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EXTREMAL BEHAVIOUR IN MODELS OF SUPER-POSITION OF RANDOM VARIABLES *

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Abstract:

- Let $\mathbf{X}^{(i)} = \{X_{g_i(n)}\}_{n\geq 1}$, $i=1,2$, be sequences of random variables, where $\{g_i(n)\}_{n\geq 1}$ are disjoint and strictly increasing sequences of integer numbers such that $\{g_1(n)\}_{n\geq 1} \cup \{g_2(n)\}_{n\geq 1} = \mathbb{N}$. Using superposition of point processes, we study the extremal behaviour of a superposed sequence

$$\{X_n\}_{n\geq 1} \;=\; \{X_{g_1(n)}\}_{n\geq 1} \cup \{X_{g_2(n)}\}_{n\geq 1}\ ,$$

where we consider the proportion of variables superposed from each sequence asymptotically constant and $\{X_n\}_{n\geq 1}$ verifying some dependence conditions. We apply the obtained results in the computation of the bivariate extremal index.

Key-Words:

- *extreme value; nonstationarity; extremal index; superposition of point processes.*

AMS Subject Classification:

- 60G55, 60G70.

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1. INTRODUCTION

Let $\mathbf{X}^{(i)} = \{X_{g_i(n)}\}_{n\geq 1}$ be stationary sequences of random variables on the same probability space $(\Omega, \Im, P)$ with common distribution function $F^{(i)}$, $i=1,2$, respectively. Let us suppose that $\{g_i(n)\}_{n\geq 1}$, $i=1,2$, are disjoint and strictly increasing sequences of integer numbers, such that

$$\{g_1(n)\}_{n\geq 1} \cup \{g_2(n)\}_{n\geq 1} \;=\; \mathbb{N}\ .$$

In this paper we consider sequences that arise from the superposition of the variables of the sequences $\mathbf{X}^{(i)}$, $i=1,2$, when considering asymptotically constant the proportion of variables to superpose from each of the sequences, that is,

$$\frac{s_i(n)}{n} \xrightarrow[n\to\infty]{} L_i\ , \qquad i=1,2, \quad L_1+L_2=1\ , \tag{1.1}$$

where $s_i(n) = \#\big\{g_i(j)\colon 1\leq j\leq n \,\wedge\, 1\leq g_i(j)\leq n\big\}$, $i=1,2$.

We study the extremal limiting behaviour of the superposed sequence

$$\{X_n\}_{n\geq 1} \;=\; \{X_{g_1(n)}\}_{n\geq 1}\,\{X_{g_2(n)}\}_{n\geq 1}\ ,$$

usually a nonstationary sequence. Such a behaviour is derived from the convergence in distribution of the sequence, $\{S_n\}_{n\geq 1}$, of the point processes of exceedances of real numbers u_n, $n\geq 1$, generated by the sequence $\{X_n\}_{n\geq 1}$, defined by

$$S_n(B) \;=\; S_n[X_i, u_n](B) \;=\; \sum_{i=1}^{n} 1_{\{X_i>u_n\}}\,\delta_{\frac{i}{n}}(B)\ , \qquad n\geq 1\ ,$$

where B is a Borel subset of $[0,1]$, $\delta_x(\cdot)$ denotes the Dirac measure at $x\in\mathbb{R}$ and 1_A the indicator function of the event A.

By considering, for each $i=1,2$,

$$S_n^{(i)}(B) \;=\; S_n[X_{g_i(j)}, u_n](B) \;=\; \sum_{j=1}^{n} 1_{\{X_{g_i(j)}>u_n\}}\,\delta_{\frac{g_i(j)}{n}}(B)\ ,$$

then

$$S_n(B) = S_n^{(1)}(B) + S_n^{(2)}(B)\ ,$$

that is, the sequence of point processes $\{S_n\}_{n\geq 1}$ is the superposition of the point processes $\{S_n^{(i)}\}_{n\geq 1}$, $i=1,2$.

We briefly present, in what follows, some important results concerning the theory of exceedances point processes generated by dependent sequences, both stationary and nonstationary.

Recall that the type of long range dependence condition appropriate for studying the convergence in distribution of $\{S_n\}_{n\geq 1}$ is the condition $\Delta(u_n)$ defined by Hsing *et al.* (1988), in the following way.

Definition 1.1. Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables and $\{u_n\}_{n\geq 1}$ a sequence of real numbers. For each $1 \leq i \leq j$, set $\mathcal{B}_i^j(u_n)$ as the σ-field generated by the events $\{X_s \leq u_n\}$, $i \leq s \leq j$, and, for $1 \leq l \leq n-1$,

$$\alpha_{n,l} = \sup_{1\leq k\leq n-l} \left\{ \left|P(A\cap B) - P(A)\,P(B)\right| \colon\ A \in \mathcal{B}_1^k(u_n),\ B \in \mathcal{B}_{k+l}^n(u_n) \right\} . \tag{1.2}$$

The condition $\Delta(u_n)$ is said to hold if there exists a sequence $l_n = o(n)$, as $n \to \infty$, such that

$$\alpha_{n,l_n} \xrightarrow[n\to\infty]{} 0 .$$

Note that by taking in (1.2) only events of the form $A = \{X_{i_1} < u_n, ..., X_{i_p} < u_n\}$ and $B = \{X_{j_1} < u_n, ..., X_{j_q} < u_n\}$ with

$$1 \leq i_1 < \cdots < i_p < i_p + l < j_1 < \cdots < j_q \leq n ,$$

we obtain Leadbetter's $D(u_n)$ condition.

