COUNTED DATA CONTROL CHARTS SENSITIVE TO SUDDEN INCREASES OF THE SUMMARY STATISTIC

CARTAS DE CONTROLO PARA O NÚMERO ESPERADO DE DEFEITOS NUMA AMOSTRA SENSÍVEIS A AUMENTOS BRUSCOS DA ESTATÍSTICA

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ABSTRACT:

- The usual practice in maintaining an upper one-sided control chart is to give an out-of-control signal whenever the summary statistic exceeds the upper control limit. However, intuition would suggest that a sudden increase in the summary statistic should also be taken as an indication of a deterioration in the production process, even if that summary statistic is not above the upper control limit.

  Using previously established results it is shown that: a) the decision rule proposed here speeds up the detection of increases in the expected count of defects per sample of fixed size s by an upper one-sided *Poisson CUSUM* chart; and b) the associated run length has interesting stochastic monotonicity properties.

KEY-WORDS:

- *C-charts, CUSUM charts, Markovian approach, stochastically monotone matrices, stochastic ordering, run length, first passage times.*

RESUMO:

- É prática usual emitir sinal de alarme assim que o valor observado da estatística sumária de uma carta unilateral superior exceda o limite superior de controlo. No entanto, parece intuitivamente razoável que um aumento brusco da estatística sumária seja encarado como indicação de uma deterioração do processo de produção, mesmo que a estatística sumária não tenha excedido o limite superior de controlo.
  Usando resultados previamente estabelecidos demonstra-se que: a) a regra de decisão aqui proposta aumenta a velocidade de detecção de aumentos do número esperado de defeitos em amostras de dimensão fixa s, por parte de uma carta *CUSUM Poisson* unilateral superior; e b) o "run length" associado possui relevantes propriedades de monotonia estocástica.

PALAVRAS-CHAVE:

- *Cartas CUSUM, abordagem markoviana, matrizes estocasticamente monótonas, ordenação estocástica, "run length", tempos de primeira passagem.*

1. INTRODUCTION

The quality characteristic behind the widely used $c-chart$ (DeVor *et al.*, 1992, p.452) is the count of defects per random sample of constant size s, represented from now on by Y. Since it is usually assumed that the defects occur independently and the maximum possible number of defects in an item is much larger than the expected number of defects per item produced (Collani, 1989), Y is frequently assumed to have a distribution belonging to the uniparametric model $\{Poisson(l), l > 0\}$.

An increase in the expected number of defects per random sample of constant size s, l, implies a deterioration in the quality of the items produced. Thus, any increase in l should be detected as soon as possible.

Upper one-sided *CUSUM* (cumulative sum) control charts tend to give earlier indication of increases in process parameters than the classical *Shewhart* control schemes. The upper one-sided *Poisson CUSUM* chart is planned to detect increases from the nominal value l_0 (in-control situation) to a larger expected value of the defects count $l_0 + q,\ q > 0$ (out-of-control situation), by using the summary statistic:

$$X_n = k \times I_{\{0\}}(n) + \max\{0, X_{n-1} + Y_n - g\} \times I_{\{1,2,\ldots\}}(n) \qquad (1)$$

where: Y_n is the defects count for the nth sample, and also the summary statistic of the upper one-sided $c-$chart (Morais and Natário (1998)); the initial value given to the summary statistic is denoted here by k; g is a positive real number usually called reference-value. Suitable values for these parameters are usually chosen by the user to produce a desired performance for the control chart, for both in-control and out-of-control situations. A relevant reference for this control chart is Lucas (1985).

An out-of-control signal is given at time n if x_n, the observed value of the *CUSUM* statistic, exceeds the upper control limit $UCL = x$. Such a signal indicates that a corrective action should be taken to prevent the production of nonconforming items. But it seems reasonable that this standard decision rule should also depend on the increase of the summary statistic: for example, a large increment in the summary statistic, say exceeding y $(0 \leq y \leq x)$, should also be taken as a sign of an increase in the parameter being controlled, even if the summary statistic does not exceed $UCL = x$. This combined procedure yields to a non standard decision rule, that will be called henceforth the "increment decision rule". One believes that this decision rule has not been proposed in the literature before.

