepsilon_{t-j}\,,\qquad t\in\mathbf{Z}\ , \tag{2.5}$$

where $\alpha_j\geq 0,\ j=1,...,p,$ and $\theta_t=1-q_t=I(\varepsilon_{t-1}^2\leq c)$. Obviously, this representation is close to linear ARMA time series, except that it has the indicators of noise (ε_t) in its own structure. They indicate the realizations of noise which have statistically significant weights in "previous" time. These "temporary" components change the ARMA structure of GSB model (2.5). In this way, they make some difficulties in the usage of well known procedures in investigation of the properties of our model .

On the other hand, similarly to the basic STOPBREAK process, the equality (2.4) enables that the sequence (y_t) can be presented in the form of additive decomposition

$$y_t\ =\ m_t+\varepsilon_t\,,\qquad t\in\mathbf{Z}\ , \tag{2.6}$$

where

$$m_t\ =\ \sum_{j=1}^{p}\alpha_j\big(y_{t-j}-\theta_{t-j}\,\varepsilon_{t-j}\big)\ =\ \sum_{j=1}^{p}\alpha_j\big(m_{t-j}+q_{t-j}\,\varepsilon_{t-j}\big) \tag{2.7}$$

is the sequence of random variables which we named *the martingale means*. It is general case of analogous equality of Engle and Smith [4], which is obtained from (2.7), for $p=\alpha_1=1$. According to (2.6) it follows

$$E\big(y_t\,|\,\mathcal{F}_{t-1}\big)\ =\ m_t+E\big(\varepsilon_t\,|\,\mathcal{F}_{t-1}\big)\ =\ m_t\ , \tag{2.8}$$

from which it follows $E(y_t)=E(m_t)=\mu$ (const.), i.e. the means of these two sequences are equal and constant. The variance of GSB process can be determined in a similar way. As

$$\mathrm{Var}\big(y_t\,|\,\mathcal{F}_{t-1}\big)\ =\ E\big(y_t^2\,|\,\mathcal{F}_{t-1}\big)-m_t^2\ =\ \sigma^2\ , \tag{2.9}$$

we can conclude that the conditional variance (volatility) of the sequence (y_t) is a constant and it is equal to the variance of the noise (ε_t). Let us remark that the equalities (2.8) and (2.9) explain the stochastic nature of (2.4). As the sequence (m_t) is predictable, it will be a component which demonstrates the stability of the process itself. Contrary, the sequence (ε_t) is the factor which represents the deviations (or random fluctuations) from values (m_t). On the other hand, the variances of sequences (m_t) and (y_t) satisfy the relation

$$\mathrm{Var}(y_t) \;=\; \mathrm{Var}(m_t) + \sigma^2$$

and, in the non-stationary case, these are not constants, i.e., depend on the observation time t.

In the following, we shall describe stochastic structure of the increments

$$X_t \stackrel{\mathrm{def}}{=} A(L)\, y_t\,, \qquad t \in \mathbf{Z}\,,$$

which, according to (2.4) and (2.5), we can write in the form of recurrent relation

$$X_t \;=\; \varepsilon_t - \sum_{j=1}^{p} \alpha_j\, \theta_{t-j}\, \varepsilon_{t-j}\,, \qquad t \in \mathbf{Z}\,. \tag{2.10}$$

Obviously, the sequence (X_t) has the multi–regime structure, which depends on the realizations of indicators (θ_t). If all fluctuations of the white noise in time $t-j$ are large, an increment X_t will be equal to the white noise. On the other hand, the fluctuations of the white noise which do not exceed the critical value c will produce a "part of" MA(p) representation of (X_t). In this way, the similarity of this model to the standard linear MA model is noticeable, and the sequence (X_t) we shall call *the general Split-MA model* (*of order* p) or, simply, *Split-MA*(p) *model*. It represents the generalization of the model defined in Stojanović *et al.* [16], and the threshold integrated moving average (TIMA) model introduced by Gonzalo and Martinez [7]. The main properties of this process can be expressed as follow.

Theorem 2.1. *The sequence* (X_t)*, defined by* (2.10)*, is stationary, with mean* $E(X_t) = 0$ *and covariance* $\gamma_X(h) = E(X_t X_{t+h})$*,* $h \ge 0$*, which satisfies the equality*

$$\gamma_X(h) \;=\; \begin{cases} \sigma^2\left(1 + b_c \sum\limits_{j=1}^{p} \alpha_j^2\right), & h = 0 \\ \sigma^2\, b_c\left(\sum\limits_{j=1}^{p-h} \alpha_j\, \alpha_{j+h} - \alpha_h\right), & 1 \le h \le p-1 \\ -\sigma^2\, b_c\, \alpha_p\,, & h = p \\ 0\,, & h > p\,. \end{cases} \tag{2.11}$$

Proof: Elementary. □

Similarly to the basic STOPBREAK model, we can, under some conditions, show the invertibility of increments (X_t). This property is analyzed from different aspects by many authors who explored the STOBREAK models. We shall do the same concerning our model in order to proceed estimation of the unknown parameters of the model and apply our model to real data. As we shall see later, only realizations of invertible Split-MA process can give strong consistent and asymptotically normal estimates.

Theorem 2.2. *The sequence* (X_t)*, defined by* (2.10)*, is invertible iff the roots* r_j*,* $j=1,...,p$*, of characteristic polynomial*

$$Q(\lambda) \;=\; \lambda^p - b_c \sum_{j=1}^{p} \alpha_j \, \lambda^{p-j}$$

satisfy the condition $|r_j| < 1$*,* $j=1,...,p$*, or, equivalently,* $b_c \sum\limits_{j=1}^{p} \alpha_j < 1$*. Then,*

$$\varepsilon_t \;=\; \sum_{k=0}^{\infty} \omega_k(t) \, X_{t-k} \,, \qquad t \in \mathbf{Z} \;, \tag{2.12}$$

where $\big(\omega_k(t)\big)$ *is the solution of stochastic difference equation*

$$\omega_k(t) \;=\; \theta_{t-k} \sum_{j=1}^{p} \alpha_j \, \omega_{k-j}(t) \,, \qquad k \geq p \,, \quad t \in \mathbf{Z} \;, \tag{2.13}$$

with the initial conditions $\omega_0(t) = 1$*,* $\omega_k(t) = \theta_{t-k} \sum_{j=1}^{k} \alpha_j \, \omega_{k-j}(t)$*,* $1 \leq k \leq p-1$*. Morever, the representation* (2.12) *is almost surely unique and the sum converges with probability one and in the mean-square sense.*

Proof: First of all, for any $t \in \mathbf{Z}$, we define the vectors and matrices

$$\mathbf{V}_t \;=\; \big(\varepsilon_t \;\; \varepsilon_{t-1} \;\; \cdots \;\; \varepsilon_{t-p+1} \big)' , \qquad \mathbf{X}_t \;=\; \big(X_t \;\; 0 \;\; \cdots \;\; 0 \big)' ,$$

$$\mathbf{A}_t \;=\; \begin{pmatrix} \alpha_1 \theta_t & \alpha_2 \theta_{t-1} & \cdots & \alpha_{p-1} \theta_{t-p+2} & \alpha_p \theta_{t-p+1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} ,$$

and we can write the model (2.10) in the form of stochastic difference equation of order one

$$\mathbf{V}_t \;=\; \mathbf{A}_{t-1} \mathbf{V}_{t-1} + \mathbf{X}_t \,, \qquad t \in \mathbf{Z} \;. \tag{2.14}$$

