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NOTES ON THE REGULAR E-OPTIMAL SPRING BALANCE WEIGHING DESIGNS WITH CORRELATED ERRORS

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Abstract:

- The paper deals with the estimation problem of individual weights of objects in E-optimal spring balance weighing design. It is assumed that errors are equal correlated. The topic is focused on the determining the maximal eigenvalue of the inverse of information matrix of estimators. The constructing methods of the E-optimal spring balance weighing design based on the incidence matrices of balanced and partially balanced incomplete block designs are given.

Key-Words:

- *balanced incomplete block design; E-optimal design; partially incomplete block design; spring balance weighing design.*

AMS Subject Classification:

- 62K05, 62K15.

1. INTRODUCTION

Let us suppose we want to estimate the weights of v objects by weighing them b times using a spring balance, $v \le b$. Suppose, that the results of this experiment can be written as

$$\mathbf{y} = \mathbf{X}\mathbf{w} + \mathbf{e} \,, \tag{1.1}$$

where $\mathbf{y}$ is an $b \times 1$ random vector of the observations, $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$, where $\mathbf{\Phi}_{b\times v}(0,1)$ denotes the class of $b \times v$ matrices $\mathbf{X} = (x_{ij})$ of known elements $x_{ij} = 1$ or 0 according as in the ith weighing operation the jth object is placed on the pan or not. Next, $\mathbf{w}$ is a $v \times 1$ vector of unknown measurements of objects and $\mathbf{e}$ is a $b \times 1$ random vector of errors. We assume, that $\mathrm{E}(\mathbf{e}) = \mathbf{0}_b$ and $\mathrm{Var}(\mathbf{e}) = \sigma^2\mathbf{G}$, where $\mathbf{0}_b$ denotes the $b \times 1$ vector with zero elements everywhere, $\mathbf{G}$ is the known $b \times b$ diagonal positive definite matrix of the form

$$\mathbf{G} = g\Big[(1-\rho)\,\mathbf{I}_v + \rho\,\mathbf{1}_v\mathbf{1}_v'\Big]\,, \qquad g > 0, \;\; \frac{-1}{b-1} < \rho < 1 \,. \tag{1.2}$$

It should be noticed that the conditions on the values of g and ρ are equivalent to the matrix $\mathbf{G}$ being positive definite. From now on, we will consider $\mathbf{G}$ on the form 1.2 only. Moreover, let note, $\mathbf{G}^{-1} = \frac{1}{g(1-\rho)}\Big[\mathbf{I}_b - \frac{\rho}{1+\rho(b-1)}\,\mathbf{1}_b\mathbf{1}_b'\Big]$. For the estimation of $\mathbf{w}$ we use the normal equations $\mathbf{M}\mathbf{w} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$, where $\mathbf{M} = \mathbf{X}'\mathbf{G}^{-1}\mathbf{X}$ is called the information matrix of $\hat{\mathbf{w}}$. A spring balance weighing design is singular or nonsingular, depending on whether the matrix $\mathbf{M}$ is singular or nonsingular, respectively. From the assumption that $\mathbf{G}$ is positive definite it follows that matrix $\mathbf{M}$ is nonsingular if and only if matrix $\mathbf{X}$ is full column rank. If matrix $\mathbf{M}$ is nonsingular, then the generalized least squares estimator of $\mathbf{w}$ is given by formula $\hat{\mathbf{w}} = \mathbf{M}^{-1}\mathbf{X}'\mathbf{G}^{-1}\mathbf{y}$ and $\mathrm{Var}(\hat{\mathbf{w}}) = \sigma^2\mathbf{M}^{-1}$. Some considerations apply to determining the optimal weighing designs are shown in many books [1]. Some problems related to optimality of the designs are presented in several papers [2] for $\mathbf{G} = \mathbf{I}_n$, whereas in Katulska and Rychlińska ([9]) for the diagonal matrix $\mathbf{G}$.

In this paper, we emphasize a special interest of the existence conditions for E-optimal design, i.e. minimizing the maximum eigenvalue of the inverse of the information matrix. The statistical interpretation of E-optimality is the following: the E-optimal design minimizes the maximum variance of the component estimates of the parameters. It can be described in terms of the maximum eigenvalue of the matrix $\mathbf{M}^{-1}$ as $\lambda_{\max}(\mathbf{M}^{-1})$ or equivalently as $\lambda_{\min}(\mathbf{M})$. Hence, for the given variance matrix of errors $\sigma^2\mathbf{G}$, any design $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ is E-optimal if $\lambda_{\max}(\mathbf{M}^{-1})$ is minimal. Moreover, if $\lambda_{\max}(\mathbf{M}^{-1})$ attains the lowest bound, then $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ is called regular E-optimal. Notice that if the design

[1] See, Raghavarao ([13]), Banerjee ([1]), Shah and Sinha ([15]), Pukelsheim ([12]).