Table 1 and Figure 1 illustrate the use of an upper one-sided *CUSUM* for *Poisson* counted data with the standard decision rule and with the increment decision rule. The chart has parameters: $s = 4$ items per sample; nominal value $l_0 = 4$; reference value $g = 5$; upper control limit equal to $UCL = x = 10$; critical level for the summary statistic increment $y = 4$; and $k = 0$ as initial value of the summary statistic, that is, no head start has been given to the chart.

Table 1 — Observed values of the summary statistics and their increments
($s = 4$; $l_0 = 4$; $g = 5$; $x = 10$; $y = 4$; $k = 0$)

n	Y_{1n}	Y_{2n}	Y_{3n}	Y_{4n}	$Y_n = \sum_{i=1}^{4} Y_{in}$	X_n	$X_n - X_{n-1}$
0	—	—	—	—	—	$k = 0$	—
1	1	0	0	1	2	0	0
2	0	1	1	1	3	0	0
3	2	0	0	0	2	0	0
4	1	2	1	0	4	0	0
5	0	1	0	0	1	0	0
6	2	6	3	1	12	7	7**
7	3	6	1	2	12	14*	7
8	2	3	3	6	14	23	9
9	3	2	3	4	12	30	7
10	5	2	4	3	14	39	9

* the *CUSUM* summary statistic is above the upper control limit
** the increment of *CUSUM* summary statistic exceeds the critical value $y = 4$

Each row of columns 2-5 of Table 1 have the observed defects count in 4 items. The first 5 groups of four observations are from a process at the desired mean level $l = l_0 = 4$. The next 5 groups of observations were taken from a process out-of-control with $l = l_0 + q = 6$.

Figure 1 — Control charting using the *CUSUM* summary statistic X_n and its increment $X_n - X_{n-1}$

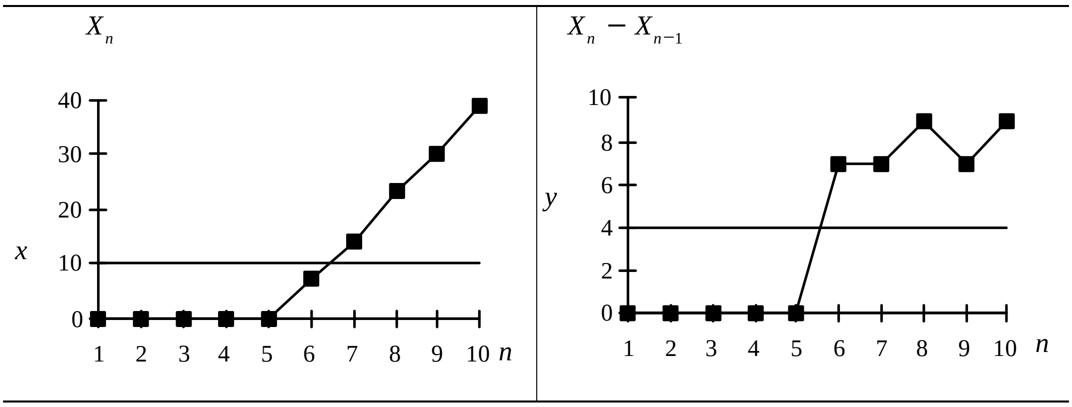

Table 1 and Figure 1 illustrate the benefit of the adoption of the increment decision rule: the upper one-sided *Poisson CUSUM* chart with the standard decision rule gives an out-of-control signal at the second observation with $l = l_0 + q = 6$ (observation 7), whereas if one adopts the increment decision rule the same chart gives an out-control at the first observation with l at the abnormal level (observation 6).

Let $RL_k(x;q)$ and $RL_k(x,y;q)$ be the run lengths of the upper one-sided *Poisson CUSUM* chart with the standard and the increment decision rules, respectively. These random variables are the number of samples taken before the chart produces an out-of-control signal, and give an indication of the ability to detect shifts in l when each of the decision rules is used.

The Section 2 of this paper consists of some preliminaries, including the survival function and moments of both performance measures $RL_k(x;q)$ and $RL_k(x,y;q)$. In Section 3, some stochastic monotonicity properties of these random variables are established and illustrated. In Section 4, one investigates closely the stochastic consequences of the substitution of the standard decision rule by the increment decision rule in the performance of the upper one-sided *Poisson CUSUM* chart, as well as the corresponding effect of changing the critical value y. Finally, in Section 5 some concluding remarks are presented; in particular, some considerations on how the increment decision rule can be implemented with other control charts are made.

