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FORECASTING IN INAR(1) MODEL

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Abstract:

- In this work we consider the problem of forecasting integer-valued time series, modelled by the INAR(1) process introduced by McKenzie (1985) and Al-Osh and Alzaid (1987). The theoretical properties and practical applications of INAR and related processes have been discussed extensively in the literature but there is still some discussion on the problem of producing coherent, i.e. integer-valued, predictions. Here Bayesian methodology is used to obtain point predictions as well as confidence intervals for future values of the process. The predictions thus obtained are compared with their classic counterparts. The proposed approaches are illustrated with a simulation study and a real example.

Key-Words:

- *INAR models; Bayesian prediction; integer prediction; Markov Chain Monte Carlo algorithm.*

1. INTRODUCTION

In applications we are frequently faced with time series whose characteristics are not compatible with a continuous modelling approach. Discrete variate time series occur in many contexts, often as counts of events or individuals in consecutive intervals or at consecutive points in time. Examples of these are the number of costumers waiting to be served, the daily number of absent workers in a firm, the number of busy lines in a telephone network noted every hour, the number of accidents in a manufacturing plant each month, etc. Several models that take the discreteness of the data explicitly into account have been developed in the literature. Cox (1981) proposed dividing them into two categories: observation-driven and parameter-driven models. MacDonald and Zucchini (1997), Cameron and Trivedi (1998) and the review by McKenzie (2003) provide an excellent overview of the literature in this area.

In this work we are interested in a special class of observation-driven models, the so-called integer-valued autoregressive (INAR) process introduced by McKenzie (1985) and Al-Osh and Alzaid (1987). The theoretical properties and practical applications of INAR and related processes have been discussed extensively in the literature. Silva *et al.* (2005) consider independent replications of count time series modelled by INAR(1) and proposed several estimation methods using the classical and Bayesian approaches in time and frequency domains. Nevertheless, there is still little consensus on which processes or model classes are best used in practice in contrast to the role played by the Box-Jenkins Gaussian ARMA methodology for continuous variables. This is partly due to the lack of reliable techniques for estimation, testing and prediction. In particular, the lack of forecasting methods that are coherent in the sense of producing only integer-valued predictions, seems to render useless the effort of using more complex models.

Usually forecasts are obtained from the conditional expectations, which have the optimality property but rarely will generate integer values. In order to produce coherent forecasts Freeland and McCabe (2003) use the median of the k-step-ahead conditional distribution to emphasize the intention of preserving the integer structure of the data in generating the forecasts. McCabe and Martin (2005) develop a general methodology for producing coherent predictions of low count data. In contrast to the usual applications of the model INAR(1), in which the arrival process is usually Poisson, they allow the arrivals to follow any distribution in the integer class. The forecasts are based on an estimate of the k-step–ahead predictive probability mass function. To eliminate unwanted values from the conditioning set of the predictive function, Bayesian methods are used. Jung and Tremayne (2006) extend some of the ideas used by Freeland and McCabe in higher order dependence structure by proposing a computer intensive method for generating coherent, integer out-of-sample predictions, particularly

obtaining the h-step-ahead predictor for the INAR(2). They propose a Monte Carlo approach using bootstrap methods to estimate the sampling distributions as a mean of generating one and multi-step ahead forecasts which respect the integer structure of the data.

The purpose of this paper is to obtain coherent forecasts for the Poisson INAR(1) process. Bayesian methodology is used to obtain point predictions as well as credibility intervals for future values of the process which are compared with their classic counterpart.

The remainder of the paper is divided into four main sections. Section 2 provides the theoretical results in order to obtain the point forecasts. Section 3 presents methods for producing confidence intervals or highest posterior predictive density intervals for forecasts. In Section 4 we conduct a simulation study to compare the performance of the classical and Bayesian approaches, considering point and interval predictions. Section 5 gives an example of forecasting a count data series using PoINAR(1) model. The data are the number of claimants receiving wage loss benefits due to injuries from burns, supplied by the Workers Compensation Board of the Province of British Columbia, Canada. The proposed methodology presented in this work is applied to this data set and compared with classical inference and forecasting procedures from Freeland (1998).

2. POINT PREDICTION

Consider a non negative integer-valued random variable X and $\alpha \in [0,1]$, the generalized thinning operation, hereafter denoted by '$\circ$', is defined as

$$\alpha \circ X = \sum_{j=1}^{X} Y_j \ , \tag{2.1}$$

where $\{Y_j\}$, $j = 1, ..., X$, is a sequence of independent and identically distributed non-negative integer-valued random variables, independent of X, with finite mean α and variance σ^2. This sequence is called the counting series of $\alpha \circ X$. When $\{Y_j\}$ is a sequence of Bernoulli random variables, the thinning operation is called binomial thinning operation and was defined by Steutel and van Harn (1979).