[2] Jacroux and Notz ([8]), Neubauer and Watkins ([11]).

$\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$ is regular E-optimal then is also E-optimal. But the inverse implication may not be true. Moreover, the E-optimal design in the set of all design matrices $\mathbf{\Phi}_{b\times v}(0,1)$ exists but the regular E-optimal design may not exist.

The problem presented in this paper is focused to determining such matrix that $\lambda_{\max}(\mathbf{M}^{-1})$ takes the minimal value over all possible matrices in $\mathbf{\Phi}_{b\times v}(0,1)$ for given matrix $\mathbf{G}$.

2. REGULAR E-OPTIMAL SPRING BALANCE WEIGHING DESIGN

In this section we give some new results concerning the lower bound for $\lambda_{\max}(\mathbf{M}^{-1})$ depending on ρ and number of objects v is even or odd. Additionally, let $\mathbf{\Pi}$ be the set of all $v\times v$ permutation matrices. We shall denote by $\bar{\mathbf{M}}$ the average of $\mathbf{M}$ over all elements of $\mathbf{\Pi}$, i.e. $\bar{\mathbf{M}} = \frac{1}{v!}\sum_{\mathbf{P}\in\mathbf{\Pi}} \mathbf{P}'\mathbf{M}\,\mathbf{P}$. It is not difficult to see that

$$\bar{\mathbf{M}} = \frac{v\,\mathrm{tr}(\mathbf{M}) - \mathbf{1}_v'\mathbf{M}\mathbf{1}_v}{v(v-1)}\,\mathbf{I}_v + \frac{\mathbf{1}_v'\mathbf{M}\mathbf{1}_v - \mathrm{tr}(\mathbf{M})}{v(v-1)}\,\mathbf{1}_v\mathbf{1}_v' ,$$

moreover, $\mathrm{tr}(\mathbf{M}) = \mathrm{tr}(\bar{\mathbf{M}})$ and $\mathbf{1}_v'\mathbf{M}\mathbf{1}_v = \mathbf{1}_v'\bar{\mathbf{M}}\mathbf{1}_v$. The matrix $\bar{\mathbf{M}}$ has two eigenvalues $\mu_1 = \frac{v\,\mathrm{tr}(\mathbf{M}) - \mathbf{1}_v'\mathbf{M}\mathbf{1}_v}{v(v-1)}$ with the multiplicity $v-1$ and $\mu_2 = \frac{\mathbf{1}_v'\mathbf{M}\mathbf{1}_v}{v}$ with the multiplicity 1. Let

$$\mathbf{M} = \frac{1}{g(1-\rho)}\left[\mathbf{X}'\mathbf{X} - \frac{\rho}{1+\rho(b-1)}\,\mathbf{X}'\mathbf{1}_b\mathbf{1}_b'\mathbf{X}\right]. \tag{2.1}$$

For $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ and $\mathbf{G}$ we have $\mathrm{tr}(\mathbf{M}) = \frac{1}{g(1-\rho)}\left[\mathbf{1}_v'\mathbf{r} - \frac{\rho}{1+\rho(b-1)}\,\mathbf{r}'\mathbf{r}\right]$ and $\mathbf{1}_v'\mathbf{M}\mathbf{1}_v = \frac{1}{g(1-\rho)}\left[\mathbf{k}'\mathbf{k} - \frac{\rho}{1+\rho(b-1)}\left(\mathbf{1}_b'\mathbf{k}\right)^2\right]$, where $\mathbf{X}\mathbf{1}_v = \mathbf{k}$, $\mathbf{X}'\mathbf{1}_b = \mathbf{r}$, $\mathbf{1}_v'\mathbf{r} = \mathbf{1}_b'\mathbf{k}$. From above, eigenvalues of $\bar{\mathbf{M}}$ are

$$\mu_1 = \frac{1}{v(v-1)\,g(1-\rho)}\left[v\mathbf{1}_b'\mathbf{k} - \mathbf{k}'\mathbf{k} + \frac{\rho}{1+\rho(b-1)}\left((\mathbf{1}_b'\mathbf{k})^2 - v\,\mathbf{r}'\mathbf{r}\right)\right],$$

$$\mu_2 = \frac{1}{v\,g(1-\rho)}\left[\mathbf{k}'\mathbf{k} - \frac{\rho}{1+\rho(b-1)}\left(\mathbf{1}_b'\mathbf{k}\right)^2\right].$$

Thus the matrix $\bar{\mathbf{M}}^{-1}$ has also two eigenvalues $\lambda_1 = \frac{1}{\mu_1}$ and $\lambda_2 = \frac{1}{\mu_2}$. Next, comparing these two eigenvalues we become following lemma.