Under condition $\Delta(u_n)$ and additional assumptions of equicontinuity and asymptotic negligibility, Nandagopalan (1990) characterized the possible distributional limits for $\{S_n\}_{n\geq 1}$, as stated in Proposition 1.1.

Let $J_1, ..., J_{k_n}$, $n \geq 1$, be a sequence of partitions of $[0,1]$ such that for each $i = 1, 2, ..., k_n$, $P(S_n(J_i) > 0) > 0$, after certain order n_0. For each $n \geq n_0$ define the following sequences of measures:

$$\nu_n(B) = \sum_{i=1}^{k_n} P\big(S_n(J_i) > 0\big)\, \frac{m(B\cap J_i)}{m(J_i)} , \qquad B \in \mathcal{B}([0,1]) ,$$

where m denotes the Lebesgue measure,

$$\Pi_{n,x}(k) = \sum_{i=1}^{k_n} \Pi_{n,i}(k)\, \delta_x(J_i) , \qquad k \in \mathbb{N},\quad x \in [0,1] ,$$

where

$$\Pi_{n,i}(k) = P\Big(S_n(J_i) = k \mid S_n(J_i) > 0\Big) , \qquad k \in \mathbb{N} .$$

Finally for each $a \in \mathbb{R}_+$ define the functions

$$g_{n,a}(x) = \int_{\mathbb{N}} \big(1 - \exp(ak)\big)\, d\Pi_{n,x}(k) .$$

Proposition 1.1. *Let $\{X_n\}_{n\geq 1}$ be a sequence of random variables verifying condition $\Delta(u_n)$,*

$$g(\epsilon_n) = \sup\Big\{P\big(S_n(I)>0\big): I\subset[0,1],\ m(I)\leq\epsilon_n\Big\} \xrightarrow[n\to\infty]{} 0 \quad \textit{if}\ \epsilon_n \xrightarrow[n\to\infty]{} 0 \tag{1.3}$$

and

$$\liminf_{n\to\infty} P\big(S_n([0,1])=0\big) > 0\ . \tag{1.4}$$

If $\{k_n\}_{n\geq 1}$ is a sequence of integer numbers such that

$$k_n\left(\alpha_{n,l_n} + g\left(\frac{l_n}{n}\right)\right) \xrightarrow[n\to\infty]{} 0 \tag{1.5}$$

and $J_1, ..., J_{k_n}$, is a partition of $[0,1]$ satisfying

$$\max\Big\{m(J_i):\ i=1,2,...,k_n\Big\} \xrightarrow[n\to\infty]{} 0$$

and

for each $a\in G$, where G is some nonempty open subset of $\mathbb{R}_+$,
the sequence $\{g_{n,a}\}_{n\geq 1}$ is equicontinuous,

then the following propositions are equivalent

(1) *The sequence of point processes $\{S_n\}_{n\geq 1}$ converges in distribution to some point process S with Laplace transform, L_S, given by*

$$L_S(f) = \exp\left(-\int_{[0,1]}\int_{\mathbb{N}}\Big(1-\exp\big(-kf(x)\big)\Big)\,d\Pi_x(k)\ d\mu(x)\right)\ , \tag{1.6}$$

for each non-negative measurable function, f, on $[0,1]$, where μ is a finite measure on $[0,1]$ and Π_x is a probability measure on $\mathbb{N}$.

(2) *ν_n converges weakly to a finite measure μ and $\Pi_{n,x}$ converges weakly to a probability measure Π_x on $\mathbb{N}$, for each $x\in[0,1]$.*

Furthermore, Nandagopalan (1990) proves that under conditions (1.3), (1.4) and (1.5) for some partition $J_1, ..., J_{k_n}$ of $[0,1]$ such that $\max\{m(J_i): i=1,2,...,k_n\} \xrightarrow[n\to\infty]{} 0$, if $S_n \xrightarrow[n\to\infty]{d} S$, the Laplace Transform L_S is given by (1.6).

The result of Hsing *et al.* (1988) which gives the convergence in distribution of exceedances point processes of a stationary random sequence is contained in the preceeding proposition. In fact, in the case of stationary sequences for normalized levels and sequences of integer numbers $\{k_n\}_{n\geq 1}$, such that

$$k_n \xrightarrow[n\to\infty]{} \infty\ , \qquad k_n\alpha_{n,l_n} \xrightarrow[n\to\infty]{} 0\ , \qquad \frac{k_n l_n}{n} \xrightarrow[n\to\infty]{} 0\ , \tag{1.7}$$

the assumptions established in the above proposition are verified and, furthermore the multiplicity distribution does not depend on the position of the atom, $\Pi_x = \Pi$, for each $x \in [0,1]$, and the intensity measure μ is equal to a constant times the Lebesgue measure, $\mu(\cdot) = \nu\, m(\cdot)$.

For the class of stationary sequences verifying condition $\Delta(u_n)$, if there exists the extremal index $\theta \in [0,1]$ (Leabetter (1974)), then such a parameter is given by the inverse of the limiting mean cluster size of exceedances. Indeed, if

$$P\big(S_n([0,1])=0\big) \xrightarrow[n\to\infty]{} e^{-\nu}$$

and

$$ES_n([0,1]) = nP(X_1 > u_n) \xrightarrow[n\to\infty]{} \tau > 0$$

then

$$\begin{aligned}\theta &= \Big(\lim_{n\to\infty} E\,\Pi_n\Big)^{-1} \\ &= \lim_{n\to\infty} \frac{P\Big(S_n\big([0,k_n^{-1}]\big) > 0\Big)}{E\,S_n([0,1])} \\ &= \frac{\nu}{\tau}\ .\end{aligned}$$

In section 2 we introduce a condition that guarantees, locally, the asymptotic independence among the maxima of the variables of the sequences, $X^{(i)}$, $i = 1,2,$ to superpose. Under this condition, for each non-negative integer k, the probability of occurrence of k exceedances of the level u_n by the variables of the superposed sequence $\{X_n\}_{n\geq1}$, in intervals of length $[\frac{n}{k_n}]$, is asymptotically equal.

