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AUTOREGRESSIVE SEQUENCES VIA LÉVY PROCESSES

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Abstract:

- We use Lévy processes to develop a family of first-order autoregressive sequences of random variables with values in $\mathbf{R}_+$, called C-AR(1) processes. We obtain various distributional and regression properties for these processes and we establish a limit theorem that leads to the property of stationarity. A connection between stationarity of C-AR(1) processes and the notion of C-self-decomposability of van Harn and Steutel (1993) is discussed. A number of stationary C-AR(1) processes with specific marginals are presented and are shown to generalize several existing $\mathbf{R}_+$-valued AR(1) models. The question of time reversibility is addressed and some examples are discussed.

Key-Words:

- *stationarity; semigroup of cumulant generating functions; self-decomposability; stability; time-reversibility.*

AMS Subject Classification:

- 60G10, 62M10.

1. INTRODUCTION

In recent years, several authors proposed generalized first-order autoregressive (or AR(1)) models with marginal distributions on $\mathbf{R}_+ := [0, \infty)$. Lewis *et al.* (1989) constructed gamma AR(1) processes with random coefficients based on the beta-gamma transformation. As an application, they analyzed inter-failure times of a computer system. Sim (1990) introduced a generalized multiplication based on a conditional compound Poisson distribution to construct a gamma AR(1) processes. Al-Osh and Alzaid (1993) provided extensions of the gamma models in Lewis *et al.* (1989) via the Gamma-Dirichlet transformation. Grunwald *et al.* (2000) introduced the family of conditional linear AR(1) (CLAR(1)) models. A CLAR(1) process is a Markov process $(X_n,\, n \geq 0)$ such that the conditional expectation $E(X_n|X_{n-1})$ is linear affine. The authors fitted a CLAR(1) model to rainfall data. Zhu (2002) introduced a class of generalized AR(1) (GAR(1)) processes with marginal distributions on $\mathbf{R}_+$. As an application, Zhu (2002) fitted a GAR(1) process with a gamma marginal distribution to ozone data. Darolles *et al.* (2006) introduced a general class of compound autoregressive AR(1) (CAR(1)) processes for non-Gaussian time series. CAR(1) processes are specified in terms of their conditional Laplace transforms.

The aim of this paper is to develop a class of AR(1) sequences of random variables (rv's) with values in $\mathbf{R}_+$ by way of Lévy processes, or processes with stationary independent increments. Our starting point is a continuous convolution semigroup of cumulant generating functions denoted by $C = (C_t,\, t \geq 0)$ and its related operator $\odot_C$ (the definition is recalled below) introduced by van Harn and Steutel (1993). The equation governing our models (equation (2.1)) is analogous to the one describing the standard AR(1) process, with the operator $\odot_C$ replacing the standard multiplication. We obtain various distributional and regression properties for these models and we discuss conditions that lead to stationarity and time reversibility. A number of stationary C-AR(1) processes with specific marginals are presented and are shown to generalize several existing models. The paper is organized as follows. In Section 2 we introduce C-AR(1) processes and give their representation in terms of independent sequences of $\mathbf{R}_+$-valued Lévy processes. We describe their various properties and obtain a limit theorem that leads to the property of stationarity for C-AR(1) processes. We also discuss a connection between the concept of C-self-decomposability of van Harn and Steutel (1993) and stationarity of C-AR(1) processes. In Section 3 we present a number of specific stationary solutions for C-AR(1) processes. Characterizations of their marginal distributions are obtained and some examples are discussed. The question of time reversibility of C-AR(1) processes is addressed in Section 4.

In the rest of this section we recall some definitions and results that are needed in the sequel. For proofs and further details we refer to Hansen (1989), van Harn and Steutel (1993), and Steutel and van Harn (2004).

The Laplace–Stieltjes transform (LST) of an $\mathbf{R}_+$-valued rv X with distribution function $F(x)$ is denoted by ϕ_X:

$$\phi_X(\tau) = \int_0^\infty e^{-\tau x}\, dF(x) \qquad (\tau \geq 0) .$$

$C = (C_t;\, t \geq 0)$ will denote a continuous composition semigroup of cumulant generating functions (cgf's): for every $t \geq 0$, $C_t = -\ln L_t$ for some infinitely divisible LST L_t, $C_t \not\equiv 1$, and $\delta_C = -\ln(-L_1'(0)) > 0$. For any $\tau \geq 0$,

$$C_0(\tau) = \tau;\;\; C_s \circ C_t(\tau) = C_{s+t}(\tau)\;\; (s, t \geq 0);\;\; \lim_{t\downarrow 0} C_t(\tau) = \tau;\;\; \lim_{t\to\infty} C_t(\tau) = 0 . \tag{1.1}$$

The infinitesimal generator U of the semigroup C is defined by

$$U(\tau) = \lim_{t\downarrow 0}\big(C_t(\tau) - \tau\big)/t \qquad (\tau \geq 0) ,$$

and satisfies $U(0) = 0$ and $U(\tau) < 0$ for $\tau > 0$. U admits the representation

$$U(\tau) = a\tau - \frac{1}{2}\sigma^2\tau^2 - \int_0^\infty \big(e^{-\tau x} - 1 + \tau x/(1+x^2)\big)\, dN(x) ,$$

where a is real, $\sigma \geq 0$, and $N(dx)$ is a Lévy spectral function such that $\int_0^y x^2\, dN(x) < \infty$ for every $y > 0$. Moreover, the following non-explosion condition holds:

$$\left|\int_{0^+}^y U(x)^{-1}\, dx\right| = \infty \qquad \text{for sufficiently small } \; y > 0 .$$

A related function, called the A-function, is defined by

$$A(\tau) = \exp\left\{\int_\tau^1 \big(U(x)\big)^{-1}\, dx\right\} \qquad \big(\tau \geq 0;\;\; A(0) = 0\big) . \tag{1.2}$$

The functions $U(\tau)$ and $A(\tau)$ satisfy for every $t \geq 0$ and $\tau \geq 0$,

$$\frac{\partial}{\partial t} C_t(\tau) = U\big(C_t(\tau)\big) = U(\tau)\, C_t'(\tau) \qquad \text{and} \qquad A\big(C_t(\tau)\big) = e^{-t} A(\tau) . \tag{1.3}$$

Moreover,

$$\delta_C = -U'(0) \qquad \text{and} \qquad C_t'(0) = e^{-\delta_C t} \quad (t \geq 0) . \tag{1.4}$$

The infinite divisibility of $L_t(\tau)$ and the second part of (1.4) imply that for any $\tau > 0$ and $t > 0$,

$$C_t(\tau) < \tau . \tag{1.5}$$

For an $\mathbf{R}_+$-valued rv X and $\alpha \in (0,1)$, the generalized multiplication $\alpha \odot_C X$ is defined by

$$\alpha \odot_C X = Y(X) \;, \tag{1.6}$$

where $(Y(s),\, s \geq 0)$ is an $\mathbf{R}_+$-valued Lévy process, independent of X, such that $\phi_{Y(1)}(\tau) = \exp(-C_t(\tau))$, $t = -\ln\alpha$. The LST of $\alpha \odot_C X$ is given by

$$\phi_{\alpha\odot_C X}(\tau) = \phi_X\big(C_t(\tau)\big) \;, \qquad t = -\ln\alpha \;. \tag{1.7}$$

If $E(X) < \infty$, then

$$E(\alpha \odot_C X) = \alpha^{\delta_C} E(X) \;. \tag{1.8}$$

2. C-AR(1) PROCESSES

Definition 2.1. A sequence $(X_n,\, n \in \mathbf{Z})$ of $\mathbf{R}_+$-valued rv's is said to be a C-AR(1) process if for any $n \in \mathbf{Z}$,

$$X_n = \alpha \odot_C X_{n-1} + \epsilon_n \;, \tag{2.1}$$

where $0 < \alpha < 1$ and $(\epsilon_n, n \in \mathbf{Z})$ is an iid sequence of $\mathbf{R}_+$-valued rv's that is assumed independent of the Y variables that define the operator $\odot_C$ (see below). $(\epsilon_n,\, n \in Z)$ is called the innovation sequence of $(X_n,\, n \in \mathbf{Z})$.