Lemma 2.1. *For any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$*, the matrix* $\bar{\mathbf{M}}^{-1}$ *has two eigenvalues* λ_1 *and* λ_2 *and moreover* (**i**) $\lambda_1 > \lambda_2$ *if and only if* $\rho < \frac{\mathbf{k}'\mathbf{k}-\mathbf{1}_b'\mathbf{k}}{(b-1)\,(\mathbf{1}_b'\mathbf{k}-\mathbf{k}'\mathbf{k})+(\mathbf{1}_b'\mathbf{k})^2-\mathbf{r}'\mathbf{r}}$,

(**ii**) $\lambda_1 = \lambda_2$ *if and only if* $\rho = \frac{\mathbf{k}'\mathbf{k}-\mathbf{1}_b'\mathbf{k}}{(b-1)\,(\mathbf{1}_b'\mathbf{k}-\mathbf{k}'\mathbf{k})+(\mathbf{1}_b'\mathbf{k})^2-\mathbf{r}'\mathbf{r}}$,

(**iii**) $\lambda_1 < \lambda_2$ *if and only if* $\rho > \frac{\mathbf{k}'\mathbf{k}-\mathbf{1}_b'\mathbf{k}}{(b-1)\,(\mathbf{1}_b'\mathbf{k}-\mathbf{k}'\mathbf{k})+(\mathbf{1}_b'\mathbf{k})^2-\mathbf{r}'\mathbf{r}}$.

Proof: We have

$$\begin{aligned}\mu_1-\mu_2 &= \tfrac{1}{gv(v-1)(1-\rho)}\,\cdot\\ &\cdot\Big[v\mathbf{1}_b'\mathbf{k}-\mathbf{k}'\mathbf{k}+\tfrac{\rho}{1+\rho(b-1)}\Big((\mathbf{1}_b'\mathbf{k})^2-v\mathbf{r}'\mathbf{r}\Big)-\tfrac{1}{gv(1-\rho)}\Big(\mathbf{k}'\mathbf{k}-\tfrac{\rho}{1+\rho(b-1)}\,(\mathbf{1}_b'\mathbf{k})^2\Big)\Big]\\ &= \tfrac{1}{g(v-1)\,(1-\rho)\,(1+\rho(b-1))}\Big[\Big(\mathbf{1}_b'\mathbf{k}-\mathbf{k}'\mathbf{k}\Big)\Big(1+\rho(b-1)\Big)+\rho\Big((\mathbf{1}_b'\mathbf{k})^2-\mathbf{r}'\mathbf{r}\Big)\Big].\end{aligned}$$

Because $g(v-1)(1-\rho)(1+\rho(b-1)) > 0$ then $\mu_1-\mu_2 > 0$ if and only if $\rho\Big[(b-1)\,(\mathbf{1}_b'\mathbf{k}-\mathbf{k}'\mathbf{k})+(\mathbf{1}_b'\mathbf{k})^2-\mathbf{r}'\mathbf{r}\Big] > \mathbf{k}'\mathbf{k}-\mathbf{1}_b'\mathbf{k}$ and we obtain the Lemma. □

Lemma 2.1 imply that in order to determine E-optimal design we have to delimit the lowest bound of eigenvalues of $\bar{\mathbf{M}}^{-1}$ according to the value of ρ.

Theorem 2.1. *Let v be even. In any nonsingular spring balance weighing design $\mathbf{X}\in\mathbf{\Phi}_{b\times v}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$*

(**i**) *if $\rho\in\Big(\frac{-1}{b-1},\,\frac{v-2}{b+v-2}\Big)$ then $\lambda_{\max}(\mathbf{M}^{-1}) \geq \frac{4g(v-1)(1-\rho)}{bv}$ and the equality is fulfilled if and only if $\mathbf{X}\mathbf{1}_v = \frac{v}{2}\,\mathbf{1}_b$,*

(**ii**) *if $\rho\in\Big(\frac{v-2}{b+v-2},\,1\Big)$ then $\lambda_{\max}(\mathbf{M}^{-1}) > \frac{g(1+\rho(b-1))}{bv}$.*