For each sequence of this class we can apply the results stated in Proposition 1.1, obtaining a compound Poisson limit $S[\nu,\Pi]$ to $\{S_n\}_{n\geq1}$. The sequence $\{S_n\}_{n\geq1}$ behaves asymptotically as though the sequence $\{X_n\}_{n\geq1}$ is stationary, that is, the multiplicity distribution does not depend on x, $\Pi_x = \Pi$, for each $x \in [0,1]$, and the intensity measure is equal a constant times the Lebesgue measure, $\mu(\cdot) = \nu\, m(\cdot)$.

The relations between the intensity measure $\nu\, m(\cdot)$, the distribution of multiplicities $\Pi(\cdot)$ and the corresponding measures $\nu^{(i)} m(\cdot)$ and $\Pi^{(i)}(\cdot)$, for each of the sequences to superpose, will be analyzed in section 3. We prove that $\nu = \nu^{(1)} + \nu^{(2)}$ and $\Pi(k) = \sum_{i=1}^{2} \frac{\nu^{(i)}}{\nu}\, \Pi^{(i)}(k)$, with $\nu^{(i)} = \theta^{(i)}\, \tau^{(i)} L_i$, $\tau^{(i)} = \lim_{n\to\infty} nP(X_{g_i(1)} > u_n)$ and L_i is given in (1.1), $i = 1,2$.

In section 4 we will apply the results in the computation of the bivariate extremal index.

2. LIMIT DISTRIBUTION OF THE NUMBER OF EXCEEDANCES IN THE SUPERPOSED SEQUENCE

We define a new condition that guarantees locally, that the maxima of the random variables of the sequences to superpose are asymptotically independent. This condition will be essential to obtain the results in this section.

Definition 2.1. The sequence $\{X_n\}_{n\geq 1}$ verifies the condition $\overset{\bullet}{D}(u_n)$ if

$$k_n\,\beta_n \xrightarrow[n\to\infty]{} 0$$

where

$$\beta_n = \sup\left\{\left|P\Big(M_n^{(1)}(J)\leq u_n,\, M_n^{(2)}(J)\leq u_n\Big) - P\Big(M_n^{(1)}(J)\leq u_n\Big)P\Big(M_n^{(2)}(J)\leq u_n\Big)\right| :\right.$$
$$\left. J\subset [0,+\infty[,\;\; m(J) = \left[\frac{n}{k_n}\right]\right\}\;,$$

$M_n^{(i)}(J) = \max\big\{X_{g_i(j)}\colon 1\leq g_i(j)\leq n,\; g_i(j)\in J\big\}$ and $\{k_n\}_{n\geq 1}$ is a sequence of integer numbers that verifies (1.7).

Under condition $\overset{\bullet}{D}(u_n)$ for the superposed sequence $\{X_n\}_{n\geq 1}$ we can, for each non-negative integer k, approach

$$P\big(S_n(J_j)=k\big) \quad\text{by}\quad P\big(S_n(J_l)=k\big)$$

where $J_i = \big[(i-1)[\frac{n}{k_n}]\frac{1}{n},\, i[\frac{n}{k_n}]\frac{1}{n}\big],\;\; j,l\in\{1,2,...,k_n\}$ and $j\neq l$.

Proposition 2.1. *Suppose that the sequence $\{X_n\}_{n\geq 1}$ resulting from the superposition of the variables of the stationary sequences $\{X_{g_i(n)}\}_{n\geq 1}$, $i=1,2$, verifies condition $\overset{\bullet}{D}(u_n)$, where $\{u_n\}_{n\geq 1}$ is a sequence of real numbers such that*

$$nP\big(X_{g_i(1)}>u_n\big) \xrightarrow[n\to\infty]{} \tau^{(i)}\;, \qquad i=1,2\;. \tag{2.1}$$

Then, for each non-negative integer k, we have

$$k_nP\big(S_n(J_i)=k\big) \;=\; k_nP\big(S_n(J_1)=k\big) + o(1)\;.$$

Proof: Since $\{X_{g_i(n)}\}_{n\geq 1}$, $i=1,2$, are stationary sequences

$$\begin{aligned}
k_nP\big(S_n(J_i)=k\big) \;&=\; k_nP\big(S_n^{(1)}(J_i)=k\big) + k_nP\big(S_n^{(2)}(J_i)=k\big)\\
&\quad + k_n\sum_{\substack{s_1+s_2=k\\ s_1>0,\, s_2>0}} P\Big(S_n^{(1)}(J_i)=s_1,\, S_n^{(2)}(J_i)=s_2\Big)\\
&=\; k_nP\big(S_n^{(1)}(J_1)=k\big) + k_nP\big(S_n^{(2)}(J_1)=k\big)\\
&\quad + k_n\sum_{\substack{s_1+s_2=k\\ s_1>0,\, s_2>0}} P\Big(S_n^{(1)}(J_i)=s_1,\, S_n^{(2)}(J_i)=s_2\Big)\;.
\end{aligned}$$