The well-known INAR(1) process $\{X_t;\ t = 0, \pm 1, \pm 2, ...\}$ is defined on the discrete support $\mathbb{N}_0$ by the equation

$$X_t = \alpha \circ X_{t-1} + \epsilon_t \ ,$$

where $0 < \alpha < 1$, $\{\epsilon_t\}$ is a sequence of independent and identically distributed integer-valued random variables, with $E[\epsilon_t] = \mu_\epsilon$ and $\mathrm{Var}[\epsilon_t] = \sigma^2_\epsilon$.

In this paper we consider only Poisson INAR(1) process, i.e., $\{\epsilon_t\}$ is a sequence of independent Poisson distributed variables with parameter λ, independent of all counting series $\{Y_j\}$. Note that, assuming $\epsilon_t \frown Po(\lambda)$ it is straightforward to show that $X_t \frown Po\big(\lambda/(1-\alpha)\big)$. The Poisson INAR(1) process will henceforth be denoted PoINAR(1).

Given that we have observed the series up through time n, i.e., $\mathbf{x}_n = (x_1, x_2, ..., x_n)$ is known, the most common procedure for constructing predictions in time series models is to use conditional expectations. The following theorem states and important result in this context.

Theorem 2.1 (Freeland, 1998, pp. 30)**.** *The moment generation function of X_{n+h} given X_n is*

$$\varphi_{X_{n+h}|X_n}(s) \;=\; \big[\alpha^h e^s + (1-\alpha^h)\big]^{X_n} \exp\left[\lambda\,\frac{1-\alpha^h}{1-\alpha}\,(e^s-1)\right]. \tag{2.2}$$

Expression (2.2) shows that the distribution of $X_{n+h}|X_n$ is a convolution of a binomial distribution with parameters α^h and X_n and a Poisson distribution with parameter $\lambda(1-\alpha^h)/(1-\alpha)$. That is, the probability function of $X_{n+h}|X_n$ is given by

$$\begin{aligned} f(x_{n+h}|x_n) \;&=\; P\big(X_{n+h}=x_{n+h}\,|\,X_n=x_n\big) \\ &=\; \exp\left\{-\lambda\,\frac{1-\alpha^h}{1-\alpha}\right\} \sum_{i=0}^{M_h} \frac{1}{(x_{n+h}-i)!} \\ &\quad\times \left(\lambda\,\frac{1-\alpha^h}{1-\alpha}\right)^{x_{n+h}-i} \binom{x_n}{i} (\alpha^h)^i\,(1-\alpha^h)^{x_n-i}\,, \qquad x_{n+h}=0,1,\ldots\,, \end{aligned} \tag{2.3}$$

where $M_h = \min(X_{n+h}, X_n)$. Consequently, we have the following corollary:

Corollary 2.1. *The INAR(1) model satisfies the properties*

a) $E\big[X_{n+h}|X_n\big] = \alpha^h\left[X_n - \dfrac{\lambda}{1-\alpha}\right] + \dfrac{\lambda}{1-\alpha}\,, \quad h=1,2,3,...,$

b) $\mathrm{Var}\big[X_{n+h}|X_n\big] = \alpha^h(1-\alpha^h)\,X_n + \lambda\,\dfrac{1-\alpha^h}{1-\alpha}\,, \quad h=1,2,3,...,$

c) *As $h\to+\infty$, $X_{n+h}|X_n$ is a Poisson distribution with parameter $\lambda/(1-\alpha)$.*

So, we can conclude that for α constant,

$$\lim_{h\to+\infty} E\big[X_{n+h}|X_n\big] \;=\; \lim_{h\to+\infty} \mathrm{Var}\big[X_{n+h}|X_n\big] \;=\; \lambda/(1-\alpha)\,,$$

i.e., as $h\to\infty$ and $0<\alpha<1$, the mean and the variance of $X_{n+h}|X_n$ remain equal and approach the mean of the process.

2.1. Classical methodology

The h-step-ahead predictor based on the conditional expectation of INAR(1),

$$\hat{X}_{n+h}|\mathbf{x}_n = E\big[X_{n+h}|X_n\big] = \alpha^h\left[X_n - \frac{\lambda}{1-\alpha}\right] + \frac{\lambda}{1-\alpha}\ , \quad h = 1,2,3,... \tag{2.4}$$

was obtained by Bränäs (1994) and Freeland and McCabe (2003), but it will hardly produce integer-valued forecasts. In order to obtain coherent predictions for X_{n+h} Freeland and McCabe (2003) suggest using the value which minimizes the expected absolute error given the sample, i.e., the value that minimizes $E\big[|X_{n+h}-\hat{X}_{n+h}|\,\big|\,X_n\big]$. So, they concluded that $\hat{X}_{n+h} = \hat{m}_{n+h}$ is the median of the h-step-ahead conditional distribution $f(x_{n+h}|x_n)$.