Proof: In order to determine regular E-optimal spring balance weighing design we have to give the lowest bound of the maximal eigenvalue of the matrix $\mathbf{M}^{-1}$. Let $\bar{\mathbf{M}}$ denote the average of $\mathbf{M}$ over all elements of $\mathbf{\Pi}$ for the design $\mathbf{X}\in\mathbf{\Phi}_{b\times v}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$. From the monotonicity theorem given by Rao and Rao (2004) it follows $\lambda_{\max}(\bar{\mathbf{M}}^{-1}) \leq \lambda_{\max}(\mathbf{M}^{-1})$. The proof falls naturally in two parts according to the value ρ in 1.2. If $\rho\in\Big(\frac{-1}{b-1},\,\frac{v-2}{b+v-2}\Big)$ then $\lambda_{\max}(\bar{\mathbf{M}}^{-1}) = \frac{gv(v-1)(1-\rho)(1+\rho(b-1))}{v(1+\rho(b-1))\mathbf{1}_b'\mathbf{k}-(1+\rho(b-1))\mathbf{k}'\mathbf{k}+\rho(\mathbf{1}_b'\mathbf{k})^2-\rho v\mathbf{r}'\mathbf{r}}$. As we want to minimize $\lambda_{\max}(\bar{\mathbf{M}}^{-1})$, we should find the maximum value for

$$A \,=\, v\big(1+\rho(b-1)\big)\,\mathbf{1}_b'\mathbf{k}-\big(1+\rho(b-1)\big)\,\mathbf{k}'\mathbf{k}+\rho(\mathbf{1}_b'\mathbf{k})^2-\rho\,v\,\mathbf{r}'\mathbf{r}\ .$$

If p is even

$$(2.2)\quad A \le v\big(1+\rho(b-1)\big)\,\mathbf{1}_b^{'}\mathbf{k} - \big(1+\rho(b-1)\big)\,bk^2 + \rho(\mathbf{1}_b^{'}\mathbf{k})^2 - \rho\, v^2 r^2 =$$
$$(2.3)\quad v\big(1+\rho(b-1)\big)\,bk - \big(1+\rho(b-1)\big)\,bk^2 + \rho bk^2 - \rho v^2 r^2 \le \frac{1}{4}\,bv^2\big(1+\rho(b-1)\big)\,.$$

Hence $\lambda_{\max}(\mathbf{M}^{-1}) \ge \frac{4g(v-1)(1-\rho)}{bv}$. The equality in inequality 2.2 holds if and only if $k_1 = k_2 = \cdots = k_b = k$ and $r_1 = r_2 = \cdots = r_v = r$, whereas the equality in 2.3 is fulfilled if and only if $k = \frac{v}{2}$. If $\rho \in \left(\frac{v-2}{b+v-2},\ 1\right)$ then $\lambda_{\max}(\bar{\mathbf{M}}^{-1}) = \frac{gv(1-\rho)(1+\rho(b-1))}{\mathbf{k}'\mathbf{k}+\rho(b-1)\mathbf{k}'\mathbf{k}-\rho(\mathbf{1}_b^{'}\mathbf{k})^2}$. So, we obtain $\lambda_{\max}(\mathbf{M}^{-1}) > \frac{g(1+\rho(b-1))}{bv}$. Thus the result. □

Theorem 2.2. *Let v be odd. In any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$*

(i) *if $\rho \in \left(\frac{-1}{b-1},\ \frac{v(v-3)}{b(v+1)+v(v-3)}\right)$ then $\lambda_{\max}(\mathbf{M}^{-1}) \ge \frac{4gv(1-\rho)}{b(v+1)}$ and the equality is satisfied if and only if $\mathbf{X}\mathbf{1}_v = \frac{v-1}{2}\mathbf{1}_b$ or $\mathbf{X}\mathbf{1}_v = \frac{v+1}{2}\mathbf{1}_b$,*

(ii) *if $\rho \in \left(\frac{v(v-3)}{b(v+1)+v(v-3)},\ \frac{v}{b+v}\right)$ then $\lambda_{\max}(\mathbf{M}^{-1}) \ge \frac{4gv(1-\rho)}{b(v+1)}$ and the equality is satisfied if and only if $\mathbf{X}\mathbf{1}_v = \frac{v+1}{2}\mathbf{1}_b$,*

(iii) *if $\rho \in \left(\frac{v}{b+v},\ 1\right)$ then $\lambda_{\max}(\mathbf{M}^{-1}) > \frac{g(1+\rho(b-1)}{bv}$.*

Proof: The proof is similar to given in Theorem 2.1 one. □

Now, we can formulate the definition of the regular E-optimal spring balance weighing design. So, we have

Definition 2.1. Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$ is regular E-optimal if the eigenvalues of the information matrix attains the bounds of Theorems 2.1 and 2.2, i.e.