2. SURVIVAL FUNCTION AND MOMENTS OF THE RUN LENGTH

The Markov chain approach provides a simple way of examining detailed properties of the control chart proposed here, such as

- the exact distribution function (or survival function) and the percentage points of its run length and
- the average detection speed of the control chart,

as the parameter l changes. The simplicity of this approach arises from the fact that the evaluation of all these properties involves no more than the use of trivial operations such as matrix multiplication (to obtain the survival function) and inversion (to evaluate the average run length and other factorial moments).

The evolution of the summary statistic of the upper one-sided *CUSUM* chart for *Poisson* counted data forms a discrete time Markov chain $X(q)=\{X_n(q), n\geq 0\}$. If g is a positive integer, the Markov chain has infinite state space $\mathrm{N}_0=\{0,1,2,...\}$ and transition matrix $\mathbf{P}(q)=\lfloor p_{ij}(q)\rfloor_{i,j\in \mathrm{N}_0}$ which depends on the parameter $q\geq 0$. Following Morais and Pacheco (1998b), one has, for this particular chart,

$$p_{ij}(q)=a_{ij}(q)-a_{i\,j-1}(q),\; i,j\in \mathrm{N}_0, \tag{2}$$

with $a_{i\,-1}(q)=0,\; i\in \mathrm{N}_0$ and

$$a_{ij}(q)=\sum_{l=0}^{j} p_{il}(q)=P[X_{n+1}(q)\leq j|X_n(q)=i]$$

$$=F_{Poisson\,(l_0+q)}(j-i+g),\; i,j,n\in \mathrm{N}_0. \tag{3}$$

One can also add that the run lengths $RL_k(x;q)$ and $RL_k(x,y;q)$ correspond to the first passage times

$$\min\{n\geq 0:\; X_n(q)>x|X_0(q)=k\} \tag{4}$$

$$\min\{n\geq 0:\; X_n(q)>x \;\text{ or }\; X_n(q)-X_{n-1}(q)>y|X_0(q)=k\} \tag{5}$$

(respectively) which have survival functions given by

$$\overline{F}_{RL_k(x;q)}(m) = I_{(-\infty,1)}(m) + \underline{\mathbf{e}}'_{k+1} \times \left[\tilde{\mathbf{Q}}(q)\right]^{[m]} \times \underline{\mathbf{1}} \times I_{[1,+\infty)}(m) \tag{6}$$

$$\overline{F}_{RL_k(x,y;q)}(m) = I_{(-\infty,1)}(m) + \underline{\mathbf{e}}'_{k+1} \times \left[\overline{\mathbf{Q}}(q)\right]^{[m]} \times \underline{\mathbf{1}} \times I_{[1,+\infty)}(m) \tag{7}$$

where: I_A is the characteristic function of the real set A; $\underline{\mathbf{e}}_{k+1}$ denotes the $(k+1)$th vector of the orthonormal basis for R^{x+1}; $\underline{\mathbf{1}}$ is a column vector of $(x+1)$ ones; $[m]$ is the integer part of m; $\tilde{\mathbf{Q}}(q) = \left[p_{ij}(q)\right]_{i,j=0}^{x}$; and $\overline{\mathbf{Q}}(q)$ has entries

$$\overline{q}_{ij}(q) = \begin{cases} p_{ij}(q), i \in \{0,1,\ldots,x\}, \, j \in \{0,1,\ldots,\min\{i+y,x\}\} \\ 0, \text{otherwise}, \end{cases} \tag{8}$$

that is, $\overline{\mathbf{Q}}(q)$ is equal to

$$\begin{bmatrix} p_{00}(q) & p_{01}(q) & \cdots & p_{0\,y}(q) & 0 & \cdots & \cdots & 0 \\ p_{10}(q) & p_{11}(q) & \cdots & \cdots & p_{1\,y+1}(q) & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ p_{x-y-1\,0}(q) & p_{x-y-1\,1}(q) & \cdots & \cdots & \cdots & \cdots & p_{x-y-1\;x-1}(q) & 0 \\ p_{x-y\,0}(q) & p_{x-y\,1}(q) & \cdots & \cdots & \cdots & \cdots & \cdots & p_{x-y\;x}(q) \\ p_{x-y+1\,0}(q) & p_{x-y+1\,1}(q) & \cdots & \cdots & \cdots & \cdots & \cdots & p_{x-y+1\;x}(q) \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots \\ p_{x\,0}(q) & p_{x\,1}(q) & \cdots & \cdots & \cdots & \cdots & \cdots & p_{x\,x}(q) \end{bmatrix}. \tag{9}$$

It is interesting to notice that the survival functions in (6) and (7) are equal if $y = x$, i.e., $RL_k(x,x;q) =_{st} RL_k(x;q)$.

