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# STATISTICAL PROPERTIES AND SENSITIVITY OF A NEW ADAPTIVE SAMPLING METHOD FOR QUALITY CONTROL

Authors: Manuel do Carmo
– Universidade Europeia, Lisbon
and CIMA-UE, University of Évora, Portugal
manuel.carmo@universidadeeuropeia.pt

Paulo Infante
– ECT/DMAT and CIMA-UE, University of Évora,
Évora, Portugal
pinfante@uevora.pt

Jorge M. Mendes
– NOVA IMS, Universidade Nova de Lisboa,
Lisbon, Portugal
jmm@novaims.unl.pt

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Abstract:

- We present a new adaptive sampling method for statistical quality control. In this method, called LSI (***Laplace*** **s**ampling **i**ntervals), we use the probability distribution function of the *Laplace* standard distribution to obtain the sampling instants, depending on a $k$ parameter that allows control of sampling costs. Several algebraic expressions concerning the statistical properties of the LSI method are presented. We compare the LSI method with fixed sampling intervals (FSI) and variable sampling intervals (VSI) methods using a *Shewhart* $X$-bar control chart and evaluate the sensitivity of these sampling methods when the lower sampling interval is truncated. The results obtained show that the new method is a viable alternative in various critical contexts and situations.

Key-Words:

- *adaptive sampling; sensitivity; simulation; Laplace sampling intervals.*

AMS Subject Classification:

- 62N10, 62P30.

## 1. INTRODUCTION

The success of a statistical quality control method is directly related to the type of control chart and especially to the sampling method used. The variability of the process is due to random causes (inherent to the process) or to the presence of assignable causes. The former cannot be economically identified and corrected, whereas the latter should be detected and eliminated. The choice of the control chart depends upon the characteristic being controlled. The quantitative characteristics are controlled using variable control charts ($\bar{X}$-charts, $R$-charts, or $s$-charts, for example) or special control charts (EWMA or CUSUM charts for continuous random variables, for example). For a long time the control charts used had fixed parameters (sampling intervals, sample sizes, and control limits). However, since final of the 1980s, new adaptive control charts have been developed for improved performance. In terms of their implementation, these charts can be classified in two broad categories. The first category encompasses control charts with adaptive parameters (sampling intervals, sample size, and control limits, depending on the sample information; see, for example, Reynolds *et al.* (1988), Daudin (1992), Prabhu *et al.* (1993), Costa (1994), Prabhu *et al.* (1994), Stoumbos & Reynolds (1997), Costa (1999), Rodrigues Dias (1999), Carot *et al.* (2002), Mahadik & Shirke (2009)). The second category encompasses control charts with predetermined parameters (parameters determined before the beginning of the process to be controlled; see for example, Banerjee & Rahim (1988), Rahim & Banerjee (1993), Lin & Chou (2005) and Rodrigues Dias & Infante (2008)).

Several measures have been developed to assess the statistical quality controls method's performance across time regarding to how quickly they detect assignable causes. The frequency of false alarms and the number of samples and analysed items are two examples. The $ARL$ ("*average run length*") is perhaps the most widely used statistical measure for assessing the performance of a statistical control chart. The $ARL$ is defined as the average number of samples that needs to be drawn before an out-of-control indication is given. If the control methods have constant and equal sampling intervals, then the time interval up to the detection of a change is directly proportional to the $ARL$. In the case of non-constant sampling intervals, the proportionality above fails and the $ARL$ is not a measure of the efficiency of the control method. The $AATS$ ("*adjusted average time to signal*"), also known in the literature as "*steady-state performance*", is defined as the average interval of time from the instant at which a failure occurs in the system to the instant at which the control chart detects the failure. In the case of a *Shewhart* control chart with variable sampling intervals, $AATS = E(G) + E(D) \times (ARL - 1)$, where $E(D)$ is the average sampling interval and $G$ represents the time interval between the instant at which the system fails and the instant at which the first sample, after the failure, is drawn. The $AATS$

is a measure that suits most practical situations. Morais (2002), Carmo (2004) and Rodrigues Dias & Carmo (2009) are important sources on the previously described approaches. In Morais & Pacheco (2001), stochastic order relations are established using the $RL$ ("*run length*"), allowing comparison of different quality control methods without numerical computation of their performance.

In the following sections, we present a new sampling method called LSI ("*Laplace sampling intervals*"), an adaptive and continuous sampling method in which the sampling intervals are obtained on the basis of the probability density function of the *Laplace* standard distribution and depends on a scale parameter, $k$. The $AATS$ will be used in section 3 to examine the statistical properties of this method and to compare its effectiveness with that of the FSI ("*fixed sampling intervals*") and VSI (*variable sampling intervals*) methods. In section 4, the sensitivity of this new method is compared to the sensitivity of the above mentioned methods. Finally, in section 5, conclusions are drawn and future work is proposed.

## 2. NEW SAMPLING METHOD: LSI (*LAPLACE* SAMPLING INTERVALS)

### 2.1. Methodology

Let $X$ be a continuous quality variable so that when the system is in control state, $X$ is a random variable with expected value $\mu = \mu_0$ and standard deviation $\sigma = \sigma_0$. If $x_1, x_2, ..., x_n$ are identical and independently distributed random variables with the same distribution of $X$, where $n$ is the sample size, then $\bar{X}$ has the same expected value $\mu_0$ and standard deviation $\sigma_0/\sqrt{n}$. As a consequence of the one assignable cause, corresponding to a failure of the system, the process state may change and then $\mu$ and $\sigma$ may assume new values $\mu_1 = \mu_0 \pm \lambda\sigma_0$, and $\sigma_1 = \sigma_0$, with $\lambda > 0$. If $t_i$ denotes a sampling instant of order $i$ and $\overline{x}_i$ is the sample mean value of order $i$, according to the LSI method, the next sampling at the instant of order $i+1$ is given by

$$t_{i+1} = t_i + k\,.\,l(u_i)\,, \qquad i = 0, 1, 2, ...\,, \tag{2.1}$$

where $u_i = \frac{\overline{x}_i - \mu_0}{\sigma_0}\sqrt{n}$, $t_0 = 0$, $\overline{x}_0 = \mu_0$, $l(u_i) = \dfrac{1}{2}\,e^{-|u_i|}$, $n$ is the sample size, $k$ is a convenient scale constant and $l(\cdot)$ is the density function of the standard *Laplace* variable. Therefore, according to (2.1), this sampling method considers consecutive sampling intervals $\delta_i = t_i - t_{i-1} = k\,.\,l(u_{i-1}) = k \times 0.5 \times e^{-|u_{i-1}|}$, $i = 1, 2, 3, ....$ These are values from independently and identically distributed continuous random variables $D_i$, $i = 1, 2, 3, ...$, with the same distribution of a generic variable $D$. When we obtain the value of $k$ we have only sampling intervals, $D_i$, under control