Attending now to condition $\overset{\bullet}{D}(u_n)$ we can write

$$\begin{aligned} k_n \sum_{\substack{s_1+s_2=k \\ s_1>0,\, s_2>0}} P\Big(S_n^{(1)}(J_i)=s_1,\, S_n^{(2)}(J_i)=s_2\Big) \;&\leq \\ &\leq\; k_n P\Big(S_n^{(1)}(J_i)>0,\, S_n^{(2)}(J_i)>0\Big) \\ &\leq\; k_n\beta_n + k_n P\big(S_n^{(1)}(J_i)>0\big)\, P\big(S_n^{(2)}(J_i)>0\big) \\ &=\; o(1)\;. \end{aligned}$$

So

$$k_n P\big(S_n(J_i)=k\big) \;=\; k_n P\big(S_n(J_1)=k\big) + o(1)\;. \qquad \square$$

We will prove next that when the superposed sequence verifies conditions $\overset{\bullet}{D}(u_n)$ and $\Delta(u_n)$ we can apply to it the results stated in Proposition 1.1. Furthermore, and as said before, the sequence $\{S_n\}_{n\geq 1}$ behaves asymptotically as though the sequence $\{X_n\}_{n\geq 1}$ is stationary, that is, the multiplicity distribution does not depend on x, $\Pi_x = \Pi$, for each $x \in [0,1]$, and the intensity measure is equal a constant times the Lebesgue measure, $\mu(\cdot) = \nu\, m(\cdot)$.

Proposition 2.2. *Suppose that the superposed sequence $\{X_n\}_{n\geq 1}$ verifies conditions $\Delta(u_n)$ and $\overset{\bullet}{D}(u_n)$, where $\{u_n\}_{n\geq 1}$ is a sequence of real numbers verifying (2.1). If the sequence $\{S_n\}_{n\geq 1}$ converges, then we have $S_n \xrightarrow[n\to\infty]{d} S[\nu,\Pi]$, with $\nu = \lim_{n\to\infty} k_n P\big(S_n([0,k_n^{-1}])>0\big)$ and Π is a probability measure such that $\Pi(k) = \lim_{n\to\infty} P\big(S_n([0,k_n^{-1}])=k \mid S_n([0,k_n^{-1}])>0\big)$, $k \in \mathbb{N}$.*

Proof: We are going to prove that the superposed sequence satisfies the assumptions of Proposition 1.1 with $J_i = \big((i-1)[\frac{n}{k_n}]\frac{1}{n},\, i[\frac{n}{k_n}]\frac{1}{n}\big]$, $i=1,2,...,k_n$.

For $I \subset [0,1]$ with $m(I) \leq \epsilon_n$ and $\epsilon_n \xrightarrow[n\to\infty]{} 0$ we have

$$P\big(S_n(I)>0\big) \;\leq\; n\,\epsilon_n \max\Big(P(X_{g_1(1)}>u_n),\, P(X_{g_2(1)}>u_n)\Big) \;=\; o(1)\;,$$

since, for each $i=1,2$, the sequence $\{X_{g_i(n)}\}_{n\geq 1}$ verifies (2.1).

For each set $I \subset [0,1]$ with Lebesgue measure not greater than $\frac{l_n}{n}$ we also have

$$k_n P\big(S_n(I)>0\big) \;\leq\; k_n \frac{l_n}{n}\, n \max\Big(P(X_{g_1(1)}>u_n),\, P(X_{g_2(1)}>u_n)\Big) \;=\; o(1)\;,$$

because $\{k_n\}_{n\geq 1}$ is a sequence of integer numbers verifying (1.7).

Since

$$\liminf_{n\to\infty} P\big(S_n([0,1])=0\big) \;=\; 1 - \limsup_{n\to\infty} P\big(S_n([0,1])>0\big)$$

and

$$\begin{aligned}
\limsup_{n\to\infty} P\big(S_n([0,1])>0\big) \;&\le\; \limsup_{n\to\infty} \max\Big(P\big(S_n^{(1)}([0,1])>0\big),\, P\big(S_n^{(2)}([0,1])>0\big)\Big)\\
&\le\; \max\big(e^{-\nu^{(1)}}, e^{-\nu^{(2)}}\big)\\
&<\; 1 \;,
\end{aligned}$$

we obtain (1.4).

For each $a\in\mathbb{R}_+$, the sequence $\{g_{n,a}\}_{n\ge1}$ is equicontinuous since if $|x-x'|<\varepsilon$ we have from a certain order

$$\big|g_{n,a}(x) - g_{n,a}(x')\big| \;\le\; \sum_{k\ge1} \big|\Pi_{n,i}(k) - \Pi_{n,j}(k)\big| \;,$$

for some pair of indexes i and j in $\{1,...,k_n\}$ such that J_i and J_j are separated by a length not greater than ε.