2.2. Bayesian methodology

The Bayesian predictive probability function is based on the assumption that, both, the future observation, X_{n+h} and the vector of unknown parameters $\boldsymbol{\theta} = (\alpha, \lambda)$ are random. As we know the information about $\boldsymbol{\theta}$ is given by the observed sample $\mathbf{x}_n$ and quantified in the posterior predictive, $\pi(\boldsymbol{\theta}|\mathbf{x}_n)$.

Definition 2.1. Let $\boldsymbol{\theta} \in \Theta$ be the vector of unknown parameters. The h-step-ahead Bayesian posterior predictive distribution is given by

$$\begin{aligned} f(x_{n+h}|\mathbf{x}_n) &= \int_\Theta f(x_{n+h};\boldsymbol{\theta}|\mathbf{x}_n)\, d\boldsymbol{\theta} \\ &= \int_\Theta f(x_{n+h}|\mathbf{x}_n;\boldsymbol{\theta})\, \pi(\boldsymbol{\theta}|\mathbf{x}_n)\, d\boldsymbol{\theta}\ , \end{aligned} \tag{2.5}$$

where $\pi(\boldsymbol{\theta}|\mathbf{x}_n)$ is the posterior probability function of $\boldsymbol{\theta}$ and $f(x_{n+h}|\mathbf{x}_n;\boldsymbol{\theta})$ is the predictive distribution (classical) given by (2.3).

The h-step-ahead predictive distribution of $X_{n+h}|\mathbf{x}_n$ given by expression (2.5) can be viewed as having all information about the future values. Once $f(x_{n+h}|\mathbf{x}_n)$ is obtained, the Bayesian h-step-ahead predictor can be given by the expected valued, the median or the mode of X_{n+h} given $\mathbf{x}_n$.

Since beta and gamma are conjugate of binomial and Poisson distributions, respectively, we use them for prior distributions of the parameters to INAR(1) model, $\alpha \frown Beta(a,b)$, $a,b>0$ and $\lambda \frown Gamma(c,d)$, $c,d>0$. Considering independence between α and λ, the prior distribution of (α,λ) is given by

$$p(\alpha,\lambda) \propto \lambda^{c-1}\exp(-d\lambda)\,\alpha^{a-1}(1-\alpha)^{b-1}\ , \qquad \lambda>0,\ \ 0<\alpha<1\ , \tag{2.6}$$

where a, b, c and d are known parameters. Note that, as $a \to 0$, $b \to 0$, $c \to 0$ and $d \to 0$ we have a vague prior distribution.

The posterior distribution of (α, λ) can be written as

$$\begin{aligned} p(\alpha,\lambda|\mathbf{x}_n) \;&\propto\; L(\mathbf{x}_n,\alpha,\lambda|x_1)\; p(\lambda,\alpha) \\ &=\; \exp\Big[-\big(d+(n-1)\big)\,\lambda\Big]\,\lambda^{c-1}\,\alpha^{a-1}(1-\alpha)^{b-1} \\ &\quad\times\; \prod_{t=2}^{n}\sum_{i=0}^{M_t}\frac{\lambda^{x_t-i}}{(x_t-i)!}\binom{x_{t-1}}{i}\alpha^i(1-\alpha)^{x_{t-1}-i}\;, \end{aligned}$$

where $L(\mathbf{x}_n|x_1)$ is the conditional likelihood function and $M_t = \min(X_t, X_{t-1})$.

Consequently for the PoINAR(1) model, the Bayesian predictive function of X_{n+h} given $\mathbf{x}_n$ is given by

$$\begin{aligned} f(x_{n+h}|\mathbf{x}_n) \;\propto\; &\int_\alpha\int_\lambda \sum_{i=0}^{M_h}\binom{x_n}{i}(\alpha^h)^i\,(1-\alpha^h)^{x_n-i}\,\frac{1}{(x_{n+h}-i)!} \\ &\times\; \exp\!\left(-\lambda\,\frac{1-\alpha^h}{1-\alpha}\right)\left(\lambda\,\frac{1-\alpha^h}{1-\alpha}\right)^{x_{n+h}-i}\exp\big[-(d+n)\,\lambda\big]\,\lambda^{c-1} \\ &\times\; \alpha^{a-1}(1-\alpha)^{b-1}\prod_{t=2}^{n}\sum_{i=0}^{M_t}\frac{\lambda^{x_t-i}}{(x_t-i)!}\binom{x_{t-1}}{i}\alpha^i(1-\alpha)^{x_{t-1}-i}\,d\alpha\,d\lambda\;. \end{aligned} \tag{2.7}$$