In the remainder of this paper we will at times refer to the single-ended C-AR(1) processes $(X_n,\, n \geq 0)$ that arises when equation (2.1) is assumed to hold only for $n \geq 0$.

The generalized multiplication $\alpha \odot_C X_{n-1}$ in (2.1) is performed independently for each n. More precisely, we assume the existence of a sequence $(Y^{(n)}(\cdot)$, $n \in \mathbf{Z})$ of iid $\mathbf{R}_+$-valued Lévy processes, independent of $(\epsilon_n,\, n \in \mathbf{Z})$, such that the LST of $Y^{(n)}(1)$ is

$$\phi_{Y^{(n)}(1)}(\tau) \,=\, \exp\big(-C_t(\tau)\big) \,, \qquad \tau \geq 0 \;, \tag{2.2}$$

where $t = -\ln\alpha$, and (see (1.6))

$$\alpha \odot_C X_{n-1} = Y^{(n-1)}(X_{n-1}) \;, \qquad n \in \mathbf{Z} \;.$$

In terms of LST's, equation (2.1) translates, by way of (1.7), into

$$\phi_{X_n}(\tau) \,=\, \phi_{X_{n-1}}\big(C_t(\tau)\big)\, \phi_\epsilon(\tau) \;, \qquad \tau \geq 0 \;, \tag{2.3}$$

where $\phi_\epsilon(\tau)$ is the marginal LST of $(\epsilon_n,\, n \in \mathbf{Z})$ and $t = -\ln\alpha$.

Some results on conditional and joint distributions of a C-AR(1) process are given next.

Proposition 2.1. *Let $(X_n, n \geq 0)$ be a C-AR(1) process for some $\alpha \in (0,1)$. Let $t = -\ln\alpha$. The following assertions hold for any $n \geq 1$.*

(i) *The conditional LST of X_n given $X_{n-1} = x$, $x \geq 0$, is*

$$\phi_{X_n|X_{n-1}=x}(\tau) \;=\; \exp\bigl(-x\,C_t(\tau)\bigr)\,\phi_\epsilon(\tau)\;, \qquad \tau \geq 0\;. \tag{2.4}$$

(ii) *The bivariate joint LST of (X_{n-1}, X_n) is given by*

$$\phi_{(X_{n-1},X_n)}(\tau_1,\tau_2) \;=\; \phi_\epsilon(\tau_2)\,\phi_{X_{n-1}}\bigl(\tau_1 + C_t(\tau_2)\bigr)\;. \tag{2.5}$$

(iii) *More generally, the joint LST of $(X_1, X_2, ..., X_n)$ can be found recursively by*

$$\phi_{(X_1,...,X_n)}(\tau_1,...,\tau_n) \;=\; \phi_\epsilon(\tau_n)\,\phi_{(X_1,...,X_{n-1})}\bigl(\tau_1,...,\tau_{n-2},\tau_{n-1}+C_t(\tau_n)\bigr)\;. \tag{2.6}$$

Proof: (i) follows from (2.1) and the fact that $\phi_{Y^{(n)}(x)}(\tau) = \exp(-xC_t(\tau))$. To show (ii), we recall that the joint LST $\phi_{(X_{n-1},X_n)}(\tau_1,\tau_2)$ of (X_{n-1}, X_n) is defined by

$$\phi_{(X_{n-1},X_n)}(\tau_1,\tau_2) \;=\; E\Bigl(e^{-(\tau_1 X_{n-1}+\tau_2 X_n)}\Bigr)\,, \qquad \tau_1 \geq 0,\;\; \tau_2 \geq 0\;.$$

It can be rewritten as

$$\phi_{(X_{n-1},X_n)}(\tau_1,\tau_2) \;=\; E\Bigl(e^{-\tau_1 X_{n-1}}\,E\bigl(e^{-\tau_2 X_n}|X_{n-1}\bigr)\Bigr)\,,$$

which, combined with (2.4), yields

$$\phi_{(X_{n-1},X_n)}(\tau_1,\tau_2) \;=\; E\Bigl(e^{-(\tau_1+C_t(\tau_2))X_{n-1}}\,\phi_\epsilon(\tau_2)\Bigr)\,,$$

which, in turn, implies (2.5). The exact same argument establishes (2.6). The details are omitted. □

We note, by definition, that any C-AR(1) process is necessarily a Markov process. Moreover, by using (2.3) recursively (and the fact that $(C_t, t \geq 0)$ is a semigroup), it can be shown that a C-AR(1) process $(X_n, n \in \mathbf{Z})$ admits the following representation for any $k \geq 1$,

$$X_n \;\overset{d}{=}\; \alpha^k \odot_C X_{n-k} + \sum_{i=0}^{k-1} \alpha^i \odot_C \epsilon_{n-i}\;, \qquad n \in \mathbf{Z}\;. \tag{2.7}$$

Basic regression properties of C-AR(1) processes are gathered in the following proposition.

Proposition 2.2. *Assume $U''(0) < \infty$. Let $(X_n, n \in \mathbf{Z})$ be a C-AR(1) process (for some $0 < \alpha < 1$) such that $E(X_n) < \infty$ and $E(X_n^2) < \infty$ for any $n \in \mathbf{Z}$, $\mu_\epsilon = E(\epsilon_0) < \infty$ and $\sigma_\epsilon^2 = \mathrm{Var}(\epsilon_0) < \infty$.*

(i) *The regression of X_n on X_{n-1} is linear:*

$$E(X_n|X_{n-1}) = \alpha^{\delta_C} X_{n-1} + \mu_\epsilon \,, \qquad n \in \mathbf{Z} \,. \tag{2.8}$$

(ii) *The conditional variance of X_n given X_{n-1} is linear:*

$$\mathrm{Var}(X_n|X_{n-1}) = B X_{n-1} + \sigma_\epsilon^2 \,, \qquad n \in \mathbf{Z} \,. \tag{2.9}$$

where $B = \left(1 - \frac{U''(0)}{U'(0)}\right) \alpha^{\delta_C} (1 - \alpha^{\delta_C})$.

(iii) *For any $n \in \mathbf{Z}$ and $k \geq 0$, the covariance at lag k, $\Gamma_n(k) = \mathrm{cov}(X_{n-k}, X_n)$ of $(X_n, n \in \mathbf{Z})$ is*

$$\Gamma_n(k) = \alpha^{k\delta_C}\, \mathrm{Var}(X_{n-k}) \,. \tag{2.10}$$

(iv) *For any $n \in \mathbf{Z}$ and $k \geq 0$,*

$$E(X_n) = \alpha^{k\delta_C} E(X_{n-k}) + \mu_\epsilon \sum_{i=o}^{k-1} \alpha^{i\delta_C} \,, \tag{2.11}$$

and

$$\mathrm{Var}(X_n) = \alpha^{2k\delta_C}\, \mathrm{Var}(X_{n-k}) + B \sum_{i=1}^{k} \alpha^{2(i-1)\delta_C} E(X_{n-i}) + \sigma_\epsilon^2 \sum_{i=1}^{k} \alpha^{2(i-1)\delta_C}, \tag{2.12}$$

where the constant B is as in (2.9) above.

Proof: We note that for $x \geq 0$

$$E(X_n|X_{n-1} = x) = -\phi'_{X_n|X_{n-1}=x}(0) \,,$$

where $\phi_{X_n|X_{n-1}=x}(\tau)$ is given by (2.4). By differentiating (2.4) and using (1.4) we obtain (2.8). By differentiating twice (w.r.t. τ) the expression $U(C_t(\tau)) = C'_t(\tau)\, U(\tau)$ $(t = -\ln \alpha)$ and letting $\tau \downarrow 0$, we obtain via (1.4), $C''_t(0) = \alpha^{\delta_C} (\alpha^{\delta_C} - 1) \cdot$ $\cdot U''(0)/U'(0)$. Moreover,

$$E(X_n^2|X_{n-1} = x) = \phi''_{X_n|X_{n-1}=x}(0) \,,$$

and

$$\mathrm{Var}(X_n|X_{n-1} = x) = \phi''_{X_n|X_{n-1}=x}(0) - \left(\phi'_{X_n|X_{n-1}=x}(0)\right)^2 .$$

Direct calculations, along with (1.4) and the formula for $C''_t(0)$ found above, leads to (2.9). Equation (2.10) is obtained by applying a conditioning argument to (2.7). Finally, (2.11) and (2.12) are easily derived from (2.8) and (2.9). □

The following result demonstrates the existence of a stationary C-AR(1) process.