By Proposition 2.1, for each Borel subset B of $[0,1]$, we have

$$\begin{aligned}
\nu_n(B) \;&=\; \sum_{i=1}^{k_n} P\big(S_n(J_i)>0\big)\,\frac{m(B\cap J_i)}{m(J_i)}\\
&=\; \sum_{i=1}^{k_n} \Big(P\big(S_n(J_1)>0\big) + o(k_n^{-1})\Big)\,\frac{m(B\cap J_i)}{m(J_i)}\\
&=\; k_n\, m(B)\, P\big(S_n(J_1)>0\big) + o(k_n^{-1})\, k_n\, m(B)\\
&=\; k_n\, m(B)\, P\big(S_n(J_1)>0\big) + o(1) \;.
\end{aligned}$$

Under condition $\Delta(u_n)$ it follows, by the Lemma of asymptotic independence of maxima over disjoint intervals (Leadbetter (1974)), that

$$\begin{aligned}
\exp(-\nu) \;&=\; \lim_n P\big(\max_{1\le i\le n} X_i \le u_n\big)\\
&=\; \lim_n \prod_{i=1}^{k_n} P\big(S_n(J_i)=0\big) \;,
\end{aligned}$$

so, by Proposition 2.1,

$$\exp(-\nu) \;=\; \lim_n P^{k_n}\big(S_n(J_1)=0\big)$$

and consequently

$$\nu \;=\; \lim_n k_n P\big(S_n(J_1)>0\big) \;.$$

Thus, $\nu_n \xrightarrow[n\to\infty]{w} \nu\, m$.

Finally, we observe that for each $i=1,2,...,k_n$,

$$\Pi_{n,i}(k) \;=\; \frac{P\big(S_n(J_i)=k\big)}{P\big(S_n(J_i)>0\big)} \;=\; \frac{k_n P\big(S_n(J_1)=k\big) + o(1)}{k_n P\big(S_n(J_1)>0\big) + o(1)}$$

and as a consequence,

$$\begin{aligned}\Pi_{n,x}(k) &= \sum_{i=1}^{k_n} \Pi_{n,i}(k)\, \delta_x(J_i) \\ &= \Pi_{n,1}(k) \sum_{i=1}^{k_n} \delta_x(J_i) \;+\; o(1) \sum_{i=1}^{k_n} \delta_x(J_i) \\ &= \Pi_{n,1}(k) + o(1)\;, \qquad \text{independent of } x\;.\end{aligned}$$

By Proposition 1.1 we can conclude that if the sequence $\{S_n\}_{n\geq 1}$ converges in distribution then the limit point process, S, has Laplace Transform given by

$$L_S(f) \;=\; \exp\left(-\nu \int_{[0,1]} \int_{\mathbb{N}} \left(1 - e^{-kf(x)}\right) d\Pi(k)\; dx\right)$$

that is, $\{S_n\}_{n\geq 1}$ converges to a compound Poisson process with intensity measure ν and multiplicity distribution Π. □

It must be noted that under condition $\Delta(u_n)$ for the superposed sequence $\{X_n\}_{n\geq 1}$ we also have the validation of such condition for the sequences to superpose, $\{X_{g_i(n)}\}_{n\geq 1}$, $i = 1, 2$, and so, if the sequence $\{S_n^{(i)}\}_{n\geq 1}$ converges then the limit point process is a compound Poisson process, $S[\nu^{(i)}, \Pi^{(i)}]$, $i = 1, 2$.

3. DESCRIPTION OF THE ASYMPTOTIC BEHAVIOUR OF THE SEQUENCE $\{S_n\}_{n\geq 1}$ FROM $\{S_n^{(i)}\}_{n\geq 1}$, $i = 1, 2$

The condition $\overset{\bullet}{D}(u_n)$ allow us to describe the asymptotic behaviour of $\{S_n\}_{n\geq 1}$ from $\{S_n^{(i)}\}_{n\geq 1}$, $i = 1, 2$, as presented in the next result.

Proposition 3.1. *Suppose that the conditions of Proposition 2.2 hold, the sequences $\{X_{g_i(n)}\}_{n\geq 1}$, $i = 1, 2$, have extremal indexes $\theta^{(i)}$, $i = 1, 2$, respectively, and the proportion of variables to superpose from each of these sequences is asymptotically constant as established in (1.1).*

If, for each $i = 1, 2$, we have

$$S_n^{(i)} \xrightarrow[n\to\infty]{d} S[\nu^{(i)}, \Pi^{(i)}]$$

then

$$k_n P\Big(S_n([0, k_n^{-1}]) > 0\Big) \xrightarrow[n\to\infty]{} \nu \;=\; \nu^{(1)} + \nu^{(2)}$$

and

$$\Pi_n(k) = P\Big(S_n([0, k_n^{-1}]) = k \mid S_n([0, k_n^{-1}]) > 0\Big) \xrightarrow[n\to\infty]{} \Pi(k) = \sum_{i=1}^{2} \frac{\nu^{(i)}}{\nu}\, \Pi^{(i)}(k)\;,$$

with $\nu^{(i)} = \theta^{(i)}\, \tau^{(i)} L_i$ and $\tau^{(i)}$ given in (2.1), $i = 1, 2$.

Proof: By using analogous arguments to the ones used in the proof of Proposition 2.1, we obtain

$$\begin{aligned}
\lim_{n\to\infty} k_n P\big(S_n([0,k_n^{-1}])>0\big) &= \\
&= \lim_{n\to\infty} k_n P\big(S_n^{(1)}([0,k_n^{-1}])>0\big) + \lim_{n\to\infty} k_n P\big(S_n^{(2)}([0,k_n^{-1}])>0\big) \\
&\quad - \lim_{n\to\infty} k_n P\Big(S_n^{(1)}([0,k_n^{-1}])>0,\ S_n^{(2)}([0,k_n^{-1}])>0\Big) \\
&= \nu^{(1)} + \nu^{(2)} \;.
\end{aligned}$$