From here, we have

$$\mathbf{V}_t \;=\; \mathbf{X}_t + \sum_{j=1}^{k}\Big(\mathbf{A}_{t-1}\cdots\mathbf{A}_{t-j}\Big)\mathbf{X}_{t-j} + \left(\prod_{j=1}^{k+1}\mathbf{A}_{t-j}\right)\mathbf{V}_{t-k-1}\ ,$$

where $k = 1, 2, \ldots$. It can be proven (see, for instance Francq *et al.* [5]) that the existence of almost sure unique, stationary solution of equation (2.14), in the form

$$\mathbf{V}_t \;=\; \mathbf{X}_t + \sum_{k=1}^{\infty}\Big(\mathbf{A}_{t-1}\cdots\mathbf{A}_{t-k}\Big)\mathbf{X}_{t-k}\,, \qquad t \in \mathbf{Z}\ , \tag{2.15}$$

is equivalent to the convergence

$$\prod_{j=1}^{k+1}\mathbf{A}_{t-j} \xrightarrow{\text{a.s.}} 0\,, \qquad k \longrightarrow \infty\ ,$$

i.e., to the fact that the eigenvalues r_j, $j = 1, ..., p$, of the matrix

$$\mathbf{A} \;=\; E(\mathbf{A}_t) \;=\; \begin{pmatrix} \alpha_1 b_c & \alpha_2 b_c & \cdots & \alpha_{p-1} b_c & \alpha_p b_c \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

satisfy the conditions $|r_j| < 1$, $j = 1, ..., p$. According to the representation

$$\det\big(\mathbf{A} - \lambda\mathbf{I}\big) \;=\; (-1)^p\, Q(\lambda)\ ,$$

it is obvious that the eigenvalues r_j, $j = 1, ..., p$, are the roots of the characteristic polynomial $Q(\lambda)$. Then, the condition $|r_j| < 1$, $j = 1, ..., p$, is necessary and sufficient for the almost sure uniqueness of the representation (2.15), and the almost sure convergence of the corresponding sum. In the similar way, we can prove that the same conditions are equivalent to the mean square convergence of the sum in (2.15). From this point on, by simple computation, we can obtain the equations (2.12) and (2.13). □

Based on the proposition above, it is clear that the presence of the sequence (θ_t) in (2.10) enables the conditions of invertibility of increments (X_t) to be weaker than corresponding conditions that are related to the stationarity of the series (y_t) and (m_t) (see, for more details [17]). In that way, even non-stationary time series (y_t) and (m_t) can form invertible Split-MA process which is always stationar. This situation is particularly interesting in the case of so called *integrated* (*standardized*) time series, where

$$\sum_{j=1}^{p}\alpha_j = 1\ . \tag{2.16}$$

If the value of parameter b_c is non-trivial, i.e., $b_c \in (0,1)$, then the sequence (X_t) will be invertible although (y_t) and (m_t) are non-stationary time series. We will further assume that the "normality condition" (2.16) is always fulfilled, because it will be of dual importance below. Primarily, the condition (2.16), which defines series (y_t) and (m_t) as the non-stationary ones, allows us, as opposed to the stationary case, that these two series have non-zero means, which is particularly important in applications (see, for instance, Section 5). Finally, another reason for introducing the "normality condition" (2.16) lays in simplifying the estimation procedure of unknown parameters of GSB model. As the sequences (y_t) and (X_t), in general, do not depend on the coefficients $\alpha_1, ..., \alpha_p$ only, but also of the critical level of reaction $c > 0$, the presumption (2.16) will be an additional "functional" relationship between the unknown parameters which allows us to compute them uniquely. In the next section there will be more discussion about such an idea of parameters estimation when GSB model is standardized GSB model.

3. PARAMETERS ESTIMATION

Procedure of estimation of the unknown parameters $\alpha_1, ..., \alpha_p, c$ of GSB model will be based on the realization of a stationary Split-MA(p) process (X_t). For this purpose, we suppose that $X_1, ..., X_T$ is the part of a realization of this time series for which we define two kinds of estimates. First, equating the covariance $\gamma_X(h),\ h = 0, ..., p$, defined by equality (2.11), with its empirical value

$$\hat{\gamma}_X(h) \,=\, \frac{1}{T-h} \sum_{t=1}^{T-h} X_t\, X_{t+h}\,, \qquad h = 0, ..., p\ ,$$

we get the, so called, *initial estimates* of unknown parameters. We denote these estimates with $\widetilde{\alpha}_1, ..., \widetilde{\alpha}_p$, respectively, i.e., in the case of critical value, with $\widetilde{c}$. Obviously, these are continuous functions of estimates $\hat{\gamma}_X(h)$, and according to the well known properties of continuity the almost sure convergence and the convergence in distribution (see, for instance Serfling [14]) it can be easily proved that $\widetilde{\alpha}_1, ..., \widetilde{\alpha}_p, \widetilde{c}$ are the consistent and the asymptotically normal estimates of $\alpha_1, ..., \alpha_p, c$.

In spite of good stochastic properties of these estimates, it can be proven that these are not the efficient estimates of unknown parameters. In order to achieve better estimates of unknown parameters we introduce *the regression estimates* $\hat{\alpha}_1, ...\hat{\alpha}_p, \hat{c}$ based on the regression of sequence

$$W_t \,=\, \sum_{j=1}^{p} \alpha_j\, \theta_{t-j+1} W_{t-j} + \varepsilon_{t-1}\,, \qquad t \in \mathbf{Z}\ . \tag{3.1}$$

For this reason, we will firstly show that the necessary and sufficient stationarity conditions of a series of (W_t) are equivalent to the conditions of invertibility of (X_t), described in Theorem 2.2.

Theorem 3.1. *The sequence* (W_t)*, defined by* (3.1)*, is the stationary and ergodic iff the roots* $r_j,\ j=1,...,p,$ *of characteristic polynomial*

$$Q(\lambda) \;=\; \lambda^p - b_c \sum_{j=1}^{p} \alpha_j\, \lambda^{p-j}$$

satisfy the condition $|r_j| < 1,\ j=1,...,p,$ *or, equivalently,* $b_c \sum\limits_{j=1}^{p} \alpha_j < 1.$

Proof: If we introduce the vectors and matrices

$$\mathbf{W}_t \;=\; \big(\, W_t \;\, W_{t-1} \;\cdots\; W_{t-p+1} \,\big)', \qquad \mathbf{E}_t \;=\; \big(\, \varepsilon_t \;\, 0 \;\cdots\; 0 \,\big)',$$

$$\mathbf{A}_t \;=\; \begin{pmatrix} \alpha_1\theta_t & \alpha_2\,\theta_{t-1} & \cdots & \alpha_{p-1}\theta_{t-p+2} & \alpha_p\,\theta_{t-p+1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

then the equality (3.1) can be written in the form of recurrent relation

$$\mathbf{W}_t \;=\; \mathbf{A}_t \mathbf{W}_{t-1} + \mathbf{E}_{t-1}\,, \qquad t \in \mathbf{Z}\;. \tag{3.2}$$

From here, completely analogously to the Theorem 2.2, it can be shown that the equation (3.2) has the strictly stationary, almost sure unique and ergodic solution

$$\mathbf{W}_t \;=\; \mathbf{E}_{t-1} + \sum_{k=1}^{\infty} \Big(\mathbf{A}_t \cdots \mathbf{A}_{t-k+1}\Big)\,\mathbf{E}_{t-k-1}\,, \qquad t \in \mathbf{Z}\;,$$

if and only if the eigenvalues $r_1,...,r_p$ of matrix $\mathbf{A} = E(\mathbf{A}_t)$ satisfy the condition $|r_j| < 1,\ j=1,...,p.$ □

Now, let us define, using the procedure described in [11], the residual sequence

$$R_t \;=\; W_t - \sum_{j=1}^{p} a_j\, W_{t-j}\,, \qquad t \in \mathbf{Z}\;, \tag{3.3}$$

where we denoted $a_j = b_c\,\alpha_j$, $j=1,...,p$. We shall prove the following proposition.