Recall that the survival function of the run length evaluated at the point m $(m \in \{1,2,\ldots\})$ essentially represents the probability of the following events:

- taking more than m samples before a false alarm is observed, when $q = 0$;
- the detection of an increase in l taking more than m samples to collect, when $q > 0$.

As for the reciprocal of the average detection speed of both charts, the average run length (ARL), one has

$$ARL_k(x;q) = \underline{\mathbf{e}}'_{k+1} \times \left[\mathbf{I} - \tilde{\mathbf{Q}}(q)\right]^{-1} \times \underline{\mathbf{1}} \tag{10}$$

$$ARL_k(x,y;q) = \underline{\mathbf{e}}'_{k+1} \times \left[\mathbf{I} - \overline{\mathbf{Q}}(q)\right]^{-1} \times \underline{\mathbf{1}}, \tag{11}$$

where $\mathbf{I}$ is the identity matrix with rank $(x+1)$. And, following Brook and Evans (1972), the vectors of s^{th} factorial moments of $RL_k(x;q)$ and $RL_k(x,y;q)$ can easily be obtained recursively using the formulas

$$\begin{aligned}\underline{m}^{(s)}(x;q) &= \left[E\{RL_k(x;q)\times[RL_k(x;q)-1]\times\cdots\times[RL_k(x;q)-s+1]\}\right]_{k=0}^{x}\\ &= s\times\left\{\left[\mathbf{I}-\tilde{\mathbf{Q}}(q)\right]^{-1}-\mathbf{I}\right\}\times\underline{m}^{(s-1)}(x;q),\ s=2,3,\ldots\end{aligned} \tag{12}$$

$$\begin{aligned}\underline{m}^{(s)}(x,y;q) &= \left[E\{RL_k(x,y;q)\times[RL_k(x,y;q)-1]\times\cdots\times[RL_k(x,y;q)-s+1]\}\right]_{k=0}^{x}\\ &= s\times\left\{\left[\mathbf{I}-\overline{\mathbf{Q}}(q)\right]^{-1}-\mathbf{I}\right\}\times\underline{m}^{(s-1)}(x,y;q),\ s=2,3,\ldots,\end{aligned} \tag{13}$$

where:

$$\underline{m}^{(1)}(x;q) = \left[ARL_k(x;q)\right]_{k=0}^{x} = \left[\mathbf{I}-\tilde{\mathbf{Q}}(q)\right]^{-1}\times\underline{\mathbf{1}}; \tag{14}$$

$$\underline{m}^{(1)}(x,y;q) = \left[ARL_k(x,y;q)\right]_{k=0}^{x} = \left[\mathbf{I}-\overline{\mathbf{Q}}(q)\right]^{-1}\times\underline{\mathbf{1}}. \tag{15}$$

3. STOCHASTIC MONOTONICITY PROPERTIES

The Markov approach also provides a form of establishing stochastic monotonicity properties concerning the run lengths associated to both decision rules.[1] But, first one must investigate if the probability transition matrices of two auxiliary Markov chains, related to the original one $X(q)$, have some special features.

The probability transition matrix

$$\tilde{\mathbf{P}}(q) = \left[\tilde{p}_{ij}(q)\right]_{i,j=0}^{x+4} = \begin{bmatrix}\tilde{\mathbf{Q}}(q) & \left[\mathbf{I}-\tilde{\mathbf{Q}}(q)\right]\times\underline{\mathbf{1}}\\ \underline{\mathbf{0}}' & 1\end{bmatrix} \tag{16}$$

(where $\underline{\mathbf{0}}$ is a column vector of $(x+1)$ zeroes) governs the absorbing Markov chain $\tilde{X}(q)=\{\tilde{X}_n(q), n\geq 0\}$ with state space $\{0,1,\ldots,x+1\}$ and has an associated first passage time to the absorbing state $(x+1)$ satisfying

$$\min\left\{n : \tilde{X}_n(q) = x+1 \,\middle|\, \tilde{X}_0(q)=k\right\} =_{st} RL_k(x;q). \tag{17}$$