Relatively to the cluster size of exceedances distribution we have

$$\begin{aligned}
\Pi_n(k) &= P\Big(S_n([0,k_n^{-1}])=k \mid S_n([0,k_n^{-1}])>0\Big) \\
&= \frac{1}{P\big(S_n([0,k_n^{-1}])>0\big)} \Big(P\big(S_n^{(1)}([0,k_n^{-1}])=k\big) + P\big(S_n^{(2)}([0,k_n^{-1}])=k\big)\Big) \\
&\quad + \frac{1}{P\big(S_n([0,k_n^{-1}])>0\big)}\, P\left(\bigcup_{\substack{s_1\geq 1,\, s_2\geq 1\\ s_1+s_2=k}} \Big(S_n^{(1)}([0,k_n^{-1}])=s_1,\ S_n^{(2)}([0,k_n^{-1}])=s_2\Big)\right) .
\end{aligned}$$

Since, for each $i=1,2$, we have

$$\frac{P\big(S_n^{(i)}([0,k_n^{-1}])=k\big)}{P\big(S_n([0,k_n^{-1}])>0\big)} = \Pi_n^{(i)}(k)\, \frac{k_n P\big(S_n^{(i)}([0,k_n^{-1}])>0\big)}{k_n P\big(S_n([0,k_n^{-1}])>0\big)} \;\xrightarrow[n\to\infty]{}\; \frac{1}{\nu}\, \Pi^{(i)}(k)\, \nu^{(i)} \;,$$

and, under condition $\overset{\bullet}{D}(u_n)$

$$\begin{aligned}
\frac{1}{P\big(S_n([0,k_n^{-1}])>0\big)}\, P\left(\bigcup_{\substack{s_1\geq 1,\, s_2\geq 1\\ s_1+s_2=k}} \Big(S_n^{(1)}([0,k_n^{-1}])=s_1,\ S_n^{(2)}([0,k_n^{-1}])=s_2\Big)\right) &\leq \\
\leq \frac{1}{P\big(S_n([0,k_n^{-1}])>0\big)}\, P\Big(S_n^{(1)}([0,k_n^{-1}])>0,\ S_n^{(2)}([0,k_n^{-1}])>0\Big) & \\
= o(1) \;, &
\end{aligned}$$

the result follows. □

Corollary 3.1. *Under the conditions of the Proposition 3.1, the extremal index of the superposed sequence verifies*

$$\theta \;=\; \frac{\theta^{(1)}\tau^{(1)}L_1 + \theta^{(2)}\tau^{(2)}L_2}{\tau^{(1)}L_1 + \tau^{(2)}L_2} \;.$$

Note that the extremal index of the superposed sequence depends on $\lim_n \sum_{i=1}^n P(X_i > u_n) = \tau^{(1)}L_1 + \tau^{(2)}L_2$, as expected since $\{X_n\}_{n\geq 1}$ is a non-stationary sequence.

We finish this section with some remarks about the results obtained previously.

Remark 3.1. Let us suppose that $\theta^{(1)}=\theta^{(2)}$, $F_1 \neq F_2$ and for each $i=1,2$, F_i belongs to the domain of attraction of an extreme value distribution, G. So we have

$$P\left(\max_{1\leq i\leq n} X_i \leq u_n(x)\right) \xrightarrow[n\to\infty]{} G^{\theta^{(1)}}(x) .$$

We have in this way found a class of non-stationary sequences for which the so-called Extremal Types Theorem of Leadbetter is still valid.

Remark 3.2. Suppose that the superposed sequence is stationary. Then, under the long range dependence condition $\Delta(u_n)$ if $\{S_n\}_{n\geq 1}$ converges in distribution to some point process S then S is necessarily a compound Poisson process and the extremal index $\theta = \lim\limits_{n\to\infty} \frac{P\left(S_n([0,k_n^{-1}])>0\right)}{ES_n([0,1])}$.

So it seems natural to ask: When the superposed sequence is stationary are there any advantages in the application of the results established in this section? We shall find an affirmative answer.

In the stationary case, the introduction of local dependence conditions (Leadbetter (1983), Leadbetter and Nandagopalan (1989), Ferreira (1994)) enables us to obtain processes with practical interest to compute the extremal index, θ.

By assuming that each sequence $X^{(i)}$, $i=1,2$, does not oscillate rapidly near high extremes in the sense of the usual local dependence conditions we have not, in general, the validition of these conditions by the superposed sequence and consequently we can not apply directly to $\{X_n\}_{n\geq 1}$ the avaiable results.

In this case, the application of Proposition 3.1 facilitates the computation of the extremal index θ of the superposed sequence since we can apply the results under local dependence conditions to each one of the sequences superposed.

4. APPLICATIONS

As an application of the results established previously we point out the computation of the extremal index of a stationary sequence of random vectors $\mathbf{X} = \{(X_n^{(1)}, X_n^{(2)})\}_{n\geq 1}$ with common distribution function, F, belonging to the domain of attraction of a bivariate extreme value distribution, G.

Let us denote by $\widehat{\mathbf{X}}$ the independent sequence associated with $\mathbf{X}$ and by $\max\limits_{1\leq j\leq n} \widehat{X}_j^{(i)}$, $n\geq 1$, $i=1,2$, the corresponding sequences of partial maxima.

We remember the definition of bivariate extremal index introduced by Nandagopalan (1990) and that is a generalization of Leadbetter's definition for unidimensional sequences.