Theorem 3.2. *If the sequence* (W_t)*, defined by* (3.1) *is stationary, then the sequence* (R_t)*, defined by* (3.3) *is the sequence of uncorrelated random variables.*

Proof: If we introduce the vectors $\mathbf{R}_t = \begin{pmatrix} R_t & 0 & \cdots & 0 \end{pmatrix}'$, $t \in \mathbf{Z}$, then it is valid

$$\mathbf{R}_t = \mathbf{W}_t - \mathbf{A}\,\mathbf{W}_{t-1}\,, \qquad t \in \mathbf{Z}\ ,$$

where $\mathbf{W}_t$ and $\mathbf{A}$ are the matrices that we defined earlier. For arbitrary $h > 0$ the covariance matrix $\mathbf{\Gamma}_{\mathbf{R}}(h) \overset{\text{def}}{=} E(\mathbf{R}_t\mathbf{R}'_{t-h})$ of the vectors $(\mathbf{R}_t)$ can be written as

(3.4) $$\mathbf{\Gamma}_{\mathbf{R}}(h) = \mathbf{\Gamma}_{\mathbf{W}}(h) - \mathbf{A}\,\mathbf{\Gamma}_{\mathbf{W}}(h+1) - \mathbf{A}\,\mathbf{\Gamma}_{\mathbf{W}}(h-1) + \mathbf{A}\,\mathbf{\Gamma}_{\mathbf{W}}(h)\mathbf{A}'\,,$$

where

$$\mathbf{\Gamma}_{\mathbf{W}}(h) = \begin{pmatrix} \gamma_W(h) & \gamma_W(h+1) & \cdots & \gamma_W(h+p-1) \\ \gamma_W(h-1) & \gamma_W(h) & \cdots & \gamma_W(h+p-2) \\ \vdots & \vdots & & \vdots \\ \gamma_W(h-p+1) & \gamma_W(h-p+2) & \cdots & \gamma_W(h) \end{pmatrix}$$

is covariance matrix of the vector series $(\mathbf{W}_t)$, and

$$\gamma_W(h) = E(W_t W_{t-h})\,, \qquad \gamma_W(-h) = \gamma_W(h)$$

is covariance of the stationary series (W_t). Using simple calculation it can be shown that there is $\mathbf{A}\,\mathbf{\Gamma}_{\mathbf{W}}(h) = \mathbf{\Gamma}_{\mathbf{W}}(h)\mathbf{A}' = \mathbf{\Gamma}_{\mathbf{W}}(h+1)$, and by substituting this equality in (3.4) immediately follows $\mathbf{\Gamma}_{\mathbf{R}}(h) = \mathbf{O}_{p\times p}$. □

Notice that in the equality (3.3) we defined the sequence $\big(W_t\big)$ as a linear autoregressive process of order p, with noise $\big(R_t\big)$. Then, by using standard regression procedure, we can obtain the estimate $\hat{\mathbf{a}}_T = (\hat{a}_1, ..., \hat{a}_p)'$ of parameter $\mathbf{a} = (a_1, ..., a_p)'$, in the form of equality

(3.5) $$\hat{\mathbf{a}}_T = \mathbf{W}_T^{-1} \cdot \mathbf{b}_T\ ,$$

where

$$\mathbf{W}_T = \begin{pmatrix} \sum\limits_{t=p+1}^{T} W_{t-1}^2 & \sum\limits_{t=p+1}^{T} W_{t-1}W_{t-2} & \cdots & \sum\limits_{t=p+1}^{T} W_{t-1}W_{t-p} \\ \sum\limits_{t=p+1}^{T} W_{t-1}W_{t-2} & \sum\limits_{t=p+1}^{T} W_{t-2}^2 & \cdots & \sum\limits_{t=p+1}^{T} W_{t-2}W_{t-p} \\ \vdots & \vdots & & \vdots \\ \sum\limits_{t=p+1}^{T} W_{t-1}W_{t-p} & \sum\limits_{t=p+1}^{T} W_{t-2}W_{t-p} & \cdots & \sum\limits_{t=p+1}^{T} W_{t-p}^2 \end{pmatrix},$$

$$\mathbf{b}_T = \left(\sum_{t=p+1}^{T} W_t W_{t-1} \ \sum_{t=p+1}^{T} W_t W_{t-2} \ \cdots \ \sum_{t=p+1}^{T} W_t W_{t-p} \right)'.$$

Now we are able to show the asymptotic properties of obtained estimates. In this order, we define the set

$$\mathcal{A} = \left\{ \mathbf{a} = (a_1, ..., a_p)' \in \mathbf{R}^p \,\middle|\, \sum_{j=1}^{p} a_j < 1 \right\}$$

which, obviously, is a set of parameter values $\mathbf{a}$ for which the invertibility condition of Split-MA process (X_t), i.e. the stationarity condition of the series (W_t) is satisfied. In the mentioned assumptions, the following assertion is valid.

Theorem 3.3. *Let, for some $T_0 > 0$ and any $T \geq T_0$, the condition $\hat{\mathbf{a}}_T \in \mathcal{A}$ is fulfilled. Then $\hat{\mathbf{a}}_T$ is strictly consistent and asymptotically normal estimate of parameter $\mathbf{a} \in \mathbf{R}^p$.*

Proof: According to equality (3.5), it is valid

$$\hat{\mathbf{a}}_T - \mathbf{a} = \left(\frac{1}{T-p} \mathbf{W}_T \right)^{-1} \cdot \left(\frac{1}{T-p} \mathbf{r}_T \right), \tag{3.6}$$

where

$$\mathbf{r}_T = \left(\sum_{t=p+1}^{T} R_t W_{t-1} \ \sum_{t=p+1}^{T} R_t W_{t-2} \ \cdots \ \sum_{t=p+1}^{T} R_t W_{t-p} \right)'.$$

According to ergodicity of (W_t), which is valid to the set $\mathcal{A}$, follows the ergodicity of residuals (R_t). Then, under conditions of the theorem, using the ergodic theorem we have

$$\frac{1}{T-p} \mathbf{W}_T \xrightarrow{\text{a.s.}} \mathbf{D}, \qquad T \to \infty, \tag{3.7}$$

where $\mathbf{D} = E(\mathbf{g}_t \mathbf{g}_t')$, $\mathbf{g}_t = \left(W_{t-1} \ \cdots \ W_{t-p} \right)'$ and the moment-matrix $\mathbf{D}$ does not depend on $t \in \mathbf{Z}$, for any $\mathbf{a}$ from the stationarity set $\mathcal{A}$. According to ergodic theorem, also, it is valid that

$$\frac{1}{T-p} \mathbf{r}_T \xrightarrow{\text{a.s.}} \mathbf{0}_{p\times 1}, \qquad T \to \infty,$$

and these two convergences, applied to equality (3.6), give

$$\hat{\mathbf{a}}_T - \mathbf{a} \xrightarrow{\text{a.s.}} \mathbf{0}_{p\times 1}, \qquad T \to \infty, \tag{3.8}$$

i.e. the strict consistency of estimate $\hat{\mathbf{a}}_T$ is proved.