(i) v is even and $\rho \in \left(\frac{-1}{b-1},\ \frac{v-2}{b+v-2)}\right)$ if $\lambda_{\max}(\mathbf{M}^{-1}) = \frac{4g(v-1)(1-\rho)}{bv}$,

(ii) v is odd and $\rho \in \left(\frac{-1}{b-1},\ \frac{v}{b+v}\right)$ if $\lambda_{\max}(\mathbf{M}^{-1}) = \frac{4gv(1-\rho)}{b(v+1)}$.

A direct consequence of above considerations is

Theorem 2.3. *Any nonsingular spring balance weighing design $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ with the variance matrix of errors $\sigma^2\mathbf{G}$ is regular E-optimal design if and only if*

(1) *v is even and $\rho \in \left(\frac{-1}{b-1},\ \frac{v-2}{b+v-2)}\right)$*
$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{1}{g(1-\rho)}\left[\frac{bv}{4(v-1)}\mathbf{I}_v + \frac{b(v-2)}{4(v-1)}\mathbf{1}_v\mathbf{1}_v^{'} - \frac{\rho b^2}{4(1+\rho(b-1))}\mathbf{1}_v\mathbf{1}_v^{'}\right]$ *and*
$\mathbf{X}\mathbf{1}_v = \frac{v}{2}\mathbf{1}_b$,

(2) *p is odd and*

(2.1) $\rho \in \left(\frac{-1}{b-1},\ \frac{v(v-3)}{b(v+1)+v(v-3)}\right)$

(2.1.1) $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{1}{g(1-\rho)}\left[\frac{b(v+1)}{4v}\mathbf{I}_v + \frac{b(v-3)}{4v}\mathbf{1}_v\mathbf{1}_v' - \frac{\rho b^2(v-1)^2}{4v^2(1+\rho(b-1))}\mathbf{1}_v\mathbf{1}_v'\right]$
and $\mathbf{X1}_v = \frac{v-1}{2}\mathbf{1}_b$, or

(2.1.2) $\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{1}{g(1-\rho)}\left[\frac{b(v+1)}{4v}\mathbf{I}_v + \frac{b(v+1)}{4v}\mathbf{1}_v\mathbf{1}_v' - \frac{\rho b^2(v+1)^2}{4v^2(1+\rho(b-1))}\mathbf{1}_v\mathbf{1}_v'\right]$
and $\mathbf{X1}_v = \frac{v+1}{2}\mathbf{1}_b$,

(2.2) $\rho \in \left(\frac{v(v-3)}{b(v+1)+v(v-3)},\ \frac{v}{b+v}\right)$
$\mathbf{X}'\mathbf{G}^{-1}\mathbf{X} = \frac{1}{g(1-\rho)}\left[\frac{b(v+1)}{4v}\mathbf{I}_v + \frac{b(v+1)}{4v}\mathbf{1}_v\mathbf{1}_v' - \frac{\rho b^2(v+1)^2}{4v^2(1+\rho(b-1))}\mathbf{1}_v\mathbf{1}_v'\right]$
and $\mathbf{X1}_v = \frac{v+1}{2}\mathbf{1}_b$.

Proof: Since the proofs for even and odd v are similar, we give the proof only for the case of even v and $\rho \in \left(\frac{-1}{b-1},\ \frac{v-2}{b+v-2}\right)$. Notice, that $\lambda_{\max}(\mathbf{M}^{-1})$ attains the lowest bound in Theorem 2.1(i) if equalities $\lambda_{\max}(\mathbf{M}^{-1}) = \lambda_{\max}(\bar{\mathbf{M}}^{-1})$ and $v = \frac{k}{2}$ hold. It follows easily that $\mathrm{tr}(\mathbf{M}) = \frac{bv(2+\rho(b-2))}{4g(1-\rho)(1+\rho(b-1))}$ and $\mathbf{1}_v'\mathbf{M1}_v = \frac{bv^2}{4g(1+\rho(b-1))}$. We apply formulas on μ_1 and μ_2 to give the form of $\mathbf{M}$. Thus the Theorem. □

It can be noted that $g = 1$, $\rho = 0$ and $\mathbf{G} = \mathbf{I}_b$, we become the equalities given by Jacroux and Notz ([8]).

Theorem 2.4. *In any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$, *if*

(i) *v is even and* $\rho \in \left(\frac{v-2}{b+v-2},\ 1\right)$ *or*

(ii) *v is odd and* $\rho \in \left(\frac{v}{b+v},\ 1\right)$,

then regular E-optimal spring balance weighing design does not exist.