Theorem 2.1. *Let $(X_n, n \geq 0)$ be a single-sided C-AR(1) process with coefficient $\alpha \in (0,1)$. Then $(X_n, n \geq 0)$ admits a proper limit distribution as $n \to \infty$ if and only if*

$$\int_0^y \frac{1-\phi_\epsilon(x)}{x - C_t(x)}\, dx < \infty\,, \qquad t = -\ln\alpha\,, \tag{2.13}$$

for some $y > 0$, and therefore for all $y > 0$.

Proof: We combine a Poisson mixture argument due to van Harn and Steutel (1993) and a convergence result for branching processes with immigration due to Foster and Williamson (1971). An induction argument based on (2.3) leads to

$$\phi_{X_n}(\tau) \;=\; \phi_{X_0}\big(C_{nt}(\tau)\big) \prod_{j=0}^{n-1} \phi_\epsilon\big(C_{jt}(\tau)\big)\,, \qquad t = -\ln\alpha, \quad \tau \geq 0, \quad n \geq 1\;.$$

Therefore, the sequence $(\phi_{X_n}(\tau), n \geq 0)$ is decreasing for every $\tau \geq 0$. It follows that

$$\phi(\tau) = \lim_{n\to\infty} \phi_{X_n}(\tau) \tag{2.14}$$

exists for each $\tau \geq 0$.

Let $\lambda > 0$. By van Harn and Steutel (1993), the functions $F^{(\lambda)} = (F_t^{(\lambda)};\, t \geq 0)$ defined by

$$F_t^{(\lambda)}(z) \;=\; 1 - \frac{1}{\lambda}\, C_t\big(\lambda(1-z)\big) \qquad (z \in [0,1])\;. \tag{2.15}$$

form a continuous composition semigroup of probability generating functions (pgf's), with

$$\frac{\partial}{\partial z} F_t^{(\lambda)}(z)\bigg|_{z=1} \;=\; e^{-\delta_c t} \tag{2.16}$$

for each $t > 0$.

Consider the branching process with immigration $\big(Y_n^{(\lambda)}, n \geq 0\big)$

$$Y_n^{(\lambda)} \;=\; \sum_{i=1}^{Y_{n-1}^{(\lambda)}} W_i^{(\lambda)} + \epsilon_n^{(\lambda)}\;, \tag{2.17}$$

where $(W_n^{(\lambda)}, n \geq 1)$ and $(\epsilon_n^{(\lambda)}, n \geq 0)$ are independent sequences of iid $\mathbf{Z}_+$-valued rv's with respective marginal pgf's $F_t^{(\lambda)}(z)$ and $P_{\epsilon^{(\lambda)}}(z) = \phi_\epsilon(\lambda(1-z))$, $0 \leq z \leq 1$

and $t = -\ln\alpha$. Moreover, $Y_0^{(\lambda)}$ has pgf $P_{Y_0^{(\lambda)}}(z) = \phi_{X_0}(\lambda(1-z))$ and is independent of $(W_n^{(\lambda)}, n \geq 1)$ and $(\epsilon_n^{(\lambda)}, n \geq 0)$. By (2.3), 2.15), (2.17), and an induction argument, we have

$$(2.18) \qquad P_{Y_n^{(\lambda)}}(z) = \phi_{X_n}\big(\lambda(1-z)\big)\,, \qquad n \geq 0, \quad 0 \leq z \leq 1\,.$$

By (2.16), $(Y_n^{(\lambda)}, n \geq 0)$ is a sub-critical branching process.

Let's now assume that (2.13) holds. Simple calculations show that

$$(2.19) \qquad \int_0^1 \frac{1 - P_{\epsilon^{(\lambda)}}(x)}{F_t^{(\lambda)}(x) - x}\, dx = \int_0^\lambda \frac{1 - \phi_\epsilon(x)}{x - C_t(x)}\, dx < \infty\,, \qquad t = -\ln\alpha\,.$$

By the main Theorem of Foster and Williamson (1971), case (iii), $(Y_n^{(\lambda)}, n \geq 0)$ has a proper limit distribution, as $n \to \infty$, whose pgf is (by (2.18))

$$(2.20) \qquad P^{(\lambda)}(z) = \lim_{n\to\infty} \phi_{X_n}\big(\lambda(1-z)\big)\,, \qquad 0 \leq z \leq 1\,.$$

It follows by (2.14) that for every $\lambda > 0$, $\phi(\lambda(1-z)) = P^{(\lambda)}(z)$, $0 \leq z \leq 1$. Therefore, by Lemma A.6 in van Harn and Steutel (1993), $\phi(\tau)$ is the LST of a distribution on $\mathbf{R}_+$.

Conversely, assume that $(X_n, n \geq 0)$ admits a proper limit distribution as $n \to \infty$. The limit LST $\phi(\tau)$ is given by (2.14). Hence, for every $\lambda > 0$, $(Y_n^{(\lambda)}, n \geq 0)$ has a proper limit distribution whose pgf is $P^{(\lambda)}(z)$ of (2.20). We deduce by the converse of the Theorem in Foster and Williamson (1971), case (iii), that (2.19) holds for every $\lambda > 0$, which in turn implies (2.13). □

Since a single-sided C-AR(1) process $(X_n, n \geq 0)$ is Markovian, it is stationary if and only if it is started with its limit distribution. By Theorem 2.1, such a limit distribution exists if condition (2.13) holds. We note that a single-sided process can be extended to a doubly-infinite stationary process (see the proof of Theorem 2.2 below).

Next, we explore the relationship between self-decomposability and stationary C-AR(1) processes. A distribution on $\mathbf{R}_+$ with LST $\phi(\tau)$ is said to be C-self-decomposable (van Harn and Steutel, 1993) if for any $t > 0$, there exists an LST $\phi_t(\tau)$ such that

$$(2.21) \qquad \phi(\tau) = \phi\big(C_t(\tau)\big)\, \phi_t(\tau)\,, \qquad \tau \geq 0\,.$$

Any C-self-decomposable distribution can arise as the marginal distribution of a stationary C-AR(1) process. More precisely, we have the following result.

Theorem 2.2. *Let $\phi(\tau)$ be the LST of a C-self-decomposable distribution. For any $\alpha \in (0,1)$, there exists a stationary C-AR(1) process $(X_n, n \in \mathbf{Z})$ whose marginal distribution has LST $\phi(\tau)$.*

Proof: Let $\alpha \in (0,1)$ and $t = -\ln\alpha$. By the Kolmogorov extension theorem (Breiman, 1968), there exists a probability space $(\Omega, \mathcal{F}, \mu)$ on which one can define an array $(Y^{(n)}(\cdot),\, n \geq 0)$ of iid $\mathbf{R}_+$-valued Lévy processes such that $Y^{(n)}(1)$ has LST (2.2), a sequence of iid rv's $(\epsilon_n,\, n \geq 0)$ with common LST $\phi_\epsilon(\tau) = \phi_t(\tau)$ of (2.21), and a rv X_0 with LST $\phi(\tau)$, with the further property that $(Y^{(n)}(\cdot),\, n \geq 0)$, $(\epsilon_n,\, n \geq 0)$, and X_0 are independent. We then construct a single-ended INAR1 process $(X_n,\, n \geq 0)$ via equation (2.1). This implies that for every $n \geq 1$, the LST $\phi_{X_n}(\tau)$ of X_n satisfies (2.3), with $\phi_{X_0}(\tau) = \phi(\tau)$. It follows by (2.3) and (2.21) that $\phi_{X_n}(\tau) = \phi(\tau)$ for every $n \geq 0$. Therefore, the X_n's are identically distributed. Since $(X_n,\, n \geq 0)$ is a Markov process, its stationarity ensues. The existence of the doubly infinite extension $(X_n,\, n \in \mathbf{Z})$ follows from Proposition 6.5, page 105, in Breiman (1968). □

Next, we state a representation theorem for stationary C-AR(1) processes. The proof follows easily from (2.7) and is omitted.