The complexity of $f(x_{n+h}|\mathbf{x}_n)$ does not allow us to work with it directly. In order to estimate X_{n+h}, we can adapt to the integer case the Tanner (1996) composition method. That is, to sample $(X_{n+h,1}, X_{n+h,2}, ..., X_{n+h,m})$, we can use the following algorithm:

Algorithm 2.1

1. From the sample $(X_1, X_2, ..., X_n)$, calculate (through the classical method) a starting estimate for α, denoted by α_0;

2. Using the adaptive rejection Metropolis sampling (ARMS) within Gibbs methodology (see Gilks *et al.*, 1995), calculate from the full conditional distributions of the parameters α and λ, a sample $(\alpha_1, \lambda_1), (\alpha_2, \lambda_2), ..., (\alpha_m, \lambda_m)$;

3. For each i $(i = 1, ..., m)$ sample $X_{n+h,i}$ from $f(x_{n+h}|x_n, \alpha_i, \lambda_i)$, using the inverse transform method adapted to integer variables, that is,

(**a**) sample u from uniform $U(0, 1)$,

(**b**) calculate the least integer-valued s: $\sum_{i=0}^{s} f(x_{n+h}|x_n, \alpha_i, \lambda_i) \geq u$,

(**c**) consider $X_{n+h,i} = s$.

After sampling $X_{n+h,1}, X_{n+h,2}, ..., X_{n+h,m}$, the h-step-ahead predictor of X_{n+h}, can be calculated from the sample mean ($\hat{X}_{n+h}$), the median ($\hat{m}_{n+h}$) or the mode ($\hat{mo}_{n+h}$).

But we can also calculate $E(X_{n+h}|\mathbf{x}_n)$ using an appropriate property of mathematical expectation. As we know $E\big[g(X_{n+h})|\mathbf{x}_n\big] = E\big[E[g(X_{n+h})|\mathbf{x}_n, \boldsymbol{\theta}]\,|\,\mathbf{x}_n\big]$; thus,

$$\begin{aligned} E(X_{n+h}|\mathbf{x}_n) &= E\Big[E(X_{n+h}|\boldsymbol{\theta}, \mathbf{x}_n)\,\Big|\,\mathbf{x}_n\Big] \\ &= E\left[\alpha^h X_n + \frac{1-\alpha^h}{1-\alpha}\,\lambda\,\middle|\,\mathbf{x}_n\right] \\ &= X_n\,E\big[\alpha^h|\mathbf{x}_n\big] + E\left[\frac{1-\alpha^h}{1-\alpha}\,\lambda\,\middle|\,\mathbf{x}_n\right]. \end{aligned}$$

These expected values can be estimated through Markov Chain Monte Carlo (MCMC) algorithms. We perform Metropolis algorithm in conjunction with Adaptive Rejection Sampling Method (ARMS) in order to sample values from full conditional distributions of α and λ; let them be noted by $(\alpha_1, \alpha_2, ..., \alpha_m)$, $(\lambda_1, \lambda_2, ..., \lambda_m)$, respectively (see Silva *et al.*, 2005). We have

$$\hat{E}\big[\alpha^h|\mathbf{x}_n\big] = \frac{1}{m}\sum_{i=1}^{m}\alpha_i^h\,,$$

$$\hat{E}\left[\frac{1-\alpha^h}{1-\alpha}\,\lambda\,\middle|\,\mathbf{x}_n\right] = \frac{1}{m}\sum_{i=1}^{m}\frac{1-\alpha_i^h}{1-\alpha_i}\,\lambda_i\,.$$

Consequently the predictor can be written as

$$\tilde{X}_{n+h} = X_n\left(\frac{1}{m}\sum_{i=1}^{m}\alpha_i^h\right) + \left(\frac{1}{m}\sum_{i=1}^{m}\frac{1-\alpha_i^h}{1-\alpha_i}\,\lambda_i\right). \tag{2.8}$$

3. INTERVAL PREDICTION

In this section we obtain interval predictions for the h-step-ahead observation, using the classical framework and Bayesian methodology.

3.1. Classical methodology

A confidence interval for the predictor $\hat{X}_{n+h}$, can be calculated through the probability function of the h-step-ahead prediction error, given by

$$e_{n+h}|\mathbf{x}_n = X_{n+h}|\mathbf{x}_n - \hat{X}_{n+h}|\mathbf{x}_n\,.$$

Replacing $\hat{X}_{n+h}|\mathbf{x}_n$ given by (2.4), we obtain

$$e_{n+h}|\mathbf{x}_n \;=\; X_{n+h} - \alpha^h x_n - \lambda\,\frac{1-\alpha^h}{1-\alpha}\;.$$