Proof: Since the proofs for even v and odd v are similar, we shall only give the proof for the case odd v. If $\rho \in \left(\frac{v}{b+v},\ 1\right)$ then the lowest bound of the maximal eigenvalue of the design matrix $\mathbf{X}$ is given in Theorem 2.2(iii). The lowest bound is attained if and only if $k = v$. It means $\mathbf{X} = \mathbf{1}_b\mathbf{1}_v'$. Such matrix is singular one. Thus the regular E-optimal design does not exist. □

Theorem 2.5. *Any nonsingular spring balance weighing design* $\mathbf{X} \in \mathbf{\Phi}_{b\times v}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho = \frac{p(k-1)}{n(p-k)+p(k-1)}$, $k = 1, 2, ..., \frac{v}{2}$ *for even v or* $k = 1, 2, ..., \frac{v+1}{2}$ *for odd v is regular E-optimal design if and only if*

$$\mathbf{X}'\mathbf{X} = \left(\frac{nk}{p} - \frac{nk(k-1)}{p(p-1)}\right)\mathbf{I}_v + \frac{nk(k-1)}{p(p-1)}\mathbf{1}_v'\mathbf{1}_v\ .$$

Proof: Let us take into consideration the case given in Lemma 2.1(ii). $\lambda_1 = \lambda_2 = \lambda = \frac{1}{\mu}$ if and only if $\rho = \frac{\mathbf{k}'\mathbf{k} - \mathbf{1}_b'\mathbf{k}}{(b-1)(\mathbf{1}_b'\mathbf{k} - \mathbf{k}'\mathbf{k}) + (\mathbf{1}_b'\mathbf{k})^2 - \mathbf{r}'\mathbf{r}}$. Now, we have to consider two cases v is even and v is odd. For both v the proofs are similar, so we give the case for even v, only. According to the proof of Theorem 2.1, the equality in 2.2 holds if and only if $k_1 = k_2 = \cdots = k_b = k$ and $r_1 = r_2 = \cdots = r_v = r$, for $k = 1, 2, ..., \frac{v}{2}$. Hence $\rho = \frac{p(k-1)}{n(p-k)+p(k-1)}$. From the above results $\bar{\mathbf{M}} = \mu \mathbf{I}_v$. Thus $\mu \mathbf{I}_v = \frac{1}{g(1-\rho)} \left(\mathbf{X}'\mathbf{X} - \frac{\rho n^2 k^2}{v^2(1+\rho(b-1))} \mathbf{1}_v \mathbf{1}_v' \right)$ and $\mathbf{X}'\mathbf{X} = \frac{nk(p-k)}{p(p-1)} \mathbf{I} + \frac{nk(k-1)}{p(p-1)} \mathbf{1}_v \mathbf{1}_v'$. Putting ρ and $\mu = \frac{nk(p-k)}{g(1-\rho)p(p-1)}$ we obtain the form of the matrix $\mathbf{X}'\mathbf{X}$. If $k > \frac{v}{2}$, then from Theorem 2.4 regular E-optimal spring balance weighing design does not exist. □

3. CONSTRUCTION OF THE REGULAR E-OPTIMAL DESIGN

For the construction of the regular E-optimal spring balance weighing design, from all possible block designs, we choose the incidence matrices of the balanced incomplete block designs and group divisible designs. The definitions of these block designs are given in Raghavarao and Padgett ([14]).

Theorem 3.1. *Let* $\mathbf{N}$ *be the incidence matrix of the balanced incomplete block design with the parameters*

(**i**) $v = 2t,\ b = 2(2t-1),\ r = 2t-1,\ k = t,\ \lambda = t-1$ *or*

(**ii**) $v = 2t,\ b = \binom{2t}{t},\ r = \binom{2t-1}{t-1},\ k = t,\ \lambda = \binom{2(t-1)}{t-2}$,

$t = 2, 3, \dots$. *Then, any* $\mathbf{X} = \mathbf{N}' \in \mathbf{\Phi}_{b\times v}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{b-1},\ \frac{v-2}{b+v-2} \right)$ *is the regular E-optimal spring balance weighing design.*

Proof: An easy computation shows that the matrix $\mathbf{X} = \mathbf{N}'$ satisfies (1) of Theorem 2.3. □

Now, let

$$\mathbf{X} = \begin{bmatrix} \mathbf{N}_1' \\ \mathbf{N}_2' \end{bmatrix}, \tag{3.1}$$

where $\mathbf{N}_u$ is the incidence matrix of the group divisible design with the same association scheme with the parameters $v,\ b_u,\ r_u,\ k = \frac{v}{2},\ \lambda_{1u},\ \lambda_{2u},\ u = 1, 2$. Furthermore, let the condition

$$\lambda_{11} + \lambda_{12} \ = \ \lambda_{21} + \lambda_{22} \ = \ \lambda \tag{3.2}$$

be satisfied. For $\mathbf{X}$ in 3.1, $b = b_1 + b_2$. The limitation on the t and s given in next Theorem 3.2 follow from the restrictions: $r,\ k \leq 10$ given in Clatworthy ([5]).