Note further that the decomposition (3.6) can be written in the form of equality

$$\sqrt{T-p}\,(\hat{\mathbf{a}}_T - \mathbf{a}) = \left(\frac{1}{T-p} \mathbf{W}_T \right)^{-1} \cdot \left(\frac{1}{\sqrt{T-p}} \mathbf{r}_T \right). \tag{3.9}$$

As for any $\mathbf{c} = (c_1 \cdots c_p)' \in \mathbf{R}^p$ the sequence

$$\mathbf{c}'\mathbf{r}_T = \sum_{t=p+1}^{T} R_t \left(\sum_{j=1}^{p} c_j W_{t-j} \right)$$

is martingale, using the central limit theorem for martingale ([1]), we have

$$\frac{1}{\sqrt{T-p}}\, \mathbf{c}'\mathbf{r}_T \xrightarrow{d} \mathcal{N}(0, \mathbf{c}'\mathbf{\Lambda}\mathbf{c}), \qquad T \to \infty ,$$

where $\mathbf{\Lambda} = E(\mathbf{u}_t \mathbf{u}_t')$, $\mathbf{u}_t = R_t \big(W_{t-1} \cdots W_{t-p}\big)'$ and $\mathbf{\Lambda}$ does not depend on t, for any $\mathbf{a} \in \mathcal{A}$. Now, using this convergence and the Cramér–Wold decomposition, we get

$$\frac{1}{\sqrt{T-p}}\, \mathbf{r}_T \xrightarrow{d} \mathcal{N}(0, \mathbf{\Lambda}), \qquad T \to \infty .$$

Finally, according to (3.7) it is valid

$$\big(T-p\big)\, \mathbf{W}_T^{-1} \xrightarrow{\text{a.s.}} \mathbf{D}^{-1}, \qquad T \to \infty ,$$

and, according to equality (3.9) and the last two convergences, we obtain

$$\sqrt{T-p}\, \big(\hat{\mathbf{a}}_T - \mathbf{a}\big) \xrightarrow{d} \mathcal{N}\big(0, \mathbf{D}^{-1}\mathbf{\Lambda}\, \mathbf{D}^{-1}\big), \qquad T \to \infty , \tag{3.10}$$

thus the theorem is completely proved. □

According to the obtained estimate $\hat{\mathbf{a}}_T$, under the condition (2.16), the estimates of unknown parameters $\alpha_1, ..., \alpha_p$, b_c of GSB processes can be expressed as

$$\begin{cases} \hat{b}_c = \sum\limits_{j=1}^{p} \hat{a}_j \\ \hat{\alpha}_j = \hat{a}_j \hat{b}_c^{-1}, \qquad j = 1, ..., p . \end{cases} \tag{3.11}$$

Using the showed properties of estimate $\hat{\mathbf{a}}_T$, we can prove that the estimates of "true" parameters have similar properties, which we formulate in the following assertion.

Theorem 3.4. *Let $\hat{\vartheta}_T = (\hat{\alpha}_1, ..., \hat{\alpha}_p, \hat{b}_c)'$ be estimate of the unknown parameter $\vartheta = (\alpha_1, ..., \alpha_p, b_c)' \in \mathbf{R}^{p+1}$, obtained according to estimate $\hat{\mathbf{a}}_T$ and equality* (3.11). *If, for some $T_0 > 0$ and any $T \geq T_0$ is satisfied the condition*

$$\hat{b}_c \sum_{j=1}^{p} \hat{\alpha}_j < 1 ,$$

then $\hat{\vartheta}_T$ is strictly consistent and asymptotically normal estimate of ϑ.

Proof: According to convergence (3.8) and the continuity properties of almost sure convergence (see, for instance Serfling [14]) it is obviously valid that

$$\hat{\vartheta}_T - \vartheta \xrightarrow{\text{a.s.}} \mathbf{0}_{p\times 1}\,, \qquad T \to \infty\ .$$

Notice that, for any $\mathbf{a} = (a_1, ..., a_p)' \in \mathbf{R}^p$, the expression (3.11) defines a mapping $g\colon \mathbf{R}^p \to \mathbf{R}^{p+1}$, by

$$\vartheta \ = \ g(\mathbf{a}) \ = \ \left(a_1 \left(\sum_{j=1}^{p} a_j \right)^{-1}, \ \cdots, \ a_p \left(\sum_{j=1}^{p} a_j \right)^{-1}, \ \sum_{j=1}^{p} a_j \right)'.$$

Then, applying the convergence (3.10) and continuity properties of asymptotically normal distributed random vectors (see, for instance Serfling [14]) we have

$$\sqrt{T-p}\,\left(\hat{\vartheta}_T - \vartheta\right) \xrightarrow{\ d\ } \mathcal{N}(0, \mathbf{V})\,, \qquad T \to \infty\ ,$$

where $\mathbf{V} = \mathbf{G}\mathbf{D}^{-1}\mathbf{\Lambda}\,\mathbf{D}^{-1}\mathbf{G}'$ and $\mathbf{G} = \left(\dfrac{\partial g(\mathbf{a})}{\partial \mathbf{a}} \right)\Big|_{\mathbf{a}=\hat{\mathbf{a}}}$. □

At the end of this section, let us remark some more facts that directly follow from the estimation procedure described above and the theorems we have just proven.

Remark 3.1. Asymptotic variances $\mathbf{D}^{-1}\mathbf{\Lambda}\mathbf{D}^{-1}$ and $\mathbf{V}$ are commonly used as measures of bias of estimates $\hat{a}_T$ and $\hat{\vartheta}_T$, compared to the true values of parameters $\mathbf{a} = (a_1, ..., a_p)'$ and $\vartheta = (\alpha_1, ..., \alpha_p, b_c)$, where $\sum_{j=1}^{p} a_j = b_c$. Based on the introduced assumptions and the proof of previous theorem we have that

$$\mathbf{G}(\mathbf{a}) \ = \ \left(\frac{\partial g(\mathbf{a})}{\partial \mathbf{a}} \right) \ = \ \begin{pmatrix} b_c^{-1} - a_1 b_c^{-2} & -a_1 b_c^{-2} & \cdots & -a_1 b_c^{-2} \\ -a_2 b_c^{-2} & b_c^{-1} - a_2 b_c^{-2} & \cdots & -a_2 b_c^{-2} \\ \vdots & \vdots & & \vdots \\ -a_p b_c^{-2} & -a_p b_c^{-2} & \cdots & b_c^{-1} - a_p b_c^{-2} \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

and taking the standard matrix norms, we conclude that for all $\mathbf{a} \in \mathbf{R}^p$ is valid

$$\|\mathbf{G}(\mathbf{a})\|_1 \ = \ \left\|\mathbf{G}(\mathbf{a})'\right\|_\infty \ = b_c^{-1} - b_c^{-2} \sum_{j=1}^{p} a_j + 1 \ = \ 1\ .$$

Then, according to the previous equalities, for any matrix norm $\|\cdot\|$ which is sub–multiplicative to $\|\cdot\|_1$ and $\|\cdot\|_\infty$, we get

$$\|\mathbf{V}\| \ = \ \left\|\mathbf{G}(\hat{\mathbf{a}})\,\mathbf{D}^{-1}\mathbf{\Lambda}\mathbf{D}^{-1}\,\mathbf{G}(\hat{\mathbf{a}})'\right\| \ \le \ \left\|\mathbf{D}^{-1}\mathbf{\Lambda}\mathbf{D}^{-1}\right\|\ .$$

Therefore, asymptotic variance of estimates $\hat{\vartheta}_T$, obtained by (3.11), does not exceed the asymptotic variance of $\hat{a}_T$, under which they were calculated. In this way, the method of parameters estimation of the GSB model is formally justified.