$(E(D|\lambda=0)=1)$; when we obtain the values of the $AATS$ we have only sampling intervals out-of-control, subject to different shifts of the sample mean. Thus, $D$ is a function of $\bar{X}$ and, consequently, of $U$, given by

$$D = \frac{k\,.\,e^{-\left|\frac{\bar{X}-\mu}{\sigma}\sqrt{n}\right|}}{2} = k\,.\,l(U)\;. \tag{2.2}$$

The constant $k$ depends on several factors and, especially, on the costs associated with the production process (not imposing, so far, any limits on the control chart for means) and $U = \left(\bar{X}-\mu\right)\sqrt{n}/\sigma$ . Using this adaptive and continuous method, the sampling frequency decreases (the sampling instants are spaced further apart in time) when the sample mean is marked close to the mean of the distribution. When the sample mean is marked close to control limits, the probability of a shift in the mean increases, and the sampling frequency increases (the sampling instants are less distant in time). Like the VSI sampling method, the LSI method is an adaptive method in which the time interval to the next sample depends on the information in the current sample. The disadvantage is that the sampling intervals function of the LSI chart is a continuous function of the chart statistic (and this implies an infinite number of possible sampling intervals). However, the sampling interval function is a very simple function of the chart statistic. It can be easy to implement in practice, particularly, in automatic monitoring. The NSI (*normal sampling intervals*) method, presented by Rodrigues Dias (1999) and studied in Infante (2004), showed limitations in practical applications. In this method the sampling instants are obtained using the density function of the standard Normal distribution; the smallest sampling interval is very small, which reduces the application of this sampling method. The idea emerged to study one analogous method in which the smallest sampling interval would be greater than in the NSI method which, therefore, allowed practical applications. Regarding the skewness and shape, the *Laplace* density function is similar to the Normal density function, as in the *Cauchy* density function, but having heavier tails. This fact addresses some of the difficulties seen in the practical application of the NSI method. In our preliminary work we simulated sampling intervals for three probability density functions ($pdf\,(x,\mu,\sigma)$): those of the Normal, *Cauchy*, and *Laplace* distributions. Considering "*3-sigma*" control limits and a time unit average sampling interval, in control, the following results were obtained:

- **a**) Normal distribution: $pdf\,(0,0,1) = 0.399$, $k = 3.535$, smallest sampling interval = 0.016, largest sampling interval = 1.410.
- **b**) *Cauchy* distribution: $pdf\,(0, \text{not defined}, \text{not defined}) = 0.318$, $k = 4.778$, smallest sampling interval = 0.152, largest sampling interval = 1.521.
- **c**) *Laplace* distribution: $pdf\,(0,0,1) = 0.500$, $k = 3.813$, smallest sampling interval = 0.095, largest sampling interval = 1.907.

Based on these results, we selected the *Laplace* distribution's probability density function. All the parameters are defined such that the sampling frequency decreases close to the central region and the smallest sampling interval is more likely to apply in practice. In addition, the smallest and largest sampling intervals are approximately equal to the sampling pair most frequently used in the VSI method ($(d_1, d_2) = (0.1, 1.9)$). We are considering general sampling interval functions that are continuous functions of the chart statistic. Stoumbos *et al.* (2001) study what function would be the optimal function in some sense.

## 2.2. Statistical Properties

In the remainder of this paper we assume that $X$ follows a normal distribution with expected value $\mu = \mu_0$ and standard deviation $\sigma = \sigma_0$. We will consider a *Shewhart* chart with $LCL$ and $UCL$, respectively, lower and upper control limits, given by:

$$LCL = \mu_0 - L\frac{\sigma_0}{\sqrt{n}}\,, \qquad UCL = \mu_0 + L\frac{\sigma_0}{\sqrt{n}}\,, \tag{2.3}$$

where $L$ is the coefficient of the control limits (in practice, typically around three units of standard deviation). As mentioned above, after shift, $\mu$ takes on the new value $\mu_1 = \mu_0 \pm \lambda\sigma_0$, where $\lambda > 0$ is the magnitude of the mean shift (in the present work, only mean shifts are considered). Therefore, if $u_i$ denotes the standard sample mean, for values to $|u_i| > L$ the process is considered to be out-of-control, although this might be a false alarm.

Considering the assumptions in (2.2) and (2.3), and that $f^*(\overline{x})$ is the corresponding conditional density function of $\overline{x}$, given by

$$f^*(\overline{x}) = \frac{\sqrt{n}}{\beta\sigma\sqrt{2\pi}}\, e^{-\frac{n(\overline{x}-\mu)^2}{2\sigma^2}}\,, \qquad \overline{x} \in \,]LCL, UCL[\,, \tag{2.4}$$

then $f^*(\overline{x})\,d\overline{x}$ is the elementary probability of $\overline{x} \in \,]\overline{x}, \overline{x}+d\overline{x}[$ and the average sampling interval is given by

$$\begin{aligned} E\,(D|\lambda, n, L) &= \int_{LCL}^{UCL} k\,.\,l(\overline{x})\,.\,f^*(\overline{x})\,d\overline{x} \\ &= \int_{LCL}^{UCL} \frac{k\,.\sqrt{n}}{2\beta\sigma_0\sqrt{2\pi}}\; e^{-\left[\left|\frac{\overline{x}-\mu_0}{\sigma_0}\sqrt{n}\right| + \frac{n(\overline{x}-\mu_0-\lambda\sigma_0)^2}{2\sigma_0^2}\right]} d\overline{x}\,, \end{aligned} \tag{2.5}$$

where $\beta$ is the probability of the sample mean lies between the control limits, and is given by

$$\beta = \Phi\big(L - \lambda\sqrt{n}\big) - \Phi\big(-L - \lambda\sqrt{n}\big)\,. \tag{2.6}$$