This matrix is such that

$$\begin{aligned}\tilde{a}_{ij}(q) &= \sum_{l=0}^{j}\tilde{p}_{il}(q) = P\left[\tilde{X}_{n+1}(q)\leq j \,\middle|\, \tilde{X}_n(q)=i\right]\\ &= F_{Poisson(l_0+q)}(j-i+g),\ i,j\in\{0,1,\ldots,x\}\end{aligned} \tag{18}$$

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[1] These stochastic monotonicity properties imply a monotonicity behaviour of the average run lengths because $Z \leq_{st} W \Rightarrow E(Z)\leq E(W)$, provided $E(Z)$ and $E(W)$ exist.

decreases with i, for all $j \in \{0, 1, \ldots, x\}$. Hence,

$$\tilde{a}_{ij}(q) \geq \tilde{a}_{i+1\,j}(q), i \in \{0,1,\ldots,x-1\}, \; j \in \{0,1,\ldots,x\}; \tag{19}$$

that is, $\tilde{\mathbf{P}}(q)$ is stochastically monotone, $\tilde{\mathbf{P}}(q) \in \mathrm{M}$. Recalling that the random variable Z is stochastically smaller than W $(Z \leq_{st} W)$ iff

$$\overline{F}_Z(x) \leq \overline{F}_W(x), \; -\infty < x < +\infty, \tag{20}$$

this special feature of $\tilde{\mathbf{P}}(q)$ essentially means that, if we associate probability functions of discrete random variables to each row of $\tilde{\mathbf{P}}(q)$, the associated random variables increase stochastically as we progress in the rows of $\tilde{\mathbf{P}}(q)$:

$$\left(\tilde{X}_{n+1}(q) \middle| \tilde{X}_n(q) = i\right) \leq_{st} \left(\tilde{X}_{n+1}(q) \middle| \tilde{X}_n(q) = i+1\right), n \in \mathrm{N}_0, i \in \{0,1,\ldots,x\}. \tag{21}$$

Moreover, since the *Poisson* random variable increases stochastically with its expected value — i.e., for $0 < b \leq b' < +\infty$,

$$\overline{F}_{Poisson(b)}(x) \leq \overline{F}_{Poisson(b')}(x), -\infty < x < +\infty \tag{22}$$

—, one concludes that:

$$d\tilde{a}_{ij}(q)/dq \leq 0, i, j \in \{0,1,\ldots,x\}. \tag{23}$$

Similarly, to study the monotonicity properties of the performance measure $RL_k(x, y; q)$ one has to investigate the features of another probability transition matrix related to $\mathbf{P}(q)$:

$$\overline{\mathbf{P}}(q) = \left[\overline{p}_{ij}(q)\right]_{i,j=0}^{x+4} = \begin{bmatrix} \overline{\mathbf{Q}}(q) & \left[\mathbf{I} - \overline{\mathbf{Q}}(q)\right] \times \underline{\mathbf{1}} \\ \underline{\mathbf{0}}' & 1 \end{bmatrix}. \tag{24}$$

This matrix rules the absorbing Markov chain $\overline{X}(q) = \{\overline{X}_n(q), n \geq 0\}$ with state space $\{0,1,\ldots,x+1\}$ and has first passage time to the absorbing state $(x+1)$ satisfying

$$\min\left\{n : \overline{X}_n(q) = x+1 \,\middle|\, \overline{X}_0(q) = k\right\} =_{st} RL_k(x, y; q). \tag{25}$$

And, noting that

$$\begin{aligned} \overline{a}_{ij}(q) &= \sum_{l=0}^{j} \overline{p}_{il}(q) = P\left[\overline{X}_{n+1}(q) \leq j \middle| \overline{X}_n(q) = i\right] = a_{i\,\min\{j, \min\{i+y, x\}\}}(q) \\ &= F_{Poisson(l_0+q)}(\min\{j, \min\{i+y, x\}\} - i + g), i, j \in \{0,1,\ldots,x\} \end{aligned} \tag{26}$$

also decreases with i (see the matrix in expression (9)), it follows:

$$\overline{a}_{ij}(q) \geq \overline{a}_{i+1\;j}(q), i \in \{0,1,\ldots,x-1\}, \; j \in \{0,1,\ldots,x\}, \tag{27}$$

that is, $\overline{\mathbf{P}}(q) \in \mathrm{M}$. Additionally,

$$d\bar{a}_{ij}(q)/dq \leq 0, i, j \in \{0,1,\ldots,x\} \tag{28}$$

for the same reason pointed out to prove (23).