Remark 3.2. If we apply estimates $\widetilde{b}_c$ and $\hat{b}_c$, we can make a modeled values of (ε_t), and thereby, we can estimate the variance σ^2 of the sequence (ε_t). To do this, we can use the sample variance, i.e. the estimates

$$\widetilde{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2(X,\widetilde{\theta}) \qquad \text{or} \qquad \hat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2(X,\hat{\theta}) \,, \tag{3.12}$$

where $\varepsilon_t(X,\widetilde{\theta})$ and $\varepsilon_t(X,\hat{\theta})$ are modeled values of the white noise which we obtained by applying estimates $\widetilde{b}_c$ and $\hat{b}_c$, respectively (see, for more details, the following section). In the case of the Gaussian noise (ε_t), these estimates are identical to those which we can get applying the maximum likelihood procedure, as it was shown in Stojanović *et al.* [16]. Namely, under the assumption that (ε_t) is the Gaussian white noise, the log-likelihood function will be

$$L(y_1,...,y_T;\sigma^2) \;=\; -\frac{T}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(y_t - m_t\right)^2 ,$$

and we can easily see that the estimated value of the variance is identical to the sample variance (3.12) of the series (ε_t). The consistency and asymptotic normality of estimates $\widetilde{\sigma}$ and $\hat{\sigma}$ can be easily shown.

4. MONTE CARLO STUDY OF THE MODEL

In this section we will demonstrate some applications of the above described estimation procedure of Split-MA(1) and Split-MA(2) models. For the white noise (ε_t) it was used a simple random sample from with Gaussian $\mathcal{N}(0,1)$ distribution, so that the elements of the sequence (ε_t^2) were χ_1^2 distributed, which has been used for solving the critical value of the reaction $\widetilde{c}$ and $\hat{c}$.

In the case of Split-MA(1) process, as it is shown in Stojanović *et al.* [16], these estimates are based on 100 independent Monte Carlo simulations for each sample size $T = 50$, $T = 100$ and $T = 500$ for the model

$$X_t = \varepsilon_t - \theta_{t-1}\varepsilon_{t-1} \,, \qquad t = 1,...,T \,, \tag{4.1}$$

where $\theta_t = I(\varepsilon_{t-1}^2 \le 1)$ and $\varepsilon_0 = \varepsilon_{-1} \overset{\text{a.s.}}{=} 0$. Firstly, according to the correlation

$$\rho(h) \;=\; \operatorname{Corr}(X_{t+h},X_t) \;=\; \begin{cases} 1\,, & h=0 \\ -b_c/(b_c+1)\,, & h=\pm 1 \\ 0\,, & \text{otherwise}\,. \end{cases} \tag{4.2}$$

it obtained the estimates $\widetilde{b}_c = -\hat{\rho}_T(1)\left(1+\hat{\rho}_T(1)\right)^{-1}$, where $\hat{\rho}_T(1)$ is the empirical first correlation of the sequence (X_t). After that, solving the equation $P\{\varepsilon_t^2 \le c\}$ $= \widetilde{b}_c$ with respect to c, we obtained the estimates for the critical value $\widetilde{c}$.

In the next estimation stage, using the equality (4.1) in the "functional" form $\varepsilon_t(X,\theta) = X_t + \theta_{t-1}\,\varepsilon_{t-1}(X,\theta)$ and $\widetilde{b}_c$ as the initial (estimated) value of the parameter $b_c \in (0,1)$, the regression estimates $\hat{b}_c$ are computed as

$$\hat{b}_c = \left[\sum_{t=0}^{T-1} W_{t+1}(X,\widetilde{\theta})\, W_t(X,\widetilde{\theta})\right] \cdot \left[\sum_{t=0}^{T-1} W_t^2(X,\widetilde{\theta})\right]^{-1}, \tag{4.3}$$

where

$$W_t(X,\widetilde{\theta}) = \widetilde{\theta}_t\, W_{t-1}(X,\widetilde{\theta}) + \varepsilon_{t-1}(X,\widetilde{\theta})\,, \qquad \widetilde{\theta}_t = I(\varepsilon_{t-1}^2 \le \widetilde{c})\ ,$$

and $\varepsilon_0(X,\widetilde{\theta}) = \varepsilon_{-1}(X,\widetilde{\theta}) \equiv 0$. Finally, the regression estimates of the (true) critical value $c = 1$ are solutions of the equation $P\{\varepsilon_t^2 \le c\} = \hat{b}_c$ with respect to c. The average values of these estimates are set, together with the correspondent estimating errors (the numeric values set in the parentheses) in the rows of Table 1.

Table 1: Estimated values of Monte Carlo simulations of the Split-MA(1) process.

Sample size	**Averages of estimated values**						
	$\hat{\rho}_T(1)$	$\widetilde{b}_c$	$\widetilde{c}$	$\hat{b}_c$	$\hat{c}$	$\widetilde{\sigma}^2$	$\hat{\sigma}^2$
$T = 50$	−0.376 (0.139)	0.614 (0.219)	0.894 (0.726)	0.647 (0.192)	0.944 (0.571)	1.216 (0.292)	1.042 (0.202)
$T = 100$	−0.386 (0.097)	0.634 (0.156)	0.894 (0.444)	0.671 (0.141)	1.039 (0.427)	1.168 (0.184)	1.016 (0.124)
$T = 500$	−0.394 (0.056)	0.664 (0.091)	0.916 (0.259)	0.676 (0.068)	0.992 (0.194)	1.135 (0.102)	0.997 (0.099)
True values	**−0.406**	**0.683**	**1.000**	**0.683**	**1.000**	**1.000**	**1.000**

The second column of Table 1 contains the estimated values of the coefficient of the first correlation $\hat{\rho}_T(1)$ of the Split-MA(1) model. The average values of that column are somewhat smaller in the absolute value of the true value, which is the case here $\rho(1) = -b_c(1+b_c)^{-1} \approx -0.406$. In that way, estimates $\widetilde{b}_c$ and $\widetilde{c}$, showed in the next two columns, will be a proper estimates if $-0,5 < \hat{\rho}_T(1) < 0$. Then, we showed the regression estimates $\hat{b}_c$ and $\hat{c}$ which average values are closer to the true value than previously mentioned, initial estimates $\widetilde{c}$. This is due to the fact, formally proved in Remark 3.1, that estimates $\hat{c}$ are more efficient than $\widetilde{c}$. The histograms of empirical distributions of the estimates $\widetilde{c}$ and $\hat{c}$ are shown in Figure 1. It can be seen that $\hat{c}$ has the asymptotically normal distribution even for the sample of a "small" sample size. Finally, the averages of estimated values of σ^2, based on modeled values of the white noise (ε_t) and equations (3.12), are displayed in the last two columns of Table 1. Their average values differ from the true value $\sigma^2 = 1$ as a consequence of two stage estimating procedure that was used. In spite of that, it can be seen that the average values of $\hat{\sigma}^2$ are closer to the true value one than the average values of $\widetilde{\sigma}^2$.