Since $e_{n+h}|\mathbf{x}_n$ is a function of discrete random variable X_{n+h}, we have

$$e_{n+h}|\mathbf{x}_n = \; k - \alpha^h x_n - \lambda\,\frac{1-\alpha^h}{1-\alpha}\;, \qquad k = 0, 1, 2, \ldots\;.$$

So,

$$\begin{aligned} & P\left(e_{n+h} = k - \alpha^h x_n - \lambda\,\frac{1-\alpha^h}{1-\alpha}\,\Big|\,\mathbf{x}_n\right) \;= \\ & \quad = \; P\Big(X_{n+h} = k \,|\, X_n = x_n\Big) \\ & \quad = \; \exp\left\{-\lambda\,\frac{1-\alpha^h}{1-\alpha}\right\} \sum_{i=0}^{M_h} \frac{1}{(k-i)!}\left(\lambda\,\frac{1-\alpha^h}{1-\alpha}\right)^{k-i}\binom{x_n}{i}(\alpha^h)^i\,(1-\alpha^h)^{x_n-i}\,. \end{aligned} \tag{3.1}$$

From the expression (3.1) we obtain a γ level confidence interval for X_{n+h}

$$\left(\hat{X}_{n+h} + e_{t_1}, \hat{X}_{n+h} + e_{t_2}\right), \tag{3.2}$$

where $\hat{X}_{n+h}$ is given by (2.4), e_{t_1} is the largest value of e_{n+h} such as $P(e_{n+h} \le e_{t_1})$ $\le (1-\gamma)/2$ and e_{t_2} is the smallest value of e_{n+h} such as $P(e_{n+h} \le e_{t_2}) \ge (1+\gamma)/2$.

3.2. Bayesian methodology

We propose an adaptive generalization of the method used to obtain Highest Posterior Density (HPD) intervals of the model parameters, in which we consider the predictive distribution instead of the posterior.

Definition 3.1. A $100\gamma\%$ predictive interval for X_{n+h} is given by

$$P\big(X_L \le X_{n+h} \le X_R\big) \;=\; \sum_{x_{n+h}=X_L}^{X_R} f(x_{n+h}|\mathbf{x}_n)\;.$$

However, since $f(x_{n+h}|\mathbf{x}_n)$ is not always symmetric[1], the intervals with a maximum posterior predictive probability are more desirable than predictive intervals (Chen *et al.*, 2000).

[1]We made a previous study with some samples from PoINAR(1) and we verified that many were neither symmetric nor unimodal.

Definition 3.2. $R(\gamma) = (X_L, X_R)$ is a $100\gamma\%$ HPD interval for X_{n+h} if

$$P\big(X_L \leq X_{n+h} \leq X_R\big) \;=\; \sum_{x_{n+h}=X_L}^{X_R} f(x_{n+h}|\mathbf{x}_n) \;\geq\; K_\gamma \;, \tag{3.3}$$

where K_γ is the largest constant such that $P\big[X_{n+h} \in R(\gamma)\big] \geq \gamma$.

Due to the complexity of the predictive probability function given by (2.7) it is not possible to calculate the exact HPD interval for X_{n+h}; we can give an approximation for $R(\gamma)$ by using the Chen and Shao (1999) algorithm, because this method does not require the knowledge of the closed form of $f(x_{n+h}|\mathbf{x}_n)$. The Chen and Shao algorithm can be described as:

Algorithm 3.1

1. Obtain an MCMC sample $(X_{n+h,1}, X_{n+h,2}, ..., X_{n+h,m})$ (Algorithm 2.1);
2. Consider $\big(X_{(n+h,1)} \leq X_{(n+h,2)} \leq ... \leq X_{(n+h,m)}\big)$;
3. Compute the $100\gamma\%$ credible intervals

$$R_i(\gamma) = \big(X_{(n+h,i)}, X_{(n+h,i+[m\gamma])}\big) \;, \qquad 1 \leq i \leq m - [m\gamma] \;,$$

   where $[m\gamma]$ is integer part of $m\gamma$;
4. The $100\gamma\%$ HPD interval to X_{n+h} is the one, denoted by $\hat{R}(\gamma)$, with the smallest amplitude among all credible intervals.

Under certain regularity conditions, $\hat{R}(\gamma) \to R(\gamma)$ a.s. as $n \to \infty$, where $R(\gamma)$ is defined in (3.3) (Chen *et al.*, 2000).

Sometimes we obtain more than one interval. For this situation, we consider for $\hat{R}(\gamma)$ the interval with greater absolute frequency, among the intervals width smaller with. When the interval is still not unique we take the one with the smallest lower limit.

4. A SIMULATION STUDY

For the simulation study we consider samples with size $n = 40, 90, 190$ generated by INAR(1) models with the parameters values $\alpha = 0.2, 0.5, 0.8$ and $\lambda = 1, 3$, and considering the hyperparameters $a = b = c = d = 10^{-4}$.