Theorem 2.3. *Any stationary C-AR(1) process $(X_n, n \in \mathbf{Z})$ admits the following (infinite order) moving average representation for some $0 < \alpha < 1$:*

$$X_n \stackrel{d}{=} \sum_{i=0}^{\infty} \alpha^i \odot_C \epsilon_{n-i}\,, \qquad n \in \mathbf{Z}\,, \tag{2.22}$$

where the convergence of the series is in the weak sense.

The mean, variance, and autocorrelation function (ACRF) of a stationary C-AR(1) process follow straightforwardly from Proposition 2.2.

Proposition 2.3. *Assume $U''(0) < \infty$. Let $(X_n, n \in \mathbf{Z})$ be a stationary C-AR(1) process (for some $0 < \alpha < 1$) such that $E(X_0) < \infty$, $E(X_0^2) < \infty$, $\mu_\epsilon = E(\epsilon_0) < \infty$ and $\sigma_\epsilon^2 = \mathrm{Var}(\epsilon_0) < \infty$. Then*

(**i**) *For any $n \in \mathbf{Z}$,*

$$E(X_n) \,=\, \mu_\epsilon (1 - \alpha^{\delta_C})^{-1}\,,$$

and

$$\mathrm{Var}(X_n) \,=\, \frac{\left(1 - \frac{U''(0)}{U'(0)}\right)\alpha^{\delta_C}\mu_\epsilon + \sigma_\epsilon^2}{1 - \alpha^{2\delta_C}}\,.$$

(**ii**) *For any $k \geq 0$ and $n \in \mathbf{Z}$, the correlation coefficient of (X_{n-k}, X_n) is*

$$\rho(k) \,=\, \alpha^{k\delta_C}\,. \tag{2.23}$$

We note that the $ACRF$ of a stationary C-AR(1) process, as given by (2.23), has the same form as that of the standard $AR(1)$ process. It decays exponentially at lag k. However, unlike the standard $AR(1)$ case, $\rho(k)$ remains always positive.

3. STATIONARY C-AR(1) PROCESSES WITH SPECIFIC MARGINAL DISTRIBUTIONS

In this section we present several stationary solutions for C-AR(1) processes.

An $\mathbf{R}_+$-valued rv X is said to have a C-stable distribution with exponent $\gamma > 0$ if there exists a sequence of iid $\mathbf{R}_+$-valued rv's $(X_i, i \ge 0)$, $X_i \overset{d}{=} X$ for all i, such that for any $n > 0$,

$$X \overset{d}{=} n^{-1/\gamma} \odot_C \sum_{i=1}^{n} X_i \ .$$

C-stable distributions are C-self-decomposable and exist only when $0 < \gamma \le \delta_C$ (van Harn and Steutel, 1993). Moreover, the LST $\phi(\tau)$ of a C-stable distribution with exponent $\gamma \in (0, \delta_C]$ admits the canonical representation

$$\phi(\tau) = \exp\big[-\lambda A(\tau)^\gamma\big] , \qquad \tau \ge 0 \tag{3.1}$$

for some $\lambda > 0$, where $A(\tau)$ is given in (1.2).

It follows by Theorem 2.2 that for every $0 < \alpha < 1$, there exists a stationary C-AR(1) process $(X_n, n \in \mathbf{Z})$ with a C-stable marginal distribution with exponent γ $(0 < \gamma \le \delta_C)$. The marginal distribution of the innovation sequence $(\epsilon_n, n \in \mathbf{Z})$, obtained by solving for ϕ_ϵ in (2.3) and by using (1.3), is also C-stable with exponent γ and has LST

$$\phi_\epsilon(\tau) = \exp\big[-\lambda(1-\alpha^\gamma)\, A(\tau)^\gamma\big] . \tag{3.2}$$

Moreover, it can be shown (see van Harn and Steutel, 1993) that stationary C-AR(1) processes whose marginal is C-stable with finite mean arise only in the case $\gamma = \delta_C$ and $A'(0) < \infty$. The process has finite variance if $A''(0) < \infty$.

We have shown via (3.2) (by letting $\alpha = e^{-t}$) that the LST $\phi(\tau)$ of the marginal distribution of a stationary C-stable C-AR(1) process satisfies the following property: for any $t > 0$, there exist $\lambda(t) > 0$ such that

$$\ln \phi(\tau) = \lambda(t) \ln \phi[C_t(\tau)] , \qquad \tau \ge 0 \ . \tag{3.3}$$

It turns out that this property characterizes such processes.

Theorem 3.1. *A function $\phi(\tau)$ on $\mathbf{R}_+$ is the LST of a C-stable distribution with some exponent $\gamma \in (0, \delta_C]$ if and only if for any $t > 0$, there exists $\lambda(t) > 0$ such that (3.3) holds for every $\tau \geq 0$. The function $\lambda(t)$ is necessarily of the form $\lambda(t) = e^{\gamma t}$.*

Proof: The 'only if' part follows from the preceding discussion. We prove only the 'if' part. Let $\psi(\tau) = \ln \phi(\tau) / \ln \phi(1)$. By (3.3), we have for any $t > 0$ and $\tau \geq 0$ (note $\lambda(t) = 1/\psi(C_t(1))$),

$$\psi\big(C_t(\tau)\big) = \psi\big(C_t(1)\big)\, \psi(\tau)\,, \qquad \tau \geq 0\,. \tag{3.4}$$

By differentiating (3.4) w.r.t. t, we obtain

$$\frac{\partial}{\partial t} C_t(\tau)\, \psi'\big(C_t(\tau)\big) = \frac{\partial}{\partial t} C_t(1)\, \psi'\big(C_t(1)\big)\, \psi(\tau)\,, \qquad \tau \geq 0\,.$$

Using $\frac{\partial}{\partial t} C_t(\tau) = U(C_t(\tau))$ and letting $t \downarrow 0$, it follows by (1.1) that

$$\frac{\psi'(\tau)}{\psi(\tau)} = \frac{U(1)}{U(\tau)}\, \psi'(1)\,, \qquad \tau \geq 0\,,$$

whose solution is $\psi(\tau) = A(\tau)^\gamma$ where $\gamma = -\psi'(1)\, U(1) > 0$. Hence, $\phi(\tau)$ has the form (3.1). Since $\phi(\tau)$ is an LST, γ must satisfy $\gamma \leq \delta_F$ (it follows by adapting to our case the argument in the proof of Lemma 4.2. in van Harn and Steutel (1993)). The form of $\lambda(t)$ results from its uniqueness and the 'only if' part. $\square$

Next, we present a stationary C-AR(1) process with a C-geometric stable marginal distribution.

An $\mathbf{R}_+$-valued rv X is said to have a C-geometric stable distribution if for any $p \in (0,1)$, there exists $\alpha(p) \in (0,1)$ such that

$$X \stackrel{d}{=} \alpha(p) \odot_C \sum_{i=1}^{N_p} X_i\,,$$

where $(X_i, i \geq 1)$ is a sequence of iid $\mathbf{R}_+$-valued rv's, $X_i \stackrel{d}{=} X$, N_p has the geometric distribution with parameter p, and $(X_i, i \geq 1)$ and N_p are independent (Bouzar, 1999). C-geometric stable distributions are C-self-decomposable and their LST's admit the canonical representation

$$\phi(\tau) = \big(1 + \lambda\, A(\tau)^\gamma\big)^{-1}\,, \qquad \tau \geq 0\,, \tag{3.5}$$

for $0 < \gamma \leq \delta_C$ and $\lambda > 0$. We will refer to a distribution with LST (3.5) as C-geometric stable with exponent γ.