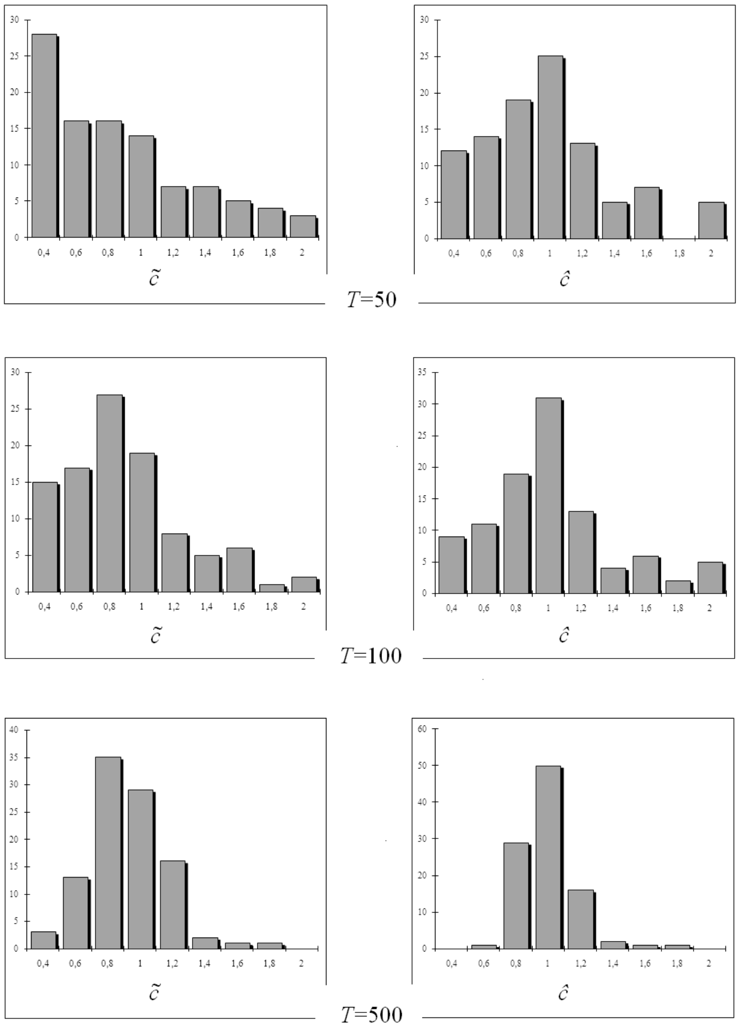

Figure 1: Empirical distributions of estimated parameters $\widetilde{c}$ and $\hat{c}$ of Split-MA(1) model

As in the case of of Split-MA(1) process, we are able to apply the above procedure for estimating the unknown parameters of Split-MA(2) process

$$X_t = \varepsilon_t - \alpha_1\,\theta_{t-1}\,\varepsilon_{t-1} - \alpha_2\,\theta_{t-2}\,\varepsilon_{t-2}\,, \qquad t = 1, ..., T\,. \tag{4.4}$$

For this purpose, we used 45 independent Monte Carlo simulations of this series of the length $T = 500$, with values of parameters $\alpha_1 = 0.6$, $\alpha_2 = 0.4$ and $c = 1$.

Firstly, by equating the correlation functions $\gamma_X(h)$ of the model above with its empirical correlations $\hat{\gamma}_X(h)$, where $h = 0, 1, 2$, we get the estimate

$$\widetilde{\alpha}_1 = \frac{-\hat{\gamma}_X(1) - \sqrt{\hat{\gamma}_X^2(1) + 4\,\hat{\gamma}_X(1)\,\hat{\gamma}_X(2)}}{2\,\hat{\gamma}_X(2)}\,,$$

$$\widetilde{\alpha}_2 = 1 - \widetilde{\alpha}_1 = \frac{2\,\hat{\gamma}_X(1) + \hat{\gamma}_X(1) + \sqrt{\hat{\gamma}_X^2(1) + 4\,\hat{\gamma}_X(1)\,\hat{\gamma}_X(2)}}{2\,\hat{\gamma}_X(2)}\,,$$

$$\widetilde{b}_c = \frac{-\hat{\gamma}_X(2)}{\left(\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2\right)\hat{\gamma}_X(2) + \tilde{\alpha}_2\,\hat{\gamma}_X(0)}\,.$$

After that, using $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$ and $\widetilde{c}$ as initial estimates we can generate the sequences

$$\begin{cases} \varepsilon_t(X,\widetilde{\theta},\widetilde{\alpha}) = X_t + \widetilde{\alpha}_1\widetilde{\theta}_{t-1}\,\varepsilon_{t-1}(X,\widetilde{\theta},\widetilde{\alpha}) + \widetilde{\alpha}_2\,\widetilde{\theta}_{t-1}\,\varepsilon_{t-2}(X,\widetilde{\theta},\widetilde{\alpha}) \\ \widetilde{\theta}_t = I\left(\varepsilon_{t-1}^2(X,\widetilde{\theta},\widetilde{\alpha}) \le \widetilde{c}\right) \\ W_t(X,\widetilde{\theta},\widetilde{\alpha}) = \widetilde{\alpha}_1\widetilde{\theta}_t W_{t-1}(X,\widetilde{\theta},\widetilde{\alpha}) + \widetilde{\alpha}_2\,\widetilde{\theta}_{t-1}W_{t-2} + \varepsilon_{t-1}(X,\widetilde{\theta},\widetilde{\alpha})\,, \end{cases}$$

where $t = 1, ..., 500$ and $\varepsilon_0 = \varepsilon_{-1} \overset{\text{a.s.}}{=} 0$. Finally, according to the equalities (3.5) and (3.11) we obtain the regression estimates of appropriate parameters $\hat{\alpha}_1$, $\hat{\alpha}_2$, $\hat{c}$ of Split-MA(2) model.

Table 2: Estimated values of Monte Carlo simulations of the Split-MA(2) process.

Estimators type	**Parameters**						
	a_1	a_2	α_1	α_2	b_c	c	σ^2
Initial estimates	— —	— —	0.597 (0.043)	0.403 (0.043)	0.673 (0.075)	0.993 (0.162)	1.016 (0.219)
Regression estimates	0.421 (0.025)	0.274 (0.025)	0.605 (0.029)	0.394 (0.029)	0.685 (0.031)	1.006 (0.145)	1.045 (0.143)
True values	**0.420**	**0.273**	**0.600**	**0.400**	**0.683**	**1.000**	**1.000**

The Table 2 shows the average values of obtained estimates, corresponding estimating errors and the true values of parameters. At the first glance, there are no major differences in the quality of the obtained estimates. Moreover, regression estimates are slightly more different from the real values of the parameters, previously obtained from the initial evaluations. However, the dispersion of the regression estimates is much smaller than the dispersion of the initial estimates, and this is one of the important advantages of this estimation method. This fact is clearly visible in Figure 2, which shows the histograms of empirical distributions of both types of estimates. Obviously, the histograms of regression estimates (panels below) have much more pronounced asymptotic tendencies in relation to the initial estimates of parameters (panels above).

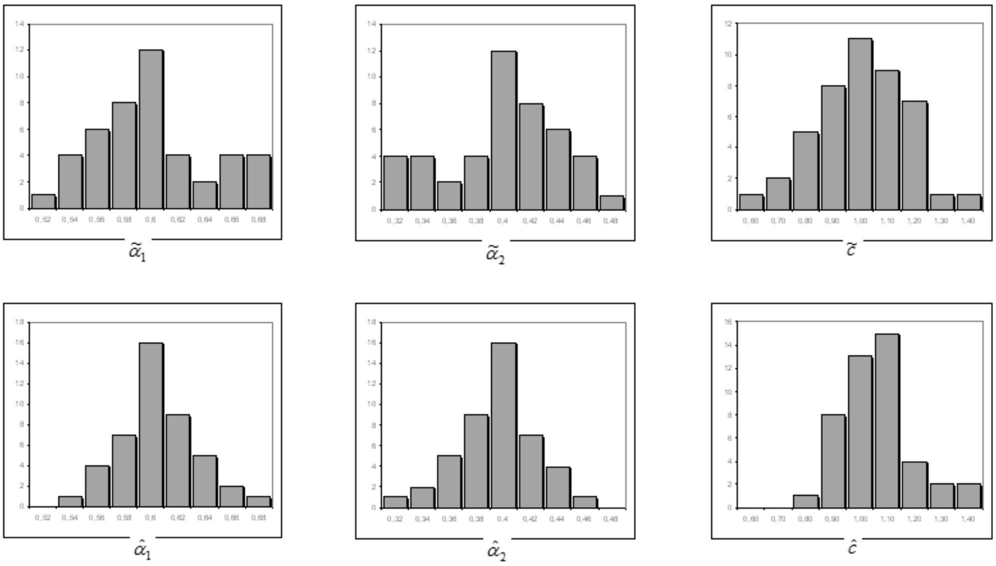

Figure 2: Empirical distributions of estimated parameters of Split-MA(2) model.