4.1. Point prediction

From the various simulated samples we conclude that large values of α and λ are related with high dispersion values. Consequently the increase in α and λ provides large values of $|x_{n+h} - x_n|$, $h > 1$. As we can see in Figure 1, independently of the prediction methodology used, the forecasts performance depends on two basic aspects: one is the difference between x_n and x_{n+h}, $h > 1$; the other is the approximation between x_n and $\hat{\lambda}/(1-\hat{\alpha})$, in particular when h increases (note that $\hat{E}(X_{n+h}|X_n) \to \hat{\lambda}/(1-\hat{\alpha})$, $h \to \infty$). These findings are illustrated in Table 1 where point predictions for 10 steps ahead for a particular model are given. The table includes the h-step ahead simulated and predicted values and the square of the deviations between x_{190} and x_{190+h}, $h = 1, ..., 10$. The last line contains the classical limiting distribution.

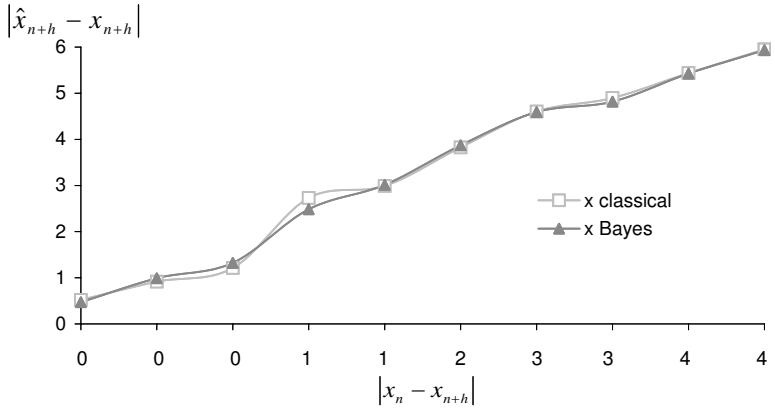

Figure 1: Values of $|\hat{x}_{n+h} - x_{n+h}|$ for a PoINAR(1) sample with $\alpha = 0.8$, $\lambda = 3$, $n = 190$ and $h = 1, 2, ..., 10$.

To confront classical and Bayesian methodologies we use the mean square error (MSE) to compare means, the mean absolute deviation (MAD) to compare medians and the "everything or nothing" lost function (FPTN), given by $1/n \sum I(x_{n+h})$ where

$$I(x_{n+h}) = \begin{cases} 1 & \text{if } |\hat{x}_{n+h} - x_{n+h}| > \delta\,, \\ 0 & \text{if } |\hat{x}_{n+h} - x_{n+h}| \leq \delta\,, \end{cases}$$

to compare modes. In this situation we consider $\delta = 1$ since we have integer values.

Table 2 shows the MSE, MAD and FPTN values from 10 one-step-ahead predictions. Values of MSE($\tilde{X}_{n+h}$) are obtained considering the Bayesian predictors given by (2.8) (values of MSE($\hat{X}_{n+h}$) are similar). Values of MAD and FPTN were calculated, respectively, through medians and modes. The indices "C" or "B" indicate which methodology is used (classical or Bayesian, respectively).

Table 1: Point predictions considering two samples of size $n = 190$ with parameters $(\lambda = 1,\ \alpha = 0.2,\ x_{190} = 0)$ and $(\lambda = 3,\ \alpha = 0.8,\ x_{190} = 16)$, respectively.

$(\lambda = 1, \quad \alpha = 0.2; \quad x_{190} = 0)$						
h	x_{190+h}	jump	classical approach		Bayesian approach	
			$\hat{x}_{190+h}$	$(x_{190+h} - \hat{x}_{190+h})^2$	$\hat{x}_{190+h}$	$(x_{190+h} - \hat{x}_{190+h})^2$
1	2	2	1.068	0.869	1.090	0.828
2	0	0	1.247	1.555	1.292	1.670
3	0	0	1.277	1.631	1.340	1.796
4	0	0	1.282	1.643	1.342	1.801
5	5	5	1.283	13.816	1.288	13.779
6	0	0	1.283	1.646	1.302	1.695
7	1	1	1.283	0.080	1.210	0.044
8	1	1	1.283	0.080	1.348	0.121
9	1	1	1.283	0.080	1.248	0.061
10	2	2	1.283	0.514	1.298	0.493
∞			1.283			

$(\lambda = 3, \quad \alpha = 0.8; \quad x_{190} = 16)$						
h	x_{190+h}	jump	classical approach		Bayesian approach	
			$\hat{x}_{190+h}$	$(x_{190+h} - \hat{x}_{190+h})^2$	$\hat{x}_{190+h}$	$(x_{190+h} - \hat{x}_{190+h})^2$
1	16	0	15.477	0.274	15.530	0.221
2	16	0	15.084	0.839	15.010	0.980
3	16	0	14.787	1.471	14.678	1.748
4	20	4	14.564	29.550	14.574	29.441
5	19	3	14.396	21.197	14.408	21.086
6	17	1	14.270	7.453	14.516	6.170
7	18	2	14.175	14.631	14.128	14.992
8	19	3	14.103	23.981	14.182	23.213
9	20	4	14.049	35.414	14.066	35.212
10	17	1	14.008	8.952	13.986	9.084
∞			13.884			

As we can see, when $\alpha = 0.8$ Bayesian methodology provides smaller values than classical methodology, so the Bayesian predictions seems to have a better performance than classical predictions.