Considering $\Phi(u)$ as the distribution function of the standard normal random variable, the following expression for the average sampling interval arises

$$E\,(D|\lambda, n, L) = \frac{k\sqrt{e}}{2\beta}\left[e^{\lambda\sqrt{n}}.\,A(L,\lambda,n) + e^{-\lambda\sqrt{n}}.\,B(L,\lambda,n)\right], \tag{2.7}$$

where

$$(2.8)\qquad \begin{aligned} A(L,\lambda,n) &= \Phi\big(-1-\lambda\sqrt{n}\big) - \Phi\big(-L-1-\lambda\sqrt{n}\big)\,,\\ B(L,\lambda,n) &= \Phi\big(L+1-\lambda\sqrt{n}\big) - \Phi\big(1-\lambda\sqrt{n}\big)\,. \end{aligned}$$

The expression (2.7) depends on the sample size, $n$, the coefficient of the control limits, $L$, the mean shifts, $\lambda$, and $\beta$ (the probability of a Type II error if the sample mean is out of the control limits). Assuming that the values of $n$, $L$ and $\lambda$ are known, then E(D) is a linear function of $k$. When the process is in control, $\lambda = 0$, the average sampling interval is given by

$$(2.9)\qquad E\,(D|L) \;=\; \frac{k\sqrt{e}}{\beta}\,\big[\Phi(L+1)-\Phi(1)\big]\,,$$

where $\beta = 2\,\Phi(L)-1$ and does not depend on the sample size, $n$. Therefore, if the average sampling interval is equal to a time unit (without loss of generality, the sampling period used in the FSI method), the constant $k$ is given by

$$(2.10)\qquad k \;=\; \frac{\beta}{\sqrt{e}\,\big[\Phi(L+1)-\Phi(1)\big]}\,,$$

which is equal to 3.8134, based on the usual "*3-sigma*" limits. This result was obtained by numerical integration using the R software.

The variance of the sampling intervals can be obtained by the equality $\mathrm{Var}(D) = E(D^2) - [E(D)]^2$. The expression for $E(D^2)$ is obtained using the same reasoning applied to derive (2.5), leading to

$$(2.11)\qquad \begin{aligned} E\left(D^2|\lambda,nL\right) &= \int_{LCL}^{UCL} \big[k\,.\,l(\overline{x})\big]^2 .\, f^*(\overline{x})\,d\overline{x}\\ &= \frac{e^2k^2}{4\beta}\,\Big[e^{2\lambda\sqrt{n}}\Big[\Phi\big(-2-\lambda\sqrt{n}\big)-\Phi\big(-L-2-\lambda\sqrt{n}\big)\Big]\\ &\qquad +\; e^{-2\lambda\sqrt{n}}\Big[\Phi\big(L+2-\lambda\sqrt{n}\big)-\Phi\big(2-\lambda\sqrt{n}\big)\Big]\Big]\,, \end{aligned}$$

which allows us to obtain the desired variance.

As mentioned above, there are different measures that are commonly used to assess the effectiveness of control charts. In this study we use the $AATS$ to compare the effectiveness of the LSI method with the effectiveness of the FSI and VSI methods. Let $RL$ (*run length*) be a random variable denoting the number of samples to be drawn before a false alarm or a failure occurrence, regardless of the sampling method used. $RL$ follows a geometric distribution with the parameter $1-\beta$, that is, with a mean and the variance, respectively, given by

$$(2.12)\qquad ARL(\lambda) \;=\; \frac{1}{1-\beta}\,,$$

and

$$\text{Var}[RL(\lambda)] = \frac{\beta}{(1-\beta)^2} . \tag{2.13}$$

In general, a process starts in control. Therefore, the time interval between occurrence of a failure and its detection is of particular importance. For example, in a production process in which the malfunction costs are high, the average total cost of a production cycle may increase. As the failure may occur in the interval between two samples, it is necessary to adjust the ATS (*average time to signal* — which is defined as the average interval of time between the beginning of the process and an out-of-control sign, eventually a false alarm, being given by the control chart). Thus, we consider $G$ to be the time interval between the occurrence of a failure and the moment when the first sample is drawn after the mean shift. The $AATS$ (*adjusted average time to signal*) is given by

$$AATS = E(G) + (ARL-1) . E(D) , \tag{2.14}$$

where the expected value of $G$ has to be determined. In the FSI method, the expected value of $G$ is, approximately, half of the inspection period used. However, in this adaptive case, we do not have a constant sampling interval. The distribution of the variable $G$ depends on when the shift of the mean occurs. Let us assume that the time when a shift occurs is uniformly distributed in each sampling interval. If a failure occurs in a sampling interval of length $d$, the average time until the next sample is drawn is $0.5 \times d$. Although the number of sampling intervals is infinite, we can assume that the probability of the shift occurring in a sampling interval of length $d$ is proportional to the product of the length of the interval and the probability of selecting this interval, as long as the process is in control, as Reynolds *et al.* (1988) and Runger & Pignatiello (1991) assumed for the VSI method. Taking into account that the variable $G$ is continuous, the expression for its expected value can be obtained using the same reasoning that Reynolds *et al.* (1988) used in the VSI case. Based on the assumptions stated above, we obtain the following expression for the expected value of $G$

$$E(G|L) = \frac{E\left(D^2|\lambda=0\right)}{2E(D|\lambda=0)} = \frac{k . e^{3/2}}{4} \frac{\Phi(L+2)-\Phi(2)}{\Phi(L+1)-\Phi(1)} , \tag{2.15}$$

which can be written as

$$E(G) = k . e^{3/2} \times C(L) , \tag{2.16}$$

with

$$C(L) = \frac{\Phi(L+2)-\Phi(2)}{4 \times \left[\Phi(L+1)-\Phi(1)\right]} . \tag{2.17}$$

Expression (2.17) depends only on the control limits and may be simplified to

$$E(G) = 0.036 \times k e^{3/2} . \tag{2.18}$$

This simplified expression will be useful in future algebraic treatments. This simplified version is originated by the data in Table 1, containing approximations for $C(L)$ for several values of $L$. It is clear that 0.036 is an excellent approximation of $C(L)$, particularly for values of $L \geq 2$. This approximation is not as good as one might expect for $L < 2$. However, this situation can be considered irrelevant in many applications, as it results in a high number of false alarms.