5. APPLICATION OF THE MODEL

Here we describe some of the possibilities of practical application the GSB process in the modeling dynamics of financial series. We observe Split-MA(1) and Split-MA(2) models, described by equalities (4.1) and (4.4), as stochastic models of dynamics the total values of stocks trading on the Belgrade Stock Exchange. As a basic financial sequence we observe the realization of log-volumes

$$y_t = \ln(S_t \cdot H_t)\,, \qquad t = 0, 1, ..., T\,, \tag{5.1}$$

where S_t is the share price and H_t is the volume of trading of the same share at time $t = 1, ..., T$. (The price is in dinars and the volume is the number of shares that were traded on the certain day. The days of trading are used as successive data.) Firstly, we applied iterative equations

$$\begin{cases} \varepsilon_t = y_t - m_t \\ m_t = m_{t-1} + \varepsilon_{t-1} I(\varepsilon_{t-2}^2 > \hat{c}) \end{cases}, \qquad t = 1, ..., T\,, \tag{5.2}$$

to generate the corresponding values of sequences (ε_t) and (m_t) of Split-BREAK process of order $p = 1$. As estimates of the critical value $\hat{c}$, we used the previous estimating procedure, and as initial values of the iterative procedure (5.2) we use $m_0 = y_0 = \overline{y}_T$, $\varepsilon_0 = \varepsilon_{-1} \overset{\text{a.s.}}{=} 0$, where $\overline{y}_T$ is the empirical mean of (y_t). We use the basic empirical series defined by (5.1) in solving the series of increments (X_t), i.e. the realized values of Split-MA(1) described above. The similar procedure can be used to estimate the parameters of Split-MA(2) model. In that case, we substitute the second equation in (5.2) with

$$m_t = \hat{\alpha_1}\big(m_{t-1} + \varepsilon_{t-1} I(\varepsilon_{t-2}^2 > \hat{c})\big) + \hat{\alpha_2}\big(m_{t-2}\,\varepsilon_{t-2} I(\varepsilon_{t-3}^2 > \hat{c})\big), \qquad t = 2, ..., T,$$

where $\hat{\alpha}_1, \hat{\alpha}_2$ is the estimated values of model's parameters, with $\hat{\alpha}_1 + \hat{\alpha}_2 = 1$ and $\varepsilon_0 = \varepsilon_{-1} \overset{\text{a.s.}}{=} 0$. Table 3 contains the number of observations for the company (T), and estimated values of Split-MA(1) and Split-MA(2) models in the case of six Serbian eminent companies.

Table 3: Estimated values of the GSB parameters of real data.

Companies	**Cities**	T	$p = 1$				
			$\hat{\rho}_T(1)$	$\widetilde{b}_c$	$\widetilde{c}$	$\hat{b}_c$	$\hat{c}$
HEMOFARM	Vršac	54	−0.346	0.530	0.582	0.613	0.836
METALAC	Milanovac	174	−0.449	0.816	4.929	0.829	5.223
SUNCE	Sombor	157	−0.424	0.735	2.836	0.784	3.132

Companies	**Cities**	T	$p = 2$					
			$\widetilde{\alpha}_1$	$\widetilde{\alpha}_2$	$\widetilde{c}$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{c}$
ALFA PLAM	Vranje	50	0.640	0.360	2.628	0.690	0.310	3.331
DIN	Niš	56	0.715	0.285	1.393	0.816	0.184	1.202
T. MARKOVIĆ	Kikinda	277	0.824	0.176	1.396	0.830	0.170	1.392

The following, Table 4 contains estimated values of means and variances of previously defined sequences: log–volumes (y_t), martingale means (m_t), the Split-MA process (X_t) and the white noise (ε_t). If we analyze empirical values of these series, we can recognize the relations that could be explained the theoretical results above. Namely, the averages of the log–volumes are close to the averages of martingale means, which is in accordance with (2.8), i.e. to the fact that the realizations of (y_t) are "close" to the sequence (m_t). On the other hand, the averages of (X_t) and (ε_t) are "close" to zero, which is consistent with previous theoretical results. Also, the estimates of the empirical variances of Split-MA series are generally higher than the corresponding values of the noise variances, which is consistent with the theoretical properties of these sequences (see Theorem 2.1).

Table 4: Estimated values of real data.

Companies	**Log-volumes**		**Mart. means**		**Split-MA**		**White noise**	
	Mean	Var	Mean	Var	Mean	Var	Mean	Var
HEMOFARM	15.250	0.814	15.310	0.694	0.022	1.786	−0.042	1.576
METALAC	13.665	2.788	13.798	2.731	0.001	3.979	−0.005	3.614
SUNCE	12.748	2.282	12.730	2.151	−0.024	1.978	−0.028	1.981
ALFA PLAM	15.320	1.505	15.354	1.457	−0.126	2.410	−0.005	1.590
DIN	14.485	4.998	14.628	6.071	−0.126	3.003	−0.139	2.868
T. MARKOVIĆ	13.816	2.295	13.830	1.977	0.001	4.002	−0.078	3.611

A high correlation between the log-volumes and the martingale means can be seen in Figure 3, which represents the realizations of these sequences. Obviously, this fact concurs with the definition of the GSB process, i.e., the equation (2.6), and justifies the application of the GSB process as an appropriate stochastic model of the dynamics of empirical time series.

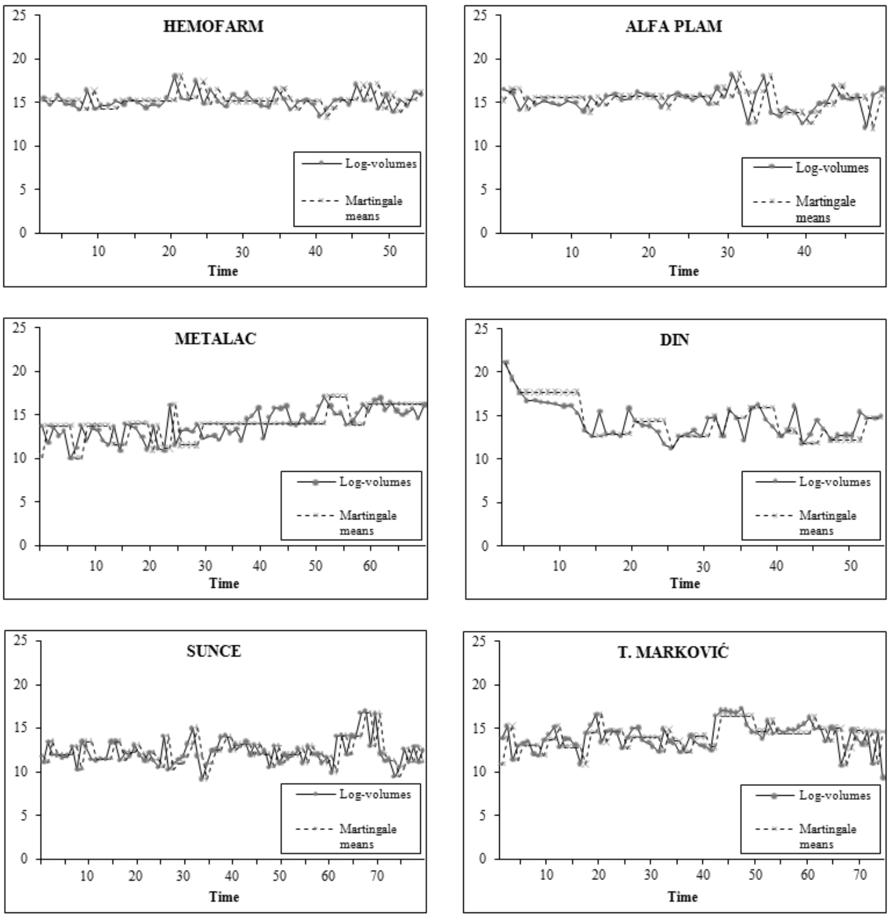

Figure 3: Comparative graphs of the real and modeled data.