By Theorem 2.2, for every $\alpha \in (0,1)$, there exists a stationary C-AR(1) process $(X_n, n \in \mathbf{Z})$ with a C-geometric stable marginal distribution with LST

(3.5). Its innovation sequence $(\epsilon_n, n \in \mathbf{Z})$ has marginal LST (obtained by solving for $\phi_\epsilon(\tau)$ in (2.3) and by using (1.3))

$$\phi_\epsilon(\tau) = \alpha^\gamma + (1-\alpha^\gamma)\big(1+\lambda\, A(\tau)^\gamma\big)^{-1}, \qquad \tau \geq 0 \,, \tag{3.6}$$

where $0 < \gamma \leq \delta_C$ and $\lambda > 0$.

It follows from (3.6) that a stationary C-AR(1) process $(X_n, n \in \mathbf{Z})$ with a C-geometric stable marginal distribution can be written as

$$X_n = \alpha \odot_C X_{n-1} + I_n E_n \;, \qquad n \in \mathbf{Z} \;, \tag{3.7}$$

where $(I_n, n \in \mathbf{Z})$ and $(E_n, n \in \mathbf{Z})$ are independent sequences of iid rv's such that I_n is Bernoulli$(1-\alpha^\gamma)$ and E_n has the same distribution as X_n.

A stationary C-AR(1) process with a C-geometric stable marginal distribution has finite mean only if $\gamma = \delta_F$ and $A'(0) < \infty$. It has a finite variance if $A''(0) < \infty$.

We have in fact shown by the above argument (and by letting $\alpha = e^{-t}$) that the LST $\phi(\tau)$ of the marginal distribution of a stationary C-geometric stable C-AR(1) process satisfies the following property: for any $t > 0$ there exists $c(t) \in (0,1)$ such that

$$\phi(\tau) = \phi\big(C_t(\tau)\big)\Big(c(t) + \big(1-c(t)\big)\,\phi(\tau)\Big), \qquad \tau \geq 0 \;. \tag{3.8}$$

We show next that the converse is true.

Theorem 3.2. *A function $\phi(\tau)$ on $\mathbf{R}_+$ is the LST of a C-geometric stable distribution with some exponent $\gamma \in (0, \delta_C]$ if and only if for any $t > 0$ there exists $c(t) \in (0,1)$ such that (3.8) holds. The function $c(t)$ is necessarily of the form $c(t) = e^{-\gamma t}$.*

Proof: The 'only if' part was established in the preceding discussion. We show the 'if' part. Rewriting $\phi(\tau) = (1+\psi(\tau))^{-1}$, it follows by (3.8) that for any $t > 0$, there exists $c(t) \in (0,1)$ such that

$$\psi\big(C_t(\tau)\big) = c(t)\,\psi(\tau) \;, \qquad \tau \geq 0 \;. \tag{3.9}$$

Using the exact same argument as the one in the proof of Theorem 3.1 (following (3.4)), we have $\psi(\tau) = \lambda\, A(\tau)^\gamma$ for some $0 < \gamma \leq \delta_C$ and $\lambda > 0$. The form of $c(t)$ follows from its uniqueness and the 'only if' part. □

We define next a compound gamma distribution and construct the corresponding stationary C-AR(1) process.

Let $0 < \gamma \leq \delta_C$, $\lambda > 0$, and $r > 0$. An $\mathbf{R}_+$-valued rv X is said to have a C-compound gamma (γ, λ, r) distribution if its LST has the form

$$\phi(\tau) = \big(1 + \lambda\, A(\tau)^\gamma\big)^{-r}, \qquad \tau \geq 0 . \tag{3.10}$$

Note that $\phi(\tau)$ indeed results from the compounding of C-stable distributions (with LST (3.1)) by a gamma distribution (with LST $\phi_1(\tau) = (1+\tau)^{-r}$). The special case $r = 1$ in (3.10) gives the C-geometric stable distribution. van Harn and Steutel (1993) showed that C-compound gamma distributions are C-self-decomposable (see also Proposition 3.1 below) and arise as solutions to stability equations for $\mathbf{R}_+$-valued processes with stationary independent increments.

Let $0 < \gamma \leq \delta_C$, $\lambda > 0$, and $r > 0$. By Theorem 2.2, for every $\alpha \in (0, 1)$, there exists a stationary C-AR(1) process $(X_n, n \in \mathbf{Z})$ with a C-compound gamma (γ, λ, r) marginal distribution. Its innovation sequence $(\epsilon_n, n \in \mathbf{Z})$ has LST

$$\phi_\epsilon(\tau) = \left(\frac{1 + \lambda\, \alpha^\gamma A(\tau)^\gamma}{1 + \lambda\, A(\tau)^\gamma}\right)^r, \qquad \tau \geq 0 . \tag{3.11}$$

It can be shown by a straightforward calculations that ϵ_n with LST (3.11) has the representation

$$\epsilon \stackrel{d}{=} \sum_{i=1}^{N} (\alpha^{U_i}) \odot_C W_i , \tag{3.12}$$

where $(W_i, i \geq 0)$ is a sequence of iid C-geometric stable rv's (with LST (3.5)), $(U_i, i \geq 0)$ are iid uniform $(0, 1)$ rv's, and N is Poisson with mean $-r\gamma \ln \alpha$, with all these variables being independent. This allows for a shot-noise interpretation of the process that is similar to the one given by Lawrance (1982) for the gamma AR(1) process. A shot-noise process is defined by

$$X(t) = \sum_{m=N(-\infty)}^{N(t)} \alpha^{t-\tau_m} \odot_C W_m , \tag{3.13}$$

where $(W_m, m \geq 0)$ are $\mathbf{R}_+$-valued iid rv's (amplitudes of the shots) and $(N(t), t \geq 0)$ is a Poisson process with occurrence times at τ_m. If the W_m's have their common LST given by (3.5) and $N(t)$ has rate $-r\gamma \ln \alpha$, then $X(t)$ of (3.13) sampled at $n = 0, \pm 1, \pm 2, \ldots$ gives another representation of the stationary C-AR(1) process (2.1) with a C-compound gamma (γ, λ, r) marginal distribution. The proof of this fact is an adaptation of Lawrance's (1982) argument and the details are omitted.

Other representations of the innovation variable ϵ_n for a C-AR(1) process with a C-compound gamma (γ, λ, r) marginal distribution can be obtained by adapting the ones derived by McKenzie (1987) for an integer-valued AR(1) process and by Walker (2000) for the gamma AR(1) process of Gaver and Lewis (1980).

As in the previously seen models, a stationary C-AR(1) process with a C-compound gamma (γ, λ, r) marginal distribution has finite mean only if $\gamma = \delta_C$ and $A'(0) < \infty$. It has a finite variance if $A''(0) < \infty$.

The C-self-decomposability of the C-geometric stable distributions (with LST (3.5)) and the C-compound gamma distributions (with LST (3.10)) can be derived from the following, more general, result.

Proposition 3.1. *Let $\varphi(\tau)$ be the LST of a self-decomposable distribution on $\mathbf{R}_+$ with respect to the usual multiplication. Then the compound distribution on $\mathbf{R}_+$ with LST*

$$\phi(\tau) = \varphi\big(\lambda A(\tau)^\gamma\big) , \qquad \tau \geq 0 , \tag{3.14}$$

for some $0 < \gamma \leq \delta_C$ and $\lambda > 0$, is C-self-decomposable.