In order to study and compare the estimates given by the sample mean, sample median and sample mode we use the minimum absolute percentual error (MAPE), given by

$$1/H \sum_{h=1}^{H} |\hat{X}_{n+h} - X_{n+h}|/X_{n+h} \ ,$$

where H represents the number of predictions realized. This criteria does not benefit any measure (mean, median or mode) in particular. The results are

Table 2: Values of MSE, MADE and FPTN considering 10 one-step-ahead predictions for the model $x_t = \alpha \circ x_{t-1} + \epsilon_t$, $\epsilon_t \frown \mathrm{P}(3)$ and sample sizes 40, 90 and 190.

	α	0.2			0.8		
	n	40	90	190	40	90	190
MSE	$\hat{X}_{n+h,\mathrm{C}}$	6.68	2.01	**5.56**	12.98	3.82	16.17
	$\hat{X}_{n+h,\mathrm{B}}$	**6.46**	**1.93**	6.06	**5.22**	**3.05**	**3.57**
MAD	$\hat{m}_{n+h,\mathrm{C}}$	2.27	1.18	**2.00**	3.18	1.45	3.81
	$\tilde{m}_{n+h,\mathrm{B}}$	**2.00**	1.18	2.05	**1.82**	**1.27**	**1.68**
FPTN	$\hat{mo}_{n+h,\mathrm{C}}$	**0.45**	0.45	**0.45**	0.73	0.55	1.00
	$\tilde{mo}_{n+h,\mathrm{B}}$	0.55	**0.36**	0.64	**0.36**	**0.36**	**0.55**

illustrated in Table 3 for three samples with sizes 40, 90 and 190 of the model $x_t = \alpha \circ x_{t-1} + \epsilon_t$, $\epsilon_t \frown P(3)$. MAPE minimum is always obtained with Bayesian approach. Similar results are obtained for $\lambda = 1$.

Table 3: Values of MAPE considering 10 one-step-ahead predictions for the model $x_t = \alpha \circ x_{t-1} + \epsilon_t$, $\epsilon_t \frown \mathrm{P}(3)$ and sample sizes 40, 90 and 190. The indices "C" or "B" indicate which methodology is used (classical or Bayesian, respectively).

α	0.2			0.8		
n	40	90	190	40	90	190
$\hat{X}_{n+h,\mathrm{C}}$	0.714	0.573	0.870	0.707	0.564	0.919
$\tilde{X}_{n+h,\mathrm{B}}$	0.178	0.137	0.260	0.110	0.109	0.0811
$\hat{m}_{n+h,\mathrm{C}}$	0.652	0.588	0.631	0.606	0.561	0.831
$\tilde{m}_{n+h,\mathrm{B}}$	0.187	0.120	0.209	0.115	0.103	0.091
$\hat{mo}_{n+h,\mathrm{C}}$	0.619	0.464	0.929	0.625	0.506	0.831
$\tilde{mo}_{n+h,\mathrm{B}}$	0.187	0.127	0.231	0.086	0.125	0.098

4.2. Interval prediction

Prediction intervals for future observations were calculated using expression (3.2) for classical methodology and Chen and Shao algorithm for Bayesian methodology. The performance of the intervals (with 95% of confidence or credibility) obtained by each approach is measured by the amplitudes and coverage probabilities, from 100 replicates. The simulation results for the case $\lambda = 1, 3$, $\gamma = 0.95$ and $n = 100$, are presented in Tables 4 and 5 respectively.