Under these conditions, the relaxed versions of results (15)/(17), and (16)/(18) from Morais and Pacheco (1998b) allow one to assert and illustrate the following stochastic and intuitive monotonicity properties.[2]

Property 1 — $RL_k(x;q)$ and $RL_k(x,y;q)$ stochastically decrease with the initial value given to the summary statistic —

$$RL_{k+1}(x;q) \leq_{st} RL_k(x;q), k \in \{0,1,\ldots,x-1\}, \tag{29}$$

$$RL_{k+1}(x,y;q) \leq_{st} RL_k(x,y;q), k \in \{0,1,\ldots,x-1\} \tag{30}$$

— due to the fact that $\widetilde{\mathbf{P}}(q) \in \mathrm{M}$ (result (29)) and $\overline{\mathbf{P}}(q) \in \mathrm{M}$ (result (30)). In other words, giving a head start to the upper one-sided *Poisson CUSUM* control chart for l leads to a stochastic decrease of both types of run lengths.

False alarms will be more likely to happen as the initial value of the summary statistic grows, as illustrated — for both decision rules with the set of parameters $s = 4$, $l_0 = 2$, $g = 3$, $x = 5$, $y = 3$ — by the graphs of Figure 2 and the 3D plot in Figure 3, and by columns 2-13 of Table 2. As a consequence, the expected time to a false alarm decreases as the initial value of the summary statistic grows, as one can see in the line corresponding to $q = 0$ in Table 3.

For example, a false alarm occurs within the first 5 samples with probability $1 - \overline{F}_{RL_0(5,3;0)}(5) = 0.025$ when the chart with the increment decision rule has no head start, whereas this probability increases to $1 - \overline{F}_{RL_2(5,3;0)}(5) = 0.035$ if a head start of $k = 2$ has been given to the chart.

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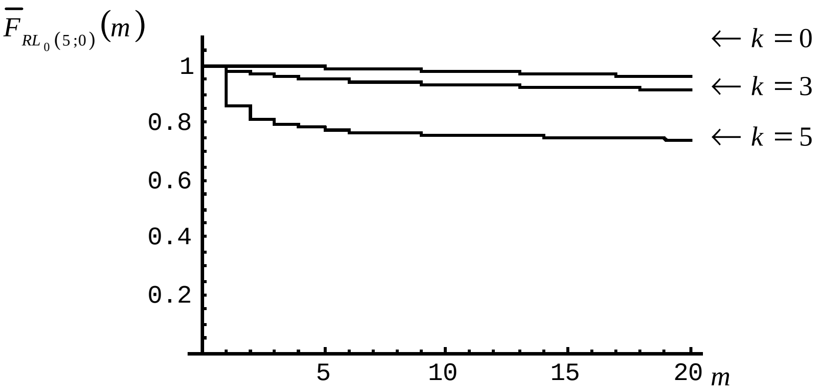
[2] Results (15) and (17) ((16) and (18)) from Morais and Pacheco (1998b) hold when $\mathbf{P}(q) \in \mathrm{M}$ ($\mathbf{P}(q)$ is a stochastically monotone convex matrix — see Li and Shaked (1997) for the definition of this type of matrix). They also hold if $\widetilde{\mathbf{P}}(q) \in \mathrm{M}$ $\left(\overline{\mathbf{P}}(q) \in \mathrm{M}\right)$, thus, relaxing the sufficient conditions.