**Table 1**: Values of $C(L)$ for different multiples $L$ of the standard deviation.

| $L$ | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
|---|---|---|---|---|---|---|---|---|---|
| $C(L)$ | 0.0394 | 0.0369 | 0.0361 | 0.0359 | 0.0359 | 0.0358 | 0.0358 | 0.0358 | 0.0358 |

The values of the $AATS$ can be obtained as

$$AATS_{\text{LSI}} = 0.036 \times k \times e^{3/2} + \left(\frac{\beta}{1-\beta}\right) \times E(D) . \tag{2.19}$$

In this case, the distribution of the sampling interval $D_i$ is the conditional distribution of the sample mean given that the process is out-of-control. $D_1, D_2, ...$ are independent of $RL$, and the variance of $TS$ (*time to signal*) can be written as

$$\operatorname{Var}(TS) = \operatorname{Var}(G) + E(RL-1)\operatorname{Var}(D) + \operatorname{Var}(RL-1)\,[E(D)]^2 , \tag{2.20}$$

for which we need the value of $\operatorname{Var}(G)$. To get $\operatorname{Var}(G)$, we begin by determining $E(G^2)$. According to Reynolds *et al.* (1988), for the VSI method, and Infante (2004), for the NSI method, the algebraic expression is given by

$$E(G^2) = \frac{E(D^3|\lambda=0)}{3\,E(D|\lambda=0)} = \frac{k^2 e^4}{12}\,\frac{\Phi(L+3)-\Phi(3)}{\Phi(L+1)-\Phi(1)} , \tag{2.21}$$

which depends only on $L$. Therefore, the variance of the variable $G$ is given by

$$\begin{aligned} \operatorname{Var}(G) &= E(G^2) - [E(G)]^2 \\ &= \frac{k^2 e^4}{12}\,\frac{\Phi(L+3)-\Phi(3)}{\Phi(L+1)-\Phi(1)} - \frac{k^2 e^3}{16}\left[\frac{\Phi(L+2)-\Phi(2)}{\Phi(L+1)-\Phi(1)}\right]^2 . \end{aligned} \tag{2.22}$$

From (2.7), (2.11), (2.21) and (2.22) we obtain (2.20).

## 3. COMPARISONS BETWEEN THE LSI METHOD AND THE FSI AND VSI METHODS

As mentioned in the previous section, comparisons of the effectiveness of the LSI sampling method with the FSI and VSI methods is made using the $AATS$.

Thus, the two sampling methods in comparison are considered to be both in control, or in other words, the average sampling intervals are equal to one time unity ($d=1$) and the control limits are "*3-sigma*" ($L=3$). Comparisons are made for mean shifts, only. Because it is assumed that the characteristic $X$ follows a normal distribution, the direction of the shift is of no importance at all. Under these assumptions the value of the parameter $k$ in the LSI method is 3.8134.

## 3.1. Comparison between the LSI and FSI sampling methods

Assuming a fixed value for the sampling interval, $d$, the expected value of $G$ (the random variable previously defined for the FSI sampling method) can be defined as half of the sampling interval, $d$. Infante & Rodrigues Dias (2002) and Carmo (2004), in independent studies, analysed this approximation for different lifetimes, and both concluded this approximation to be acceptable. Therefore, the $AATS$ of the fixed sampling method is given by

$$AATS_{FSI} = E(G) + (ARL-1)\times d \cong \frac{d}{1-\beta} - \frac{d}{2}\,. \tag{3.1}$$

To compare the effectiveness of the two sampling methods, LSI and FSI, we assume that, in control, the average sampling interval of the LSI method is equal to the sampling interval of the FSI method (without loss of the generality, $d=1$), obtaining a value of $k=3.8134$ for the LSI method. Considering (2.19) and (3.1) the ratio

$$Q_{\text{LSI/FSI}} = \frac{AATS_{\text{FSI}} - AATS_{\text{LSI}}}{AATS_{\text{FSI}}} \times 100\% \tag{3.2}$$

represents a measure of the relative variation, in %, of the $AATS$ value when $AATS_{\text{FSI}}$ is the reference. The results obtained for mean shifts with different sample sizes are illustrated in Figure 1.

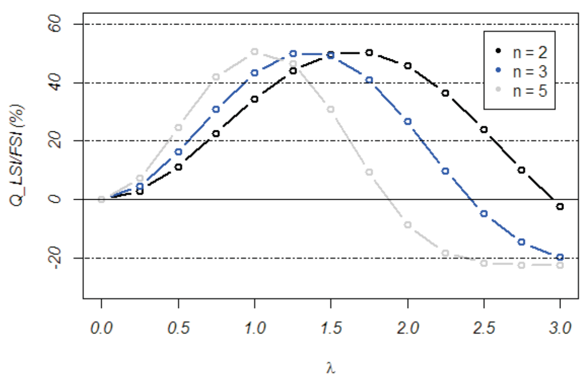


**Figure 1**: $Q_{LSI/FSI}(\%)$, as a function of $\lambda$ and different values of $n$.

From Figure 1, the following conclusions arise:

**i)** The control chart for means with the LSI method detects small and moderate mean shifts more quickly than the control chart for means with the FSI method. This means that the LSI method is more sensitive to changes whose probability of detection is low. The reductions in the $AATS$ using the LSI method can be very large.

**ii)** For shifts in which the probability of the detection is high, the FSI method performs better than the LSI method. This is not surprising, because this is true for most known adaptive methods. In the situations described above, the average number of samples taken before detection of a failure is very small. Therefore, the value of $E(G)$ is of great importance. It is approximately equal to the time of system malfunction because only a single sample is required to detect the shift. For an average sampling interval, in control, equal to unity, $E(G) \cong 0.50$ in the FSI method and $E(G) \cong 0.61$ in the LSI method. However, the reduction obtained with FSI method, in terms of the $AATS$, is limited to a maximum of 22.5% (for $n = 5$), whereas the reduction obtained with LSI method has a maximum of 50.3%.

**iii)** For the different sample sizes considered, the $Q_{\mathrm{LSI/FSI}}$ values begin with an average rate of positive variation, reaching an absolute value maximum, and then reaching an average rate of negative variation. The average rate of positive variation increases more quickly as a function of $\lambda$ when the sample size increases. The average rate of negative variation increases more quickly as a function of $\lambda$ when the sample size decreases.

**iv)** In general, when the sample size increases, the values that maximize $(\lambda)$ and the reductions obtained with the LSI method decrease. This makes sense because the probability of detection of the shift increases with the sample size.