Definition 4.1. The sequence $\mathbf{X}=\{(X_n^{(1)},X_n^{(2)})\}_{n\geq 1}$ has an extremal index $\theta(\tau^{(1)},\tau^{(2)})\in[0,1]$, $\tau=(\tau^{(1)},\tau^{(2)})\in\mathbb{R}_+^2$, when for each $\tau\in\mathbb{R}_+^2$, there are $u_n^{(\tau)}=(u_n^{(\tau^{(1)})},u_n^{(\tau^{(2)})})$, $n\geq 1$, verifying

$$nP\Big(X_1^{(i)}>u_n^{(\tau^{(i)})}\Big)\xrightarrow[n\to\infty]{}\tau^{(i)}\;,\qquad i=1,2\;,$$

$$P\bigg(\max_{1\leq j\leq n}\widehat{X}_j^{(1)}\leq u_n^{(\tau^{(1)})},\ \max_{1\leq j\leq n}\widehat{X}_j^{(2)}\leq u_n^{(\tau^{(2)})}\bigg)\xrightarrow[n\to\infty]{}G(\tau)$$

and

$$P\bigg(\max_{1\leq j\leq n}X_j^{(1)}\leq u_n^{(\tau^{(1)})},\ \max_{1\leq j\leq n}X_j^{(2)}\leq u_n^{(\tau^{(2)})}\bigg)\xrightarrow[n\to\infty]{}G(\tau)^{\theta(\tau)}\;.$$

If $\mathbf{X}$ has extremal index $\theta(\tau)$ then, for each $i=1,2$, $\{X_n^{(i)}\}_{n\geq 1}$ has extremal index $\theta^{(i)}=\lim\limits_{\substack{\tau^{(j)}\to 0^+\\ j\neq i}}\theta(\tau^{(1)},\tau^{(2)})$.

We shall assume, without loss of generality, that the common distribution F of the vectors of the stationary sequence $\mathbf{X}=\{(X_n^{(1)},X_n^{(2)})\}_{n\geq 1}$ has unit Fréchet margins, *id est*,

$$F_1(x)=F_2(x)=\exp(-x^{-1})\;,\qquad x>0\;.$$

For fixed $\tau^{(1)}$ and $\tau^{(2)}$ and normalized levels $u_n^{(\tau^{(i)})}=\frac{n}{\tau^{(i)}}$ for $\{X_n^{(i)}\}_{n\geq 1}$, $i=1,2$, we have

$$\begin{aligned}(4.1)\qquad P\bigg(\max_{1\leq j\leq n}X_j^{(1)}\leq u_n^{(\tau^{(1)})},\ \max_{1\leq j\leq n}X_j^{(2)}\leq u_n^{(\tau^{(2)})}\bigg)&=\\ =P\bigg(\max_{1\leq j\leq n}X_j^{(1)}\leq\frac{n}{\tau^{(1)}},\ \max_{1\leq j\leq n}X_j^{(2)}\leq\frac{n}{\tau^{(2)}}\bigg)&\\ =P\bigg(\max_{1\leq j\leq n}\tau^{(1)}X_j^{(1)}\leq n,\ \max_{1\leq j\leq n}\tau^{(2)}X_j^{(2)}\leq n\bigg)&\;.\end{aligned}$$

Let us consider the stationary sequences $\{X_{g_i(n)}=\tau^{(i)}X_n^{(i)}\}_{n\geq 1}$, $i=1,2$. By superposing the variables of these sequences we can form different sequences $\{X_n\}_{n\geq 1}$ but the limiting behaviour of $\{S_n\}_{n\geq 1}$ is only affected by the asymptotic proportion of variables to superpose from each one of these sequences and not by the order of the variables of the superposed sequence.

By considering, for example,

$$\{X_n\}_{n\geq 1}=\Big\{\tau^{(1)}X_1^{(1)},\tau^{(2)}X_1^{(2)},\tau^{(1)}X_2^{(1)},\tau^{(2)}X_2^{(2)},...,\tau^{(1)}X_n^{(1)},\tau^{(2)}X_n^{(2)}\Big\}_{n\geq 1}\;,$$

we can rewrite (4.1) in the following way

$$P\Big(\max_{1\le j\le 2n} X_j \le n\Big) \; . \tag{4.2}$$

Since, for each $i=1,2$,

$$\frac{s_i(n)}{n} = \frac{\frac{n}{2}}{n} \xrightarrow[n\to\infty]{} L_i = \frac{1}{2} \;,$$

then, under the conditions established in Proposition 3.1, we have

$$\lim_{n\to\infty} P\Big(\max_{1\le j\le 2n} X_j \le n\Big) = \exp\Big[-\frac{1}{2}\Big(\theta^{(1)}\tau'^{(1)} + \theta^{(2)}\tau'^{(2)}\Big)\Big] \tag{4.3}$$

with

$$\tau'^{(i)} = \lim_{n\to\infty} nP\Big(\tau^{(1)} X_1^{(i)} > n\Big) = \lim_{n\to\infty} n\Big(1 - e^{-\frac{\tau^{(i)}}{n}}\Big) = \tau^{(i)} \; . \tag{4.4}$$

By paying attention to (4.1), (4.2), (4.3) and (4.4) we can write

$$\begin{aligned} \lim_{n\to\infty} P\Big(\max_{1\le j\le n} X_j^{(1)} \le u_n^{(\tau^{(1)})},\, \max_{1\le j\le n} X_j^{(2)} \le u_n^{(\tau^{(2)})}\Big) &= \\ &= \exp\Big[-\frac{1}{2}\Big(\theta^{(1)}\tau^{(1)} + \theta^{(2)}\tau^{(2)}\Big)\Big] \; . \end{aligned} \tag{4.5}$$