Theorem 3.2. *Let $\mathbf{N}_u,\ u = 1, 2,$ be the incidence matrix of the group divisible block design with the same association scheme and with the parameters*

(**1**) $v = 4,\ k = 2$ *and*

- (**1.1**) $b_1 = 2(3t+1),\ r_1 = 3t+1,\ \lambda_{11} = t+1,\ \lambda_{21} = t,\ t = 1, 2, 3,$ *and* $b_2 = 2(3s+2),\ r_2 = 3s+2,\ \lambda_{12} = s,\ \lambda_{22} = s+1,\ s = 0, 1, 2,$
- (**1.2**) $b_1 = 2(3t+2),\ r_1 = 3t+2,\ \lambda_{11} = t+2,\ \lambda_{21} = t,\ t = 1, 2,$ *and* $b_2 = 2(3s+4),\ r_2 = 3s+4,\ \lambda_{12} = s,\ \lambda_{22} = s+2,\ s = 0, 1, 2,$
- (**1.3**) $b_1 = 2(t+3),\ r_1 = t+3,\ \lambda_{11} = t+1,\ \lambda_{21} = 1$ *and* $b_2 = 4t,$ $r_2 = 2t,\ \lambda_{12} = 0,\ \lambda_{22} = t,\ t = 1, 2, ..., 5,$
- (**1.4**) $b_1 = 16, r_1 = 8, \lambda_{11} = 0, \lambda_{21} = 4$ *and* $b_2 = 2(3s+4), r_2 = 3s+4,$ $\lambda_{12} = s+4,\ \lambda_{22} = s,\ s = 1, 2,$
- (**1.5**) $b_1 = 18, r_1 = 9, \lambda_{11} = 5, \lambda_{21} = 2$ *and* $b_2 = 6(s+2), r_2 = 3(s+2),$ $\lambda_{12} = s,\ \lambda_{22} = s+3,\ s = 0, 1,$

(**2**) $v = 6,\ k = 3$ *and*

- (**2.1**) $b_1 = 4t, r_1 = 2t, \lambda_{11} = 0, \lambda_{21} = t$ *and* $b_2 = 6t, r_2 = 3t, \lambda_{12} = 2t,$ $\lambda_{22} = t,\ t = 1, 2, 3,$
- (**2.2**) $b_1 = 2(2t+5), r_1 = 2t+5, \lambda_{11} = t+1, \lambda_{21} = t+2$ *and* $b_2 = 6t,$ $r_2 = 3t,\ \lambda_{12} = t+1,\ \lambda_{22} = t,\ t = 1, 2,$
- (**2.3**) $b_1 = 12, r_1 = 6, \lambda_{11} = 4, \lambda_{21} = 2$ *and* $b_2 = 2(5s+4), r_2 = 5s+4,$ $\lambda_{12} = 2s,\ \lambda_{22} = 2(s+1),\ s = 0, 1,$
- (**2.4**) $b_1 = 16, r_1 = 8, \lambda_{11} = 4, \lambda_{21} = 3$ *and* $b_2 = 2(5s+2), r_2 = 5s+2,$ $\lambda_{12} = 2s,\ \lambda_{22} = 2s+1,\ s = 0, 1,$

(**3**) $v = 8,\ k = 4$ *and*

- (**3.1**) $b_1 = 4(t+1), r_1 = 2(t+1), \lambda_{11} = 0, \lambda_{21} = t+1$ *and* $b_2 = 4(6-t),$ $r_2 = 2(6-t),\ \lambda_{12} = 6,\ \lambda_{22} = 5-t,\ t = 1, 2, 3,$
- (**3.2**) $b_1 = 2(3t+2),\ r_1 = 3t+2,\ \lambda_{11} = t+2,\ \lambda_{21} = t+1$ *and* $b_2 = 6(4-t),\ r_2 = 3(4-t),\ \lambda_{12} = 4-t,\ \lambda_{22} = 5-t,\ t = 1, 2,$

(**4**) $v = 10,\ k = 5$ *and* $b_1 = 8t,\ r_1 = 4t,\ \lambda_{11} = 0,\ \lambda_{21} = 2t$ *and* $b_2 = 10t,$ $r_2 = 5t,\ \lambda_{12} = 4t,\ \lambda_{22} = 2t,\ t = 1, 2,$