## 3.2. Comparison between the LSI and VSI sampling methods

Looking for improvements in the performance of classical control charts, Reynolds *et al.* (1988) divided the region of continuation, $C = ]-L, L[$, into two sub-regions, $C_1 = ]-L, -w] \cup [w, L[$ and $C_2 = ]-w, w[$, and used two sampling intervals, $d_1$ and $d_2$ , with $d_1 < d < d_2$. The VSI method allows us to anticipate the next sample (we use $d_1$ if the sample mean belongs to the $C_1$ region) or to delay it (using $d_2$ if the sample mean belongs to the $C_2$ region). Reynolds & Arnold (1989), Reynolds (1989), Runger & Pignatiello (1991), and Reynolds (1995), in different contexts, gave theoretical justifications for the use of two sampling intervals. For two intervals the average sampling interval in the VSI method is given by:

$$E(D|\lambda, n) = \frac{d_1 \times p_{11} + d_2 \times p_{12}}{\beta}, \tag{3.3}$$

where $\beta$ is given by (2.6), and

$$\begin{aligned} p_{11} &= \Phi\big(L - \lambda\sqrt{n}\big) - \Phi\big(w - \lambda\sqrt{n}\big) + \Phi\big(-w - \lambda\sqrt{n}\big) - \Phi\big(-L - \lambda\sqrt{n}\big), \\ p_{12} &= \Phi\big(w - \lambda\sqrt{n}\big) - \Phi\big(-w - \lambda\sqrt{n}\big), \end{aligned} \tag{3.4}$$

are the probabilities of a sample mean occurring in regions $C_1$ and $C_2$, respectively, when a mean shift occurs. $W$ is given by

$$W = \Phi^{-1}\left[\frac{2\Phi(L) \times (d - d_1) + d_2 - d}{2(d_2 - d_1)}\right], \tag{3.5}$$

according to the expression presented by Runger & Pignatiello (1991), when the average sampling interval in the VSI method, in control, is equal to the sampling period, $d$, in the FSI method. According to Reynolds *et al.* (1988), the average time interval between the instant when a failure occurs and the instant when the first sample is drawn after the shift occurs is given by

$$E(G) = \frac{d_1^2 p_{01} + d_2^2 p_{02}}{2(d_1 p_{01} + d_2 p_{02})}. \tag{3.6}$$

The adjusted average time to signal, $AATS$, is given by

$$AATS_{\text{VSI}} = \frac{d_1^2 p_{01} + d_2^2 p_{02}}{2(d_1 p_{01} + d_2 p_{02})} + \frac{d_1 p_{11} + d_2 p_{12}}{1 - \beta}, \tag{3.7}$$

where

$$p_{01} = 2\big[\Phi(L) - \Phi(w)\big] \qquad \text{and} \qquad p_{02} = 2\Phi(w) - 1 \tag{3.8}$$

are the probabilities of a sample mean belonging to the regions $C_1$ and $C_2$, respectively, when the process is in control.

To compare the effectiveness of the LSI and VSI methods, we assume that the average sampling intervals in both sampling methods are equal to the fixed sampling interval ($d = 1$ and $k = 3.8134$) in the expressions (2.19) and (3.7). Once again, the ratio

$$Q_{\text{LSI/VSI}} = \frac{AATS_{\text{VSI}} - AATS_{\text{LSI}}}{AATS_{\text{VSI}}} \times 100\%, \tag{3.9}$$

represents a measure in % of the relative variation of the $AATS$ value, with respect to the $AATS_{\text{VSI}}$ reference. The results obtained for mean shifts with different sample sizes are presented in Table 2. The following conclusions are immediate:

**Table 2**: $Q_{\text{LSI/VSI}}(\%)$, as a function of $\lambda$, for different values of $n$ and different sampling pairs in VSI.

| $n$ | $(d_1, d_2)$ | $\lambda$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | *0.00* | *0.25* | *0.50* | *0.75* | *1.00* | *1.25* | *1.50* | *1.75* | *2.00* | *2.25* | *2.50* | *3.00* |
| *2* | *(0.1, 1.9)* | 0.1 | −1.4 | −5.9 | −13.2 | −21.6 | −26.0 | −19.8 | −4.5 | 10.5 | 20.7 | 26.4 | 31.0 |
| | *(0.1, 1.5)* | 0.0 | −0.9 | −3.7 | −8.6 | −14.9 | −20.5 | −20.9 | −14.0 | −4.0 | 4.4 | 9.7 | 14.2 |
| | $AATS_{\text{LSI}}$ | *370.01* | *216.71* | *79.98* | *29.08* | *11.31* | *4.86* | *2.40* | *1.41* | *0.98* | *0.79* | *0.70* | *0.63* |
| *3* | *(0.1, 1.9)* | 0.1 | −2.2 | −8.9 | −19.0 | −25.9 | −18.2 | 1.1 | 17.0 | 25.5 | 29.5 | 31.2 | 32.2 |
| | *(0.1, 1.5)* | 0.0 | −1.4 | −5.7 | −12.8 | −20.1 | −20.5 | −10.6 | 1.2 | 8.8 | 12.7 | 14.4 | 15.4 |
| | $AATS_{\text{LSI}}$ | *370.01* | *175.53* | *50.46* | *15.24* | *5.27* | *2.23* | *1.22* | *0.86* | *0.71* | *0.66* | *0.63* | *0.61* |
| *5* | *(0.1, 1.9)* | 0.1 | −3.7 | −14.6 | −25.7 | −15.4 | 9.3 | 23.9 | 29.6 | 31.5 | 32.1 | 32.3 | 32.3 |
| | *(0.1, 1.5)* | 0.0 | −2.3 | −9.5 | −19.4 | −19.4 | −4.9 | 7.3 | 12.8 | 14.7 | 15.3 | 15.4 | 15.5 |
| | $AATS_{\text{LSI}}$ | *370.01* | *122.99* | *24.81* | *5.97* | *1.98* | *1.01* | *0.74* | *0.65* | *0.63* | *0.62* | *0.61* | *0.61* |

**i)** In sample sizes more widely used in the literature, $n \geq 3$, the LSI method is quicker than the VSI method in detecting shifts of magnitude $\lambda > 1.5$, i.e., in situations whose probability of detection is high.

**ii)** The effectiveness of the LSI method increases when the sample size increases for moderate and large shifts in the mean. For small shifts in the mean, the effectiveness of the LSI method decreases as the sample size increases.

**iii)** If we consider $(d_1, d_2) = (0.1, 1.9)$ in VSI, the maximum reductions obtained with the LSI method are considerable (approximately 32%); in general, the performance of the LSI improves significantly when the sample size is larger; if the probability of occurrence of a shift is equal for all $\lambda$, using the LSI method could be a competitive advantage.