Finally, Figure 4 shows that there is also a strong correlation between the white noise (ε_t) and the increments (X_t). It is clear that the concurrence of realizations of these two sequences will be better if, in addition to the great fluctuation of (X_t), the critical value of the reaction c is relatively small (see, for instance, Section 4). In fact, small values of c point out to the possibility that the true value of this parameter is $c = 0$, when increments (X_t) are equal to the noise (ε_t). In that case (y_t) is the sequence with independent increments and the whole statistical analysis is made easier. According to the previous facts about asymptotic normality of obtained estimates, testing the null hypothesis $H_0 : c = 0$, (i.e. $b_c = 0$), in the case of the "large" sample size, will be based on the normal distribution, i.e. standard, well known statistical tests.

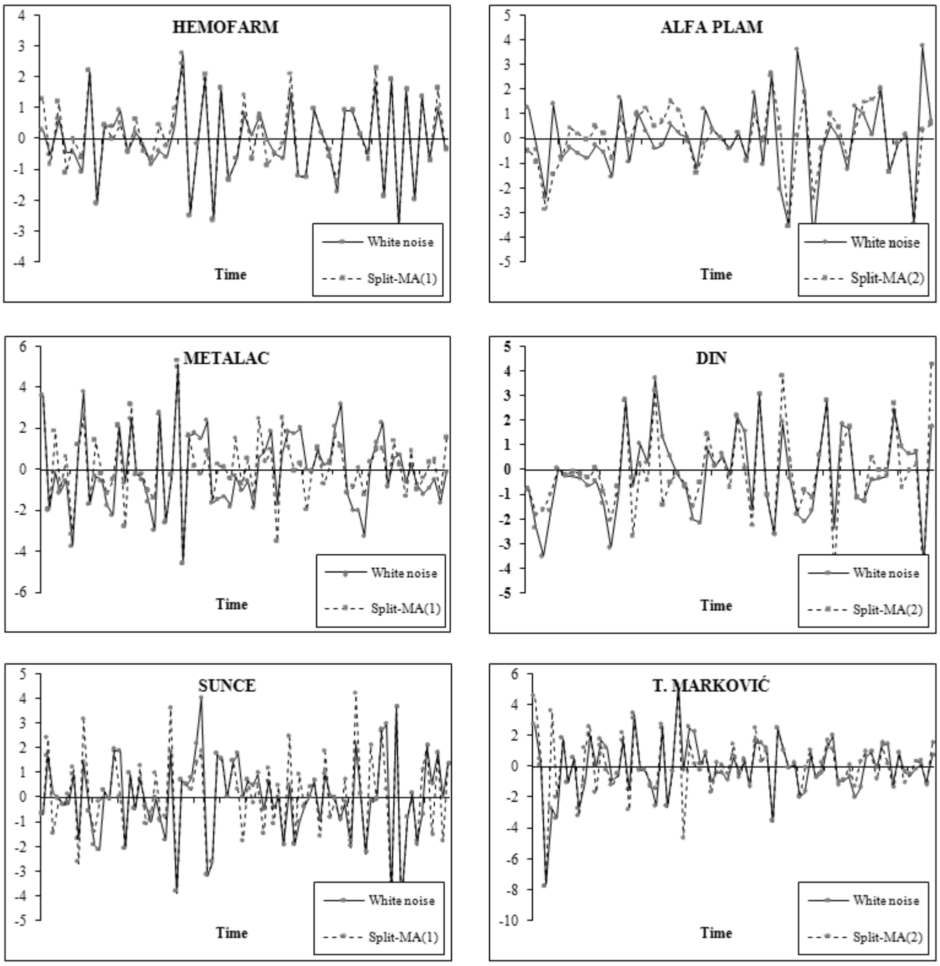

Figure 4: Comparative graphs of real and modeled data.

6. CONCLUSION

As we know, the non-linear stochastic models of financial time series usually give excellent results in explaining many aspects of their behavior. In this sense, various modifications of the STOPBREAK process enable successful description of the dynamics of financial time series with emphatic permanent fluctuations. We should point out once again Stojanović *et al.* [16], where were compared the efficiency between the simplest GSB model (named Split-BREAK model) of order $p = 1$ and some well known models which are standardly used in the real data modeling. Using the same data set as in the section above, it is shown that our process represents these time series better, and that fewer coefficients need to be estimated in comparison with well known models used so far.

ACKNOWLEDGMENTS

This research was supported in part by the Serbian Ministry of Education, Science and Technological Development (No. #OI 174007 and 174026).

REFERENCES

[1] Billingsley, P. (1961). The Lindeberg–Levy theorem for martingales, *Proceedings of the American Mathematical Society*, **12**(5), 788–792.

[2] Chan, S.K. (1993). Consistency and limiting distribution of the lest squares esimator of a thresholds autoregressive model, *Annals of Statistics*, **21**, 520–533.

[3] Diebold, F.X. (2001). Long memory and regime switching, *Journal of Econometrics*, **105**(1), 131–159.

[4] Engle, R.F. and Smith, A.D. (1999). Stochastic permanent breaks, *The Review of Economics and Statistics*, **81**, 553–574.

[5] Francq, C.; Roussignol, M. and Zakoian, M.J. (2001). Conditional heteroskedasticity driven by hidden markov chains, *Journal of Time Series Analysis*, **22**(2), 197–220.

[6] González, A. (2004). A smooth permanent surge process, *SSE/EFI Working Paper Series in Economics and Finance*, **572**, Stockholm.

[7] Gonzalo, J. and Martinez, O. (2006). Large shocks vs. small shocks (Or does size matter? May be so), *Journal of Econometrics*, **135**, 311–347.

[8] Hansen, B. (2000). Sample splitting and threshold estimation, *Econometrica*, **68**(3), 575–603.

[9] Huang, B.-N. and Fok, R.C.W. (2001). Stock market integration-an application of the stochastic permanent breaks model, *Applied Economics Letters*, **8**(11), 725–729.

[10] Kapetanios, G. and Tzavalis, E. (2010). Modeling structural breaks in economic relationships using large shocks, *Journal of Economic Dynamics and Control*, **34**(3), 417–436.

[11] Lawrence, A.J. and Lewis, P.A.W. (1992). Reversed residuals in autoregressive time series analysis, *Journal of Time Series Analysis*, **13**, 253–266.

[12] Popović, B.Č. and Stojanović, V.S. (2005). Split-ARCH, *PLISKA Studia Mathematica Bulgarica*, **17**, 201–220.

[13] Scarrott, C. and MacDonald, A. (2012). A review of extreme value threshold estimation and uncertainty quantification, *RevStat – Statistical Journal*, **10**(1), 33–60.

[14] Serfling, R.J. (1980). *Approximation Theorems of Mathematical Statistics*, John Wiley & Sons, Inc.

[15] Stojanović, V.S. and Popović, B.Č. (2007). Estimation in real data set by Split-ARCH model, *Filomat*, **21**(2), 133–152.

[16] Stojanović, V.S.; Popović, B.Č. and Popović, P.M. (2011). The Split-BREAK model, *Brazilian Journal of Probability and Statistics*, **25**(1), 44–63.

[17] Stojanović, V.S.; Popović, B.Č. and Popović, P.M. (2014). Stochastic analysis of GSB process, *Publications de l'Institut Mathématique*, **95**(109), 149–159.

[18] Tong, H. (1980). *Thresholds Models in Non-linear Time Series Analysis*, Lecture Notes in Statistics, **21**, Springer-Verlag, Berlin, Heidelberg, New York.