(**5**) $v = 2(2t+1),\ k = 2t+1$ *and* $b_1 = 4t,\ r_1 = 2t,\ \lambda_{11} = 0,\ \lambda_{21} = t$ *and* $b_2 = 2(2t+1),\ r_2 = 2t+1,\ \lambda_{12} = 2t,\ \lambda_{22} = t,\ t = 1, 2, 3, 4,$

(**6**) $v = 4(t+1),\ k = 2(t+1)$ *and* $b_1 = 2(2t+1),\ r_1 = 2t+1,\ \lambda_{11} = 2t+1,$ $\lambda_{21} = t$ *and* $b_2 = 4(t+1), r_2 = 2(t+1), \lambda_{12} = 0, \lambda_{22} = t+1, t = 1, 2, 3, 4.$

Then any $\mathbf{X} \in \mathbf{\Phi}_{b \times v}(0, 1)$ in the form 3.1 with the variance matrix of errors $\sigma^2\mathbf{G}$ for $\rho \in \left(\frac{-1}{b-1},\ \frac{v-2}{b+v-2}\right)$ is the regular E-optimal spring balance weighing design.

Proof: This is proved by checking that the matrix $\mathbf{X}$ in 3.1 satisfies (1) of Theorem 2.3. $\square$

Theorem 3.3. *Let* $\mathbf{N}$ *be the incidence matrix of balanced incomplete block design with the parameters*

(**i**) $v = 2t+1,\ b = 2(2t+1),\ r = 2(t+1),\ k = t+1,\ \lambda = t+1,$

(**ii**) $v = b = 4t^2 - 1,\ r = k = 2t^2,\ \lambda = t^2,$

(**iii**) $v = b = 8t+7,\ r = k = 4(t+1),\ \lambda = 2(t+1),$

(**iv**) $v = b = 4t-1,\ r = k = 2t,\ \lambda = t,$

(**v**) $v = 4t+1,\ b = 2(4t+1),\ r = 2(2t+1),\ k = \lambda = 2t+1,$

(**vi**) $v = 2t+1,\ b = \binom{2t+1}{t+1},\ r = \binom{2t}{t},\ k = t+1,\ \lambda = \binom{2t-1}{t-1},$

$t = 1, 2, \ldots$. *Then any* $\mathbf{X} = \mathbf{N}' \in \mathbf{\Phi}_{b\times v}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{b-1}, \frac{v}{b+v}\right)$ *is the regular E-optimal spring balance weighing design.*

Proof: For $\mathbf{X} = \mathbf{N}' \in \mathbf{\Phi}_{b\times v}(0,1)$ the equalities (2.1.2) and (2.2) of Theorem 2.3 are satisfied. $\square$

Theorem 3.4. *Let* $\mathbf{N}$ *be the incidence matrix of balanced incomplete block design with the parameters*

(**i**) $v = 2t+1,\ b = 2(2t+1),\ r = 2t,\ k = t,\ \lambda = t-1,\ t = 2, 3, \ldots$

(**ii**) $v = b = 4t^2 - 1,\ r = k = 2t^2 - 1,\ \lambda = t^2 - 1,\ t = 2, 3, \ldots$

(**iii**) $v = b = 8t+7,\ r = k = 4t+3,\ \lambda = 2t+1,\ t = 1, 2, \ldots$

(**iv**) $v = b = 4t-1,\ r = k = 2t-1,\ \lambda = t-1,\ t = 2, 3, \ldots$

(**v**) $v = 4t+1,\ b = 2(4t+1),\ r = 4t,\ k = 2t,\ \lambda = 2t-1,\ t = 1, 2, \ldots$

(**vi**) $v = 2t+1,\ b = \binom{2t+1}{t+1},\ r = \binom{2t}{t-1},\ k = t,\ \lambda = \binom{2t-1}{t-2},\ t = 2, 3, \ldots$

Then any $\mathbf{X} = \mathbf{N}' \in \mathbf{\Phi}_{b\times v}(0,1)$ *with the variance matrix of errors* $\sigma^2\mathbf{G}$ *for* $\rho \in \left(\frac{-1}{b-1}, \frac{v(v-3)}{b(v+1)+v(v-3)}\right)$ *is the regular E-optimal spring balance weighing design.*

Proof: It is obvious that for $\mathbf{X} = \mathbf{N}' \in \mathbf{\Phi}_{b\times v}(0,1)$ the equality (2.1.1) given in Theorem 2.3 is fulfilled. $\square$

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