**iv)** If we consider $(d_1, d_2) = (0.1, 1.5)$ in VSI, the maximum reductions obtained with the LSI method are significantly smaller (approximately 16%) than those obtained with the other sampling pair in the VSI method; in general, the performance of the LSI improves as the sample size gets larger.

## An example of application

In sections 3.1 and 3.2 we compared, in a critical way, the performances of the FSI and VSI methods with the performance of the LSI method in terms of the $AATS$. For a better perception in absolute terms of the LSI method, we present an example of application that allows checking the effectiveness in the detection of the shift.

Thereby, if there is a mean shift of magnitude $\lambda = 1.0$, and considering how unit of the time one hour, for the quality characteristic being monitored:

- **i)** If we use the FSI sampling method, the first sample after the mean shift is drawn, on average, after 30 minutes, and we need 240 minutes, on average, to detect the shift.
- **ii)** If we use the VSI sampling method:
  - **a)** with the sampling pair $(d_1, d_2) = (0.1, 1.9)$, the first sample after the mean shift is drawn, on average, after 54 minutes, and we need 103 minutes, on average, to detect the shift;
  - **b)** with the sampling pair $(d_1, d_2) = (0.1, 1.5)$, the first sample after the mean shift is drawn, on average, after 44 minutes, and we need 100 minutes, on average, to detect the shift.
- **iii)** If we use the LSI sampling method: the first sample after the mean shift is drawn, on average, after 37 minutes, and we need 119 minutes, on average, to detect the shift.

In this case we can conclude that the use of the LSI method allows us to reduce the out-of-control period by 121 minutes, on average, compared to the FSI method, and increase the out-of-control by either 16 minutes or 19 minutes, compared to the VSI method, depending on whether we use the sampling pair $(d_1, d_2) = (0.1, 1.9)$ or the sampling pair $(d_1, d_2) = (0.1, 1.5)$.
On the other hand, if a shift of magnitude is of $\lambda = 1.5$, for the quality characteristic being monitored:

- **i)** If we use the FSI sampling method, the first sample after the mean shift is drawn, on average, after 30 minutes, and we need 64 minutes, on average, to detect the shift.
- **ii)** If we use the VSI sampling method:
  - **a)** with the sampling pair $(d_1, d_2) = (0.1, 1.9)$, the first sample after the mean shift is drawn, on average, after 54 minutes, and we need 58 minutes, on average, to detect the shift;
  - **b)** with the sampling pair $(d_1, d_2) = (0.1, 1.5)$, the first sample after the mean shift is drawn, on average, after 44 minutes, and we need 48 minutes, on average, to detect the shift.
- **iii)** If we use the LSI sampling method: the first sample after the mean shift is drawn, on average, after 37 minutes, and we need 44 minutes, on average, to detect the shift.

We can conclude that the use of the LSI method allows us to reduce the out-of-control period by 20 minutes, on average, compared to the FSI method, and by either 14 minutes or 4 minutes, compared to the VSI method, depending on whether we use the sampling pair $(d_1, d_2) = (0.1, 1.9)$ or the sampling pair $(d_1, d_2) = (0.1, 1.5)$.
Thus, for this situation and others in which $\lambda > 1.5$, the use of the LSI method makes it possible to reduce the malfunction costs and makes the product more competitive by reducing its final price.

The influence of the sampling interval distribution on the standard deviation must be analysed as well. The values of the coefficient of variation of the $TS$ for the different sampling methods (for the conditions previously described) are presented in Table 3. The results shown there allow us to conclude that for all methods and small mean shifts, the coefficients of variation are very close to 1.

**Table 3**: Values of the coefficient of variation of $TS$ for the FSI, VSI and LSI methods.

| *CV* | | $\lambda$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | *0.00* | *0.50* | *1.00* | *1.50* | *1.75* | *2.00* | *2.50* | *3.00* |
| FSI | $d=1$ | 1.0000 | 1.0000 | 0.9998 | 0.9996 | 0.9989 | 0.9977 | 0.9798 | 0.9633 |
| VSI | *(0.1, 1.9)* | 1.0000 | 0.9968 | 0.8022 | 0.6143 | 0.6211 | 0.6261 | 0.6221 | 0.6112 |
| | *(0.1, 1.5)* | 1.0000 | 0.9981 | 0.8609 | 0.6071 | 0.6031 | 0.6087 | 0.6130 | 0.6133 |
| LSI | | 0.9986 | 0.9818 | 0.7915 | 0.6357 | 0.6715 | 0.6943 | 0.7060 | 0.7067 |

For moderate mean shifts, the coefficient of variation for the LSI method is the smallest, although it is similar to the one of the VSI method when $d_2 = 1.9$. For $\lambda \geq 1.5$, the LSI method has a slightly larger coefficient of variation than the VSI method for all sampling pairs (due to greater dispersion in the sampling intervals underlying the Laplace distribution), but a smaller coefficient of variation than the FSI method.

## 4. SENSITIVITY ANALYSIS

To evaluate the consistency of the LSI method, a sensitivity analysis was performed. In this section the lower sampling interval is truncated, as it results in a situation similar to the VSI method. On the other hand, the concern in practical applications in certain industrial contexts in which one may be physically or administratively unable to take and analyse samples at very short time intervals justifies this type of study.

$D$ is the random variable that represents the time interval between consecutive inspections, and $d_1$ is the smallest sampling interval possible. Hence, we have:

$$(4.1)\qquad D \leq d_1 \iff \frac{k}{2}.e^{-|u|} \leq d_1 \iff u \geq -\ln\left(\frac{2\times d_1}{k}\right) \vee u \leq \ln\left(\frac{2\times d_1}{k}\right),$$

where $L^* = -\ln\left(\frac{2\times d_1}{k}\right)$ is a multiple of the standard deviation that can be interpreted as $W$ in the VSI method. Let us consider $D^*$ as the time interval between consecutive samples when the sample mean is between $\mu_0 \pm L^* \sigma_0\, n^{-0{,}5}$. The distribution of $D^*$ is the conditional distribution of the mean, given that $D^*$ falls between the control limits for the given mean shifts. The probability density function of $D^*$ is given by

$$f^{**}(\bar{x}) \;=\; \frac{\sqrt{n}}{\beta^*\, \sigma_0 \sqrt{2\pi}}\; e^{-\frac{n(\bar{x}-\mu_0-\lambda\sigma_0)^2}{2\sigma_0^2}}\;, \tag{4.2}$$

with

$$\beta^* \;=\; \Phi\big(L^* - \lambda\sqrt{n}\big) - \Phi\big(-L^* - \lambda\sqrt{n}\big)\,. \tag{4.3}$$