Proof: We note first that $\phi(\tau)$ is indeed an LST. Specifically, it is the LST of the $\mathbf{R}_+$-valued rv $Y = X(T)$ where $(X(t), t \geq 0)$ is an $\mathbf{R}_+$-valued Lévy process such that $X(1)$ has LST (3.1) and T is a rv (independent of $(X(t), t \geq 0)$) with LST $\varphi(\tau)$ (see Steutel and van Harn (2004), Chapter I, Section 3, for a discussion on compound distributions of the type (3.14)). By self-decomposability with respect to the usual multiplication, we have for every $\tau \geq 0$ and $t > 0$

$$\varphi(\tau) = \varphi(e^{-\gamma t}\tau)\, \varphi_{\gamma,t}(\tau) , \tag{3.15}$$

for some LST $\varphi_{\gamma,t}(\tau)$. Combining equations (1.3), (3.14) and (3.15), yields for every $\tau \geq 0$ and $t > 0$,

$$\phi(\tau) = \varphi\big(e^{-\gamma t}\lambda\, A(\tau)^\gamma\big)\, \varphi_{\gamma,t}\big(\lambda\, A(\tau)^\gamma\big) = \varphi\big(\lambda\, A\big(C_t(\tau)\big)\big)\, \varphi_{\gamma,t}\big(\lambda\, A(\tau)^\gamma\big) .$$

Therefore, (2.21) holds for $\phi(\tau)$, with $\phi_t(\tau) = \varphi_{\gamma,t}(\lambda A(\tau)^\gamma)$. The same argument we used above to show that $\phi(\tau)$ is an LST can be repeated to conclude $\phi_t(\tau)$ is also an LST. $\square$

The LST's described by (3.5) and (3.10) are special cases of (3.14). In this case, $\varphi(\tau) = (1+\tau)^{-r}$ (with $r = 1$ for (3.5)). Steutel and van Harn (2004), Chapter 5, Section 9, offer a multitude of examples of LST's $\varphi(\tau)$ from which one can construct stationary C-AR(1) processes (by combining Proposition 3.1 and Theorem 2.2).

Next, we present a random coefficient stationary C-AR(1) process with the C-compound gamma marginal distribution of (LST) (3.10) and with an innovation sequence that is simpler than (3.12) or (3.13).

Let B be a rv taking values in $(0,1)$ and X an $\mathbf{R}_+$-valued rv independent of B. The random coefficient operator $B \odot_C X$ is defined via its LST by the equation

$$\phi_{B\odot_C X}(\tau) = \int_0^1 \phi_X\big(C_{-\ln b/\delta_C}(\tau)\big)\,dF(b)\,, \tag{3.16}$$

where $F(b)$ is the distribution function of B.

Lemma 3.1. *Let $0<s<r$, $0<\gamma\le\delta_C$, and $\lambda>0$. Define $\gamma_1=\frac{\gamma}{\delta_C}$ and note $\gamma_1\in(0,1]$. Assume that B has the probability density function*

$$f(b) \;=\; \frac{\gamma_1\,\Gamma(r)}{\Gamma(s)\,\Gamma(r-s)}\; b^{\gamma_1 s-1}(1-b^{\gamma_1})^{r-s-1}\,, \qquad 0<b<1\;, \tag{3.17}$$

and that X has the C-compound gamma (γ,λ,r) distribution. Then $B\odot_C X$ has a C-compound gamma (γ,λ,s) distribution.

Proof: Using (3.16), (3.17), and the change of variable $w^\gamma=\dfrac{1-b^{\gamma_1}}{1+\lambda A(\tau)^\gamma b^{\gamma_1}}$, we obtain

$$\phi_{B\odot_C X}(\tau) \;=\; \left[\frac{\gamma_1\Gamma(r)}{\Gamma(s)\,\Gamma(r-s)}\int_0^1 (1-w^{\gamma_1})^{s-1}\,w^{\gamma_1(r-s)-1}\,dw\right]\big(1+\lambda\,A(\tau)^\gamma\big)^{-s}\,.$$

Since $\dfrac{\gamma_1\Gamma(r)}{\Gamma(s)\,\Gamma(r-s)}\displaystyle\int_0^1 (1-w^{\gamma_1})^{s-1}\,w^{\gamma_1(r-s)-1}\,dw=1$, the conclusion follows. □

An $\mathbf{R}_+$-valued stochastic process $(X_n,\, n\ge0)$ is said to be a random coefficient C-AR(1) process if it satisfies the equation

$$X_n \;=\; B_n\odot_C X_{n-1}+\epsilon_n\;, \tag{3.18}$$

where $(B_n,\, n\ge1)$ is an iid sequence of rv's with $0<B_n<1$ and $(\epsilon_n,\, n\ge1)$ is an iid sequence of $\mathbf{R}_+$-valued rv's. Moreover, it is assumed that B_n, X_{n-1}, and ϵ_n are mutually independent.

Theorem 3.3. *Let $0<s<r$, $0<\gamma\le\delta_C$, and $\lambda>0$. Let $(X_n,\, n\ge0)$ be the random coefficient C-AR(1) process of (3.18) such that B_n has probability density function (3.17) and ϵ_n has a C-compound gamma $(\gamma,\lambda,r-s)$ distribution. If X_0 has a C-compound gamma (γ,λ,r) distribution, then $(X_n,\, n\ge0)$ is stationary with a C-compound gamma (γ,λ,r) marginal distribution.*

Proof: We have by (3.18) and Lemma 3.1,

$$\phi_{X_1}(\tau) \;=\; \phi_{B_1\odot_C X_0}(\tau)\,\phi_\epsilon(\tau) \;=\; \big(1+\lambda A(\tau)^\gamma\big)^{-s}\big(1+\lambda A(\tau)^\gamma\big)^{s-r} \;=\; \big(1+\lambda A(\tau)^\gamma\big)^{-r}\,.$$

An induction argument shows that X_n has a C-compound gamma (γ,λ,r) for all $n\ge1$. Since $(X_n,\, n\ge1)$ is a Markov process, stationarity ensues. □

We conclude the section by mentioning a family of semigroups of cgf's. For $\theta \in [0,1)$, let

$$C_t^{(\theta)}(\tau) = \frac{\overline{\theta}\, e^{-\overline{\theta}t}\tau}{\overline{\theta} + \theta(1-e^{-\overline{\theta}t})\tau}\,, \qquad t,\tau \geq 0, \quad \overline{\theta} = 1-\theta\,. \tag{3.19}$$

It is easy to verify that $C_t^{(\theta)}(\tau)$ has a completely monotone derivative and hence is a cgf. Moreover, straightforward calculations show that the properties in (1.1) hold. Therefore, $C^{(\theta)} = (C_t^{(\theta)},\, t \geq 0)$ is a continuous semigroup of cgf's. In this case

$$U^{(\theta)}(\tau) = -\tau(\overline{\theta}+\theta\tau)\,, \qquad A^{(\theta)}(\tau) = \left(\frac{\tau}{\overline{\theta}+\theta\tau}\right)^{1/\overline{\theta}}, \qquad \delta_C^{(\theta)} = \overline{\theta}\,. \tag{3.20}$$

The special case $\theta = 0$ corresponds to the ordinary multiplication. The stationary $C^{(0)}$-AR(1) process with a $C^{(0)}$-stable marginal distribution corresponds to the AR(1) process with the standard stable distribution on $\mathbf{R}_+$ as its marginal. The stationary $C^{(0)}$-AR(1) process with a $C^{(0)}$-geometric stable marginal distribution reduces to the Mittag–Leffler AR(1) process of Jayakumar and Pillai (1993). The stationary $C^{(0)}$-AR(1) process with a $C^{(0)}$-compound gamma (with LST (3.10)) becomes the gamma AR(1) process of Gaver and Lewis (1980).

4. TIME-REVERSIBILITY OF STATIONARY C-AR(1) PROCESSES

A stochastic process $(X_n,\, n \in \mathbf{Z})$ is said to be time-reversible if for any $n \in \mathbf{Z}$ and $k \geq 0$, $(X_n, X_{n+1}, ..., X_{n+k})$ and $(X_{n+k}, X_{n+k-1}, ..., X_n)$ have the same joint distribution.

Let $(X_n,\, n\in\mathbf{Z})$ be a C-AR(1) process. By the Markov property, $(X_n,\, n\in\mathbf{Z})$ is time-reversible if and only if for any $n \in \mathbf{Z}$, (X_{n-1}, X_n) and (X_n, X_{n-1}) have the same joint distribution. $(X_n,\, n \in \mathbf{Z})$ is time-reversible if and only if for every $n \in \mathbf{Z}$,

$$\phi_{(X_{n-1},X_n)}(\tau_1,\tau_2) = \phi_{(X_{n-1},X_n)}(\tau_2,\tau_1)\,, \qquad \tau_1 \geq 0, \quad \tau_2 \geq 0\,, \tag{4.1}$$

where $\phi_{(X_{n-1},X_n)}(\tau_1,\tau_2)$ is the joint LST of (X_{n-1}, X_n).