On the other hand, from the definition of extremal index $\theta(\tau^{(1)},\tau^{(2)})$,

$$\begin{aligned} \lim_{n\to\infty} P\Big(\max_{1\le j\le n} \tau^{(1)} X_j^{(1)} \le n,\, \max_{1\le j\le n} \tau^{(2)} X_j^{(2)} \le n\Big) &= \\ = \Big(\lim_{n\to\infty} P\Big(\max_{1\le j\le n} \tau^{(1)} \widehat{X}_j^{(1)} \le n,\, \max_{1\le j\le n} \tau^{(2)} \widehat{X}_j^{(2)} \le n\Big)\Big)^{\theta\left(\tau^{(1)},\tau^{(2)}\right)} \\ = \Big[\exp\Big[-\frac{1}{2}\Big(\tau^{(1)} + \tau^{(2)}\Big)\Big]\Big]^{\theta\left(\tau^{(1)},\tau^{(2)}\right)} \;, \end{aligned} \tag{4.6}$$

since under condition $\overset{\bullet}{D}(u_n)$ for $\{X_n\}_{n\ge1}$, the sequence $\{\widehat{X}_n\}_{n\ge1}$ also satisfies $\overset{\bullet}{D}(u_n)$ and, for each $i=1,2$, $\widehat{S}_n^{(i)}([0,1]) = S_n[\widehat{X}_n^{(i)}, u_n^{(\tau_i)}]([0,1])$ converges in distribution to a random variable with Poisson distribution with parameter $\tau^{(i)}$.

By attending to a (4.5) and (4.6) it follows that

$$\exp\Big[-\frac{1}{2}\Big(\theta^{(1)}\tau^{(1)} + \theta^{(2)}\tau^{(2)}\Big)\Big] = \Big[\exp\Big(-\frac{1}{2}\Big(\tau^{(1)} + \tau^{(2)}\Big)\Big)\Big]^{\theta\left(\tau^{(1)},\tau^{(2)}\right)}$$

and so

$$\theta\Big(\tau^{(1)},\tau^{(2)}\Big) = \theta^{(1)} \frac{\tau^{(1)}}{\tau^{(1)}+\tau^{(2)}} + \theta^{(2)} \frac{\tau^{(2)}}{\tau^{(1)}+\tau^{(2)}} \; . \tag{4.7}$$

This result is not surprising since under the condition $\overset{\bullet}{D}(u_n)$ for $\{X_n\}_{n\geq 1}$ we have the asymptotic independence of the maxima of the vector margins and Nandagopalan (1994) proves that in this case the bivariate extremal index is a convex linear combination of the marginal extremal indexes as in (4.7).

We finish this section by exhibiting a nonstationary sequence that verifies condition $\overset{\bullet}{D}(u_n)$.

Example 4.1. Let $\{Z_n^{(1)}\}_{n\geq 1}$ and $\{Z_n^{(2)}\}_{n\geq 1}$ be independent sequences of random variables. For $0<\lambda\leq 1$ constant, consider the autoregressive sequences of maxima defined as

$$X_n = \lambda \max\bigl(X_{n-1}, Z_n^{(1)}\bigr)$$

and

$$Y_n = \lambda \max\bigl(Y_{n-1}, Z_n^{(2)}\bigr)$$

where $X_0 = Y_0$ is independent of $\{Z_n^{(1)}\}_{n\geq 1}$.

For each $n\geq 1$ it follows that

$$X_n = \max\left(\max_{1\leq j\leq n} \lambda^j Z_{n-j+1}^{(1)},\ \lambda^n X_0\right)$$

and

$$Y_n = \max\left(\max_{1\leq j\leq n} \lambda^j Z_{n-j+1}^{(2)},\ \lambda^n Y_0\right) .$$

So, for each $J\subset [0,+\infty[$ such that $m(J) = r_n = [\frac{n}{k_n}]$, we have

$$P\Bigl(M_n^{(1)}(J)\leq u_n,\ M_n^{(2)}(J)\leq u_n\Bigr) = \\ = P\left(\bigcap_{s\in J}\bigcap_{j=1}^{s} Z_{s-j+1}^{(1)} \leq \frac{u_n}{\lambda^j}\right) P\left(X_0 \leq \frac{u_n}{\lambda^n}\right) P\left(\bigcap_{s\in J}\bigcap_{j=1}^{s} Z_{s-j+1}^{(2)} \leq \frac{u_n}{\lambda^j}\right)$$

and

$$P\Bigl(M_n^{(1)}(J)\leq u_n\Bigr) P\Bigl(M_n^{(2)}(J)\leq u_n\Bigr) = \\ = P\left(\bigcap_{s\in J}\bigcap_{j=1}^{s} Z_{s-j+1}^{(1)} \leq \frac{u_n}{\lambda^j}\right) P^2\left(X_0 \leq \frac{u_n}{\lambda^n}\right) P\left(\bigcap_{s\in J}\bigcap_{j=1}^{s} Z_{s-j+1}^{(2)} \leq \frac{u_n}{\lambda^j}\right)$$

and consequently,

$$\left|P\Bigl(M_n^{(1)}(J)\leq u_n,\ M_n^{(2)}(J)\leq u_n\Bigr) - P\Bigl(M_n^{(1)}(J)\leq u_n\Bigr) P\Bigl(M_n^{(2)}(J)\leq u_n\Bigr)\right| \leq \\ \leq \left|P\left(X_0\leq \frac{u_n}{\lambda^n}\right) - P^2\left(X_0\leq \frac{u_n}{\lambda^n}\right)\right| \xrightarrow[n\to\infty]{} 0 .$$

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