Through reasoning similar to that which has been applied in the statistical properties of the LSI method, we have

$$E\big(D^*|L^*,\lambda,n\big) \;=\; \frac{\sqrt{e}\,k^*}{2\beta^*}\Big[e^{\lambda\sqrt{n}}\times A(L^*,\lambda,n) + e^{-\lambda\sqrt{n}}\times B(L^*,\lambda,n)\Big]\,, \tag{4.4}$$

where

$$\begin{aligned} A(L^*,\lambda,n) \;&=\; \Phi\big(-1-\lambda\sqrt{n}\big) - \Phi\big(-L^*-1-\lambda\sqrt{n}\big)\,,\\ B(L^*,\lambda,n) \;&=\; \Phi\big(L^*+1-\lambda\sqrt{n}\big) - \Phi\big(1-\lambda\sqrt{n}\big)\,, \end{aligned} \tag{4.5}$$

and $k^*$ depends on the value of $L^*$. Thus, the probability of using the sampling interval $d_1$ is given by

$$\begin{aligned} p_1 \;&=\; P\big(D\!=\!d_1|\lambda\big)\\ &=\; 1 - P\bigg(\mu_0 - L^*\frac{\sigma_0}{\sqrt{n}} \le \bar{X} \le \mu_0 + L^*\frac{\sigma_0}{\sqrt{n}}\,\bigg|\; LCL \le \bar{X} \le UCL\bigg)\\ &=\; 1 - \frac{\beta^*}{\beta}\,. \end{aligned} \tag{4.6}$$

Based on the assumptions stated, the average sampling interval is given by

$$E(D) \;=\; \frac{\beta^*}{\beta}\times E(D^*) + d_1\times\bigg(1-\frac{\beta^*}{\beta}\bigg)\,. \tag{4.7}$$

Considering (4.7), "*3-sigma*" control limits and a unit average sampling interval, in control, the values of $k^*$ and $L^*$, obtained by simulation are presented in Table 4 for the considered values of $d_1$.

**Table 4**: Values of $k^*$ and $L^*$ obtained by simulation for different values of $d_1$.

| $d_1$ | $k^*$ | $L^*$ |
|---|---|---|
| *0.1* | 3.8134 | 2.9480 |
| *0.2* | 3.8099 | 2.2539 |
| *0.3* | 3.7942 | 1.8443 |
| *0.4* | 3.7591 | 1.5473 |
| *0.5* | 3.6976 | 1.3077 |

Examining the results in Table 4, we conclude that the value of $k^*$ gets smaller as the value of $d_1$ increases, reducing the multiples of the standard deviation. This feature shows how the LSI method can be equated to the VSI method because when we increase the smaller sampling interval in the VSI method, the $W$ value decreases.

To assess the impact of truncation of the lower sampling interval in terms of the $AATS$ values, we rewrite expression (2.14), adapted to the new conditions, as

$$AATS = E(G) + (ARL - 1)\, E(D) = E(G) + \frac{E(D^*)\, \beta^* + d_1 (\beta - \beta^*)}{1-\beta}\,, \tag{4.8}$$

where $E(G)$ value is obtained by simulation and is used in comparisons between the LSI method and the remaining methods. Intuitively, an increase in the value of $d_1$ leads to an increase in its probability of use. To prove that this intuition is correct, we perform a sensitivity study of the LSI method. We compare the $AATS$ values obtained using the LSI method in its original form with those obtained using the LSI method with truncation of the lower sampling interval.

The results are presented in Figure 2, using a measure of relative variation (sensitive to the lower sampling interval change) and the values of $k^*$ and $L^*$, expressed in terms of % of the $AATS$ value (being $AATS_{\text{LSI}}$ the reference)

$$Q_{\text{LSI}^*/\text{LSI}} = \frac{AATS_{\text{LSI}^*} - AATS_{\text{LSI}}}{AATS_{\text{LSI}}} \times 100\%\,. \tag{4.9}$$

Analysing this figure, one concludes that the differences in the $AATS$ values increase as the probability of detecting mean shifts increases, reaching its maximum for shifts of magnitudes of $\lambda = 1.25$. From this point onward, the effectiveness of the methods tends to converge, becoming identical for large magnitudes of mean shifts. However, for $d_1 = 0.4$ and $d_1 = 0.5$, there are strong increases in the $AATS$ for some mean shifts. Even when $d_1$ is three times greater than the initial value, the maximum relative reduction in the $AATS$ using the non-truncated LSI method is only 12.7%.

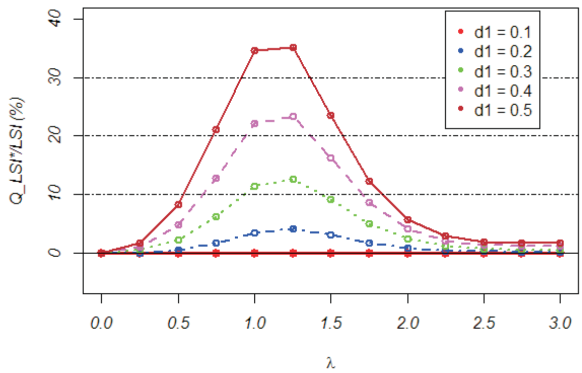


**Figure 2**: $Q_{\mathrm{LSI}^*/\mathrm{LSI}}(\%)$, as a function of $\lambda$ for different values of $d_1$, with $n = 5$.

### Comparison between the LSI* and FSI* sampling methods

For the conditions mentioned in the previous section, the $AATS$ values for the truncated LSI method and the FSI method for a sample size of five were compared. The assessment measures the effect that the change in the lower sampling interval can have on the performance of the LSI method, compared to what occurs in the FSI method. Thus, one considers the measure of performance presented in (3.2), taking into consideration the new values of the $AATS$ in LSI ($AATS_{\mathrm{LSI}^*}$). From the results of $Q_{\mathrm{LSI}^*/\mathrm{FSI}}$, only for mean shifts, we can conclude the following: when the lower sampling interval is truncated, the LSI* method is more effective than the FSI* method for the same mean shifts; the increase in the sampling interval is not proportional to the reduction in effectiveness of the method; the FSI* method detects large mean shifts more quickly than does the LSI* method, maintaining the effectiveness presented previously.