By Proposition 2.2-(i), a time-reversible C-AR(1) process $(X_n,\, n \in \mathbf{Z})$ (such that $E(X_n) < \infty$ and $E(\epsilon_n) < \infty$) possesses the property of linear backward regression. That is, there exist $c > 0$ and $d \geq 0$ such that for any $n \in \mathbf{Z}$,

$$E(X_{n-1}|X_n) = d + cX_n\,. \tag{4.2}$$

We show next that a stationary C-AR(1) process with finite mean and finite variance has the property of linear backward regression only if its LST admits a certain form.

Theorem 4.1. *Let $(X_n, n \in \mathbf{Z})$ be a stationary C-AR(1) process with finite mean and finite variance with the property of linear backward regression (4.2). Further, assume*

$$C_t(1) \sim a e^{-\delta_C t} \qquad (t \to \infty) , \tag{4.3}$$

for some constant $a > 0$. Then the marginal distribution of $(X_n, n \in \mathbf{Z})$ is infinitely divisible with LST $\phi(\tau)$ of the form

$$\phi(\tau) = \exp\left\{-\int_0^\tau \left(b - \lambda A(x)^{\delta_C}\right) dx\right\} , \tag{4.4}$$

for some $b > 0$ and $\lambda > 0$.

Proof: Let $n \in \mathbf{Z}$ and let $\phi(\tau)$, $\tau \geq 0$, and $g(\tau_1, \tau_2)$, $\tau_1, \tau_2 \geq 0$, be the LST of X_n and joint LST of (X_{n-1}, X_n), respectively. Recall that by the stationarity assumption, both $\phi(\tau)$ and $g(\tau_1, \tau_2)$ are independent of n. By Proposition 2.1-(ii) and equation (2.3), we have for any $\tau_1,\ \tau_2 \geq 0$

$$g(\tau_1, \tau_2) = \phi_\epsilon(\tau_2)\, \phi\big(\tau_1 + C_t(\tau_2)\big) = \frac{\phi\big(\tau_1 + C_t(\tau_2)\big)\, \phi(\tau_2)}{\phi\big(C_t(\tau_2)\big)} . \tag{4.5}$$

Differentiating g with respect to τ_1, then setting $\tau_1 = 0$ and $\tau_2 = \tau$, it follows that for any $n \in \mathbf{Z}$,

$$E(X_{n-1} e^{-\tau X_n}) = -\frac{\phi(\tau)}{\phi\big(C_t(\tau)\big)}\, \phi'\big(C_t(\tau)\big) , \qquad \tau \geq 0 . \tag{4.6}$$

By the property of linear backward regression (see equation (4.2)), we have for some $c > 0$ and $d \geq 0$,

$$E(X_{n-1} e^{-\tau X_n}) = E\big(e^{-\tau X_n} E(X_{n-1}|X_n)\big) = c\, E(X_n e^{-\tau X_n}) + d\, E(e^{-\tau X_n}) ,$$

for any $n \in \mathbf{Z}$ and $\tau \geq 0$. Noting that $E(X_n e^{-\tau X_n}) = -\phi'(\tau)$, it follows that

$$E(X_{n-1} e^{-\tau X_n}) = d\phi(\tau) - c\,\phi'(\tau) , \qquad \tau \geq 0 . \tag{4.7}$$

Letting $h(\tau) = \phi'(\tau)/\phi(\tau)$ and combining (4.6) and (4.7) , we obtain

$$c\, h(\tau) - d = h\big(C_t(\tau)\big) , \qquad \tau \geq 0 .$$

It follows by differentiation that $c\, h'(\tau) = C_t'(\tau)\, h'(C_t(\tau))$. Noting that $h'(0) = \mathrm{Var}(X_n) \neq 0$ (and recalling that $C_t(0) = 0$), it follows that $c = C_t'(0) = e^{-\delta_C t}$, with the second equation following from (1.4). This implies

$$h'(\tau) = e^{\delta_C t} C_t'(\tau)\, h'\big(C_t(\tau)\big) , \qquad \tau \geq 0 .$$

An induction argument yields for any $n \geq 1$,

$$h'(\tau) = e^{n\delta_C t} h'\big(C_{nt}(\tau)\big) \prod_{j=0}^{n-1} C_t'\big(C_{jt}(\tau)\big) , \qquad \tau \geq 0 .$$

By the semigroup properties (1.1) and (1.3), we have

$$C_t'\big(C_{jt}(\tau)\big) = U\big(C_{(j+1)t}(\tau)\big)/U\big(C_{jt}(\tau)\big) , \qquad j = 0, ..., n-1 .$$

Therefore,

$$h'(\tau) = e^{n\delta_C t} \frac{U\big(C_{nt}(\tau)\big)}{U(\tau)} h'\big(C_{nt}(\tau)\big) , \qquad \tau \geq 0 . \tag{4.8}$$

Calling again on the semigroup properties (1.1) and (1.3), we have for any $\tau \geq 0$,

$$\lim_{n\to\infty} C_{nt}(\tau) = 0 , \qquad \lim_{n\to\infty} \frac{U\big(C_{nt}(\tau)\big)}{C_{nt}(\tau)} = U'(0) = -\delta_C .$$

By Lemma 3.2-(i) in Hansen (1989),

$$\lim_{n\to\infty} \frac{C_{nt}(\tau)}{C_{nt}(1)} = A(\tau)^{\delta_C} , \qquad \tau \geq 0 .$$

Moreover, (4.3) implies

$$\lim_{n\to\infty} e^{n\delta_C t} C_{nt}(1) = a .$$

Therefore, by letting $n \to \infty$ in (4.8), we obtain

$$h'(\tau) = -a\,\delta_C\, h'(0) \frac{A(\tau)^{\delta_C}}{U(\tau)} , \qquad \tau \geq 0 .$$

Since by (1.2) $1/U(\tau) = -A'(\tau)/A(\tau)$, we have

$$h(\tau) - h(0) = \int_0^\tau h'(x)\,dx = a h'(0) \int_0^\tau \delta_C\, A'(x)\, A(x)^{\delta_C - 1} , \qquad \tau \geq 0 ,$$

which implies (note $A(0) = 0$)

$$h(t) = h(0) + a h'(0)\, A(\tau)^{\delta_C} , \qquad \tau \geq 0 ,$$

or $\phi'(\tau)/\phi(\tau) = h(0) + a h'(0)\, A(\tau)^{\delta_C}$. It follows

$$\ln \phi(\tau) = h(0)\tau + a h'(0) \int_0^\tau A(x)^{\delta_C}\, dx , \qquad \tau \geq 0 .$$

The representation (4.4) follows by letting $b = -h(0)$ and $\lambda = a h'(0)$. To show that $\phi(\tau)$ of (4.4) is indeed the LST of an infinitely divisible distribution, let $\psi(\tau) = b - \lambda A(\tau)^{\delta_C}$, $\tau \geq 0$. By Theorem 4.2, Chapter III, Section 4, in Steutel and van Harn (2004), it is enough to establish that $\psi(\tau)$ is completely monotone on $(0, \infty)$. Since $\phi_1(\tau) = \exp(-\lambda A(\tau)^{\delta_C})$ is the LST of a C-stable distribution, it is infinitely divisible (van Harn and Steutel, 1993). It follows that the function $\psi_1(\tau) = -\ln\phi_1(\tau) = \lambda\, A(\tau)^{\delta_C}$ has a completely monotone derivative on $(0,\infty)$ (again by Theorem 4.2 in Steutel and van Harn, 2004, quoted above). Since $\psi'(\tau) \le 0$ and for any $n \ge 2$,

$$(-1)^n\,\psi^{(n)}(\tau) = (-1)^{n-1}\,(\psi_1')^{(n-1)}(\tau)\ , \qquad \tau > 0\ ,$$

it follows that $\psi(\tau)$ is completely monotone on $(0,\infty)$. □

We note that Theorem 4.1 remains valid if the property of linear backward regression is replaced by the (stronger) assumption of time-reversibility.