Table 4: Coverage probability estimates and mean amplitudes of the intervals for the h-step-ahead future values, in INAR(1) model with $n = 100$ and $\lambda = 1$.

h	$\alpha = 0.2$				$\alpha = 0.8$			
	cov. prob. estimates		mean amplitude		cov. prob. estimates		mean amplitude	
	classical	Bayes.	classical	Bayes.	classical	Bayes.	classical	Bayes.
1	0.43	0.99	2.56	4.17	0.96	0.98	6.01	4.72
2	0.32	0.98	2.74	4.22	0.95	0.94	7.79	6.18
3	0.36	1.00	2.75	4.22	0.96	0.92	8.20	6.87
4	0.36	0.99	2.75	4.16	0.94	0.92	8.81	7.35
5	0.34	1.00	2.75	4.19	0.96	0.93	9.05	7.57
6	0.38	0.99	2.75	4.21	0.97	0.90	9.43	7.83
7	0.29	0.98	2.75	4.19	0.98	0.92	9.55	7.85
8	0.28	0.99	2.75	4.22	0.94	0.91	9.49	8.04
9	0.38	0.99	2.75	4.22	0.94	0.90	9.78	8.00
10	0.34	1.00	2.75	4.15	0.97	0.92	9.78	8.11

Table 5: Coverage probability estimates and mean amplitudes of the intervals for the h-step-ahead future values, in INAR(1) model with $n = 100$ and $\lambda = 3$.

h	$\alpha = 0.2$				$\alpha = 0.8$			
	cov. prob. estimates		mean amplitude		cov. prob. estimates		mean amplitude	
	classical	Bayes.	classical	Bayes.	classical	Bayes.	classical	Bayes.
1	0.68	0.96	6.34	7.34	0.97	0.96	10.18	9.53
2	0.78	0.99	7.16	7.79	0.96	0.94	12.63	12.00
3	0.98	0.99	8.00	7.88	0.95	0.93	13.89	13.29
4	0.99	0.94	8.00	7.77	0.95	0.89	14.81	14.31
5	0.98	0.98	8.00	7.78	0.95	0.87	15.36	14.67
6	0.96	1.00	8.00	7.86	0.96	0.92	15.58	14.92
7	0.96	0.99	8.00	7.79	0.96	0.91	15.78	14.97
8	0.95	0.99	8.00	7.79	0.92	0.94	15.83	15.30
9	0.96	0.96	8.00	7.77	0.93	0.93	15.95	15.47
10	0.93	0.98	8.00	7.84	0.96	0.91	15.97	15.54

Tables 4 and 5 indicate that when $\alpha = 0.2$ the Bayesian intervals have large coverage probability, specially when $\lambda = 1$; it can be noted that, when $h > 2$ and for small values of α and λ the classic intervals have small amplitudes and they coincide with those obtained from the asymptotic distribution. Moreover, it is worthwhile to mention that when $\alpha \geq 0.5$ the mean amplitude of the prediction intervals obtained by Bayesian methodology are lower than their classic counterpart.

5. ANALYSIS OF BURN CLAIMS DATA

We apply the proposed methodology to a data set analysed by Freeland (1998) comprising 120 monthly counts of workers collecting Wage Loss Benefits (WLB) for burn injuries. All the descriptive details of the data set can be found in Freeland (1998) who concludes that PoINAR(1) is a plausible choice for modelling the data. In order to evaluate and compare the different prediction methodologies, h-step ahead forecasts ($h = 1, 2, 3, 4, 5, 6$) are produced for the time period from July to December 1994, for which we know the observed values. The point forecasts based on the mean, median and mode and the observed values are presented in Table 6. In general, it can be noted that MAPE values of classic point predictions are smaller than those of Bayesian predictions. This result is expected in view of the simulation results presented in the last section since the estimated value for alpha is 0.4.

Interval predictions for the period July to December 1994 are obtained using the two approaches proposed given by (3.2). The intervals obtained, presented in Table 6, are analogous, although the Bayesian have smaller width.

Table 6: h-step ahead predictions of monthly claims from July to December 1994.

h	year/ month	claims of	point prediction						interval prediction	
			classical			Bayesian			classical	Bayesian
			$\hat{x}$	$\hat{m}$	$\hat{m}_0$	$\hat{x}$	$\hat{m}$	$\hat{m}_0$		
1	94/07	11	7.89	8	7	7.67	7	7	(2.1,13.0)	(3.0,13.1)
2	94/08	12	8.24	8	8	8.01	8	7	(2.0,14.0)	(3.0,14.0)
3	94/09	11	8.38	8	8	8.14	8	7	(2.0,14.0)	(3.0,14.0)
4	94/10	12	8.44	8	8	8.21	8	7	(2.0,15.0)	(3.0,14.0)
5	94/11	7	8.46	8	8	8.22	8	7	(2.0,15.0)	(3.0,14.0)
6	94/12	11	8.47	8	8	8.22	8	7	(2.0,15.0)	(3.0,14.0)
		MAPE	0.26	0.27	0.27	0.27	0.27	0.32		

6. FINAL REMARKS

Forecasting low integer values of time series of counts remains an open problem. Although conditional means do not preserve coherently the integer nature of the data, it seems there is no advantage in using median or mode values of the predictive distribution. Simulations indicate that the performance of the different approaches depend on the parameters of the model and that Bayesian methodology provides the best results when MAPE statistic is used.

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