### Comparison between the LSI* and VSI* sampling methods

Using a similar methodology, the LSI* and VSI* methods were compared for the same and for new conditions. Considering a sample size of 5 units and the same number of false alarms, we truncate the lower sampling interval in both methods to the same values. To compare the effectiveness of the two methods, the performance measure defined in (3.9) is used, replacing $AATS_{\mathrm{LSI}}$ with $AATS_{\mathrm{LSI}^*}$ and $AATS_{\mathrm{VSI}}$ with $AATS_{\mathrm{VSI}^*}$. From the results presented in Table 5, for mean shifts, we can draw the following conclusions:

- **i**) In general, the performance of the LSI* method improves when the lower sampling interval gets larger for small shifts. In particular, when $\lambda = 1$ and $d_1 \geq 0.3$, LSI* is more effective than VSI*.
- **ii**) For moderate to large shifts, the performance of LSI* is better than VSI*, except when the lower sampling interval increases.

**iii)** When we use the VSI* method, the increases obtained are significantly greater than the reductions for the different sampling pairs.

**Table 5**: $Q_{\mathrm{LSI}^*/\mathrm{VSI}^*}$ (%), as a function of $\lambda$, for different values of $d_1$ equal to the smaller sampling interval in VSI.

| $d_1$ | $(d_1, d_2)$ | $\lambda$ | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | *0.00* | *0.25* | *0.50* | *0.75* | *1.00* | *1.25* | *1.50* | *1.75* | *2.00* | *2.25* | *2.50* | *3.00* |
| *0.1* | *(0.1, 1.9)* | 0.1 | −3.7 | −14.6 | −25.7 | −15.4 | 9.3 | 23.9 | 29.6 | 31.5 | 32.1 | 32.2 | 32.3 |
| | *(0.1, 1.5)* | 0.0 | −2.3 | −9.5 | −19.4 | −19.4 | −4.9 | 7.3 | 12.8 | 14.7 | 15.3 | 15.4 | 15.5 |
| *0.2* | *(0.2, 1.9)* | 0.1 | −2.7 | −9.7 | −14.5 | −4.9 | 12.0 | 22.3 | 26.5 | 28.0 | 28.4 | 28.6 | 28.6 |
| | *(0.2, 1.5)* | 0.0 | −1.5 | −5.9 | −10.5 | −8.3 | 0.7 | 7.6 | 10.8 | 11.8 | 12.2 | 12.3 | 12.3 |
| *0.3* | *(0.3, 1.9)* | 0.1 | −1.8 | −6.3 | −8.0 | −0.6 | 11.0 | 18.6 | 22.2 | 23.6 | 24.1 | 24.3 | 24.3 |
| | *(0.3, 1.5)* | 0.0 | −1.0 | −3.5 | −5.6 | −3.6 | 1.6 | 5.6 | 7.5 | 8.3 | 8.6 | 8.6 | 8.6 |
| *0.4* | *(0.4, 1.9)* | 0.0 | −1.3 | −4.0 | −4.4 | 0.8 | 8.5 | 14.2 | 17.3 | 18.7 | 19.3 | 19.4 | 19.5 |
| | *(0.4, 1.5)* | 0.0 | −0.6 | −2.1 | −3.0 | −1.8 | 0.8 | 2.8 | 3.9 | 4.4 | 4.6 | 4.6 | 4.6 |
| *0.5* | *(0.5, 1.9)* | 0.0 | −0.8 | −2.5 | −2.5 | 0.9 | 5.7 | 9.7 | 12.1 | 13.3 | 13.9 | 14.0 | 14.1 |
| | *(0.5, 1.5)* | 0.0 | −0.4 | −1.2 | −1.8 | −1.4 | −0.6 | 0.0 | 0.2 | 0.3 | 0.3 | 0.3 | 0.3 |

For the purpose of illustration, consider the case of the lower sampling interval for the LSI* method being truncated to $d_1 = 0.2$. For the example given at the end of section 3.2 and for a mean shift of magnitude $\lambda = 1.5$, we conclude that the use of the LSI* method allows us to reduce the malfunction period by 18 minutes, on average, compared to the FSI* method, and by either 13 minutes or 4 minutes, compared to the VSI* method, depending on whether we use the sampling pair $(d_1, d_2) = (0.2, 1.9)$ or the sampling pair $(d_1, d_2) = (0.2, 1.5)$. These results demonstrate the good performance and sensitivity of the LSI* method.

Results concerning the robustness of the method have been obtained by Carmo *et al.* (2013) for a case in which the distribution of quality has a *t-Student* distribution and another case in which the distribution of quality is a mixture of two normal distributions with different standard deviations. In both cases the performance of the LSI method is better than the performance of FSI and VSI methods, and there are situations in which the LSI method detects mean shifts more quickly than the VSI method.

## 5. CONCLUSIONS

The LSI method detects small and moderate mean shifts in quality more quickly than the FSI method. For large mean shifts, FSI is more efficient. However, the gains achieved with the use of LSI are greater. In a production system in which the sampling costs are very significant (for example, in the production of a touchscreen display for the iPhone 5; 44 U.S. dollars per unit), and in which the quality changes are small or moderate, the use of LSI offers a competitive advantage in reducing sampling and malfunction costs.

When we use the sample pair $(d_1, d_2) = (0.1, 1.9)$ in VSI, the adaptive methods are subject to the same conditions. In other words, the smallest and largest sampling intervals in the LSI method are approximately the same. In LSI, the smallest interval is 0.095 and the largest interval is 1.907. For the sampling pairs considered, LSI detects moderate and large mean shifts more quickly.

We consider the LSI method to be not very sensitive because it has a similar performance to that of the non-truncated method for several mean shifts, particularly when the smallest sample interval is smaller than three times the original smallest interval.

For the reasons explained, and for simplicity, the use of the LSI method can offer a competitive advantage in automating tasks and using nano-scale measurement instruments.

Future research will involve a different approach to the calculation of $E(G)$, using different distributions for the lifetime of the system and assessing its impact. We will extend the study of the statistical properties and performance to the use of joint control charts ($\bar{X}$-chart and $S$-chart or $\bar{X}$-chart and $R$-chart) and special control charts (CUSUM and EWMA charts) to compare the LSI method with other adaptive methods (for example, VSS, VSSI, and VP).

Finally, it is our intention to conduct a study to determine the $k$ value that minimizes a cost function by production cycle.

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