For the family of semigroups $(C^{(\theta)}, \theta \in [0,1))$ of (3.19), the condition (4.3) is easily seen to be satisfied (by (3.19)–(3.20)) as $C_t^{(\theta)}(1) \sim \overline{\theta}\, e^{-\overline{\theta} t}$ $(t \to \infty)$. Applying Theorem 4.1 to the semigroup $C^{(\theta)}$ $(\theta \in [0,1))$, we obtain (via (3.20)) the LST $\phi^{(\theta)}(\tau)$ of (4.4) to be, in the case $\theta = 0$,

$$\phi^{(0)}(\tau) = \exp\left\{-b\,\tau + \frac{\lambda}{2}\,\tau^2\right\}, \qquad \tau \ge 0\ , \tag{4.9}$$

for some $b > 0$ and $\lambda > 0$, and in the case $0 < \theta < 1$,

$$\phi^{(\theta)}(\tau) = e^{-c\tau}\left(1 + \frac{\theta}{\overline{\theta}}\,\tau\right)^{-r}, \qquad \tau \ge 0\ , \tag{4.10}$$

for some $c \ge 0$ and $r > 0$. We note that if a rv X has LST $\phi^{(\theta)}(\tau)$ given by (4.10), for $0 < \theta < 1$, then $X \overset{d}{=} c + Y$, where Y admits a gamma distribution with LST

$$\varphi^{(\theta)}(\tau) = \left(1 + \frac{\theta}{\overline{\theta}}\,\tau\right)^{-r}, \qquad \tau \ge 0\ . \tag{4.11}$$

It is a simple exercise to verify that $\phi^{(0)}(\tau)$ of (4.9) is the LST of a $C^{(0)}$-self-decomposable distribution. In this case, the LST $\phi_t^{(0)}(\tau)$ in equation (2.21) is

$$\phi_t^{(0)}(\tau) = \exp\left\{-b\,(1 - e^{-t})\,\tau + \frac{\lambda}{2}\,(1 - e^{-2t})\,\tau^2\right\}, \qquad \tau \ge 0\ . \tag{4.12}$$

Assume that $(X_n, n \in \mathbf{Z})$ is a stationary $C^{(0)}$-AR(1) process with marginal LST $\phi^{(0)}(\tau)$. The marginal LST of the innovation sequence $(\epsilon_n, n \in \mathbf{Z})$ of $(X_n, n \in \mathbf{Z})$ is given by (4.12). Combining (4.5) with (4.9) and (4.12), we obtain the joint LST of (X_{n-1}, X_n) to be

$$g_0(\tau_1, \tau_2) = \exp\left\{-b(t_1 + \tau_2) + \frac{\lambda}{2}\,(\tau_1^2 + 2\,e^{-t}\tau_1\tau_2 + \tau_2^2)\right\}, \qquad \tau_1 \ge 0, \quad \tau_2 \ge 0\ .$$

Since $g_0(\tau_1, \tau_2) = g_0(\tau_2, \tau_1)$, it follows that $(X_{n-1}, X_n) \stackrel{d}{=} (X_n, X_{n-1})$. Therefore, $(X_n, n \in \mathbf{Z})$ is time-reversible (and hence, has the property of linear backward regression).

The case $0 < \theta < 1$ is slightly more involved. We need a lemma.

Lemma 4.1. *Let $0 < \theta < 1$. A distribution μ on $\mathbf{R}_+$ with LST $\phi^{(\theta)}(\tau)$ of (4.10) is $C^{(\theta)}$-self-decomposable if and only if $c = 0$ or, equivalently, if and only if μ is a gamma distribution with LST $\varphi^{(\theta)}(\tau)$ of (4.11).*

Proof: Assume that μ has LST $\varphi^{(\theta)}(\tau)$ of (4.11). Straightforward calculations show that for any $t > 0$ and $\tau \geq 0$,

$$\varphi_t^{(\theta)}(\tau) = \frac{\varphi^{(\theta)}(\tau)}{\varphi^{(\theta)}\big(C_t^{(\theta)}(\tau)\big)} = \left(1 + \frac{\theta}{\overline{\theta}}\big(1 - e^{-\overline{\theta}t}\big)\tau\right)^{-r}. \tag{4.13}$$

Clearly, $\varphi_t^{(\theta)}(\tau)$ is the LST of a gamma distribution. Therefore, μ is $C^{(\theta)}$-self-decomposable. Conversely, assume that μ is $C^{(\theta)}$-self-decomposable with LST $\phi^{(\theta)}(\tau)$ of (4.10). By Theorem 5.4 in van Harn and Steutel (1993),

$$\ln \phi^{(\theta)}(\tau) = -\int_0^\tau \frac{\ln f(x)}{U^{(\theta)}(x)}\, dx\,, \qquad \tau \geq 0\,, \tag{4.14}$$

where $f(\tau)$ is the LST of an infinitely divisible distribution on $\mathbf{R}_+$. By differentiating both sides of (4.14) and using (3.20), we deduce that for every $\tau \geq 0$,

$$-\ln f(\tau) = U^{(\theta)}(\tau)\,\frac{d}{d\tau}\ln \phi^{(\theta)}(\tau) = c\,\theta\tau^2 + (c\,\overline{\theta} + r\,\theta)\tau\ .$$

By Theorem 4.2, Chapter III, Section 4, in Steutel and van Harn (2004), the function

$$-\frac{d}{d\tau}\ln f(\tau) = 2\,c\,\theta\tau + c\,\overline{\theta} + r\,\theta$$

must be a completely monotone function on $(0, \infty)$. This can only hold if $c = 0$. □

Assume now that $(X_n, n \in \mathbf{Z})$ is a stationary $C^{(\theta)}$-AR(1) process with marginal LST $\varphi^{(\theta)}(\tau)$ of (4.11). The marginal LST of the innovation sequence $(\epsilon_n, n \in \mathbf{Z})$ of $(X_n, n \in \mathbf{Z})$ is given by (4.13). Using (4.5), along with (4.11) and (4.13), we find the joint LST of (X_{n-1}, X_n) to be

$$g_\theta(\tau_1, \tau_2) = \left(1 + \frac{\theta}{\overline{\theta}}(\tau_1 + \tau_2) + \frac{\theta^2}{\overline{\theta}^2}\,\tau_1\tau_2\right)^{-r}, \qquad \tau_1 \geq 0,\quad \tau_2 \geq 0\ .$$

Since $g_\theta(\tau_1, \tau_2) = g_\theta(\tau_2, \tau_1)$, it follows that $(X_{n-1}, X_n) \stackrel{d}{=} (X_n, X_{n-1})$. Therefore, $(X_n, n \in \mathbf{Z})$ is time-reversible (and hence, has the property of linear backward regression).

We summarize our discussion in the following proposition.

Proposition 4.1. *Let $\theta \in [0,1)$. A stationary $C^{(\theta)}$-AR(1) process with a $C^{(\theta)}$-self-decomposable marginal distribution has the property of linear backward regression if and only if its marginal LST is given by (4.9), if $\theta = 0$, or by (4.11), if $0 < \theta < 1$.*

The stationary $C^{(\theta)}$-AR(1) process with the gamma marginal distribution with LST (4.11) is equivalent to the gamma model developed by Sim (1990). Sim makes use of a generalized multiplication based on a conditional compound Poisson distribution. Sim's operator is tailored to lead to a stationary gamma AR(1) process and offers no other stationary solutions. On the other hand, and as seen in Section 3, the $\odot_{C^{(\theta)}}$-multiplication leads to a variety of stationary models.

We conclude by noting that the extension of Proposition 4.1 to an arbitrary semigroup of cgf's C is an open question. Specifically, what kind of semigroups will give rise to a C-self-decomposable distribution with LST of the form (4.4)?

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