Table 2 — Values of RL survival functions for both decision rules for different *abcissae* and some head starts

m	$\overline{F}_{RL_k(5;q)}(m)$						$\overline{F}_{RL_k(5,3;q)}(m)$					
	$k=0$			$k=2$			$k=0$			$k=2$		
	q			q			q			q		
	0.0	0.1	0.5	0.0	0.1	0.5	0.0	0.1	0.5	0.0	0.1	0.5
1	1.000	1.000	0.999	0.995	0.994	0.986	0.995	0.994	0.986	0.995	0.994	0.986
2	0.999	0.998	0.993	0.989	0.986	0.962	0.991	0.988	0.971	0.986	0.982	0.955
3	0.997	0.996	0.984	0.984	0.978	0.939	0.986	0.982	0.954	0.978	0.971	0.926
4	0.995	0.993	0.972	0.980	0.973	0.919	0.981	0.975	0.935	0.971	0.962	0.900
5	0.993	0.989	0.958	0.977	0.968	0.901	0.975	0.967	0.916	0.965	0.953	0.876
10	0.981	0.971	0.884	0.964	0.948	0.825	0.948	0.931	0.817	0.937	0.916	0.775
20	0.958	0.935	0.747	0.941	0.912	0.697	0.895	0.861	0.645	0.885	0.847	0.612
50	0.890	0.834	0.451	0.874	0.814	0.420	0.755	0.682	0.318	0.746	0.671	0.302
100	0.788	0.689	0.194	0.774	0.672	0.181	0.567	0.463	0.098	0.561	0.455	0.093
200	0.617	0.470	0.036	0.606	0.459	0.034	0.321	0.213	0.009	0.317	0.209	0.009
500	0.297	0.149	0.000	0.292	0.146	0.000	0.058	0.021	0.000	0.057	0.020	0.000

Figure 2 — Graphs of the $RL_k(5;0)$ survival function for initial states $k = 0,3,5$

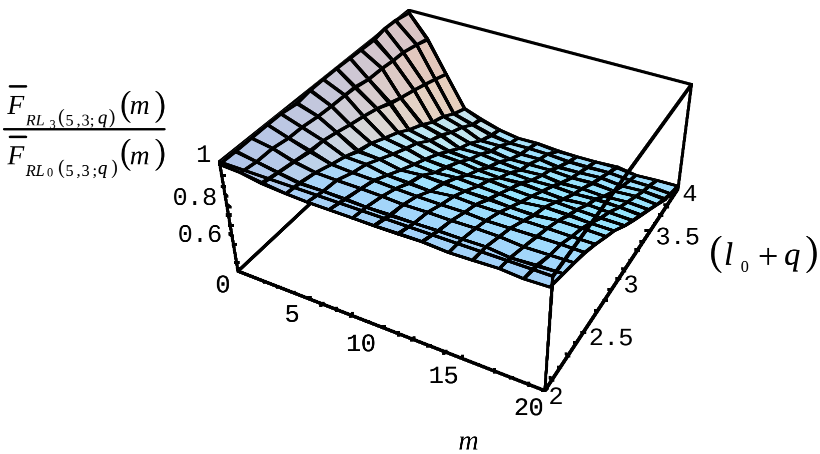


Similarly, the number of samples taken until detection of an increase in l will be stochastically reduced by increasing the initial value of the summary statistic — see, for instance, columns 3-4/6-7 and 9-10/12-13 of Table 2. With respect to the increment decision rule, Figure 3 suggests that larger values of q imply a more distinct behaviour between $RL_0(x,y;q)$ and $RL_g(x,y;q)$.

Table 3 — ARL values for the standard and the increment decision rule and $\left[1 - ARL_k(x,y;q) / ARL_k(x;q)\right] \times 100\,\%$ values, for several head starts and different q

q	$ARL_k(5;q)$			$ARL_k(5,3;q)$			$\left[1 - \frac{ARL_k(5,3;q)}{ARL_k(5;q)}\right] \times 100\,\%$		
	k			k			k		
	0	2	4	0	2	4	0	2	4
0.0	412.5	405.3	368.0	176.5	174.5	159.2	57.220	56.955	56.730
0.1	264.5	258.2	229.4	130.1	128.0	114.5	50.830	50.433	50.100
0.2	175.6	170.1	147.5	97.2	95.1	83.2	44.665	44.099	43.625
0.3	120.6	115.6	97.7	73.7	71.5	61.1	38.926	38.149	37.506
0.4	85.5	81.1	66.7	56.7	54.5	45.4	33.752	32.727	31.890
0.5	62.6	58.5	46.8	44.3	42.2	34.2	29.224	27.917	26.869
1.0	19.5	16.9	11.8	16.4	14.8	10.6	15.639	12.657	10.611
1.5	9.7	7.8	5.1	8.6	7.3	4.9	11.431	6.635	4.177
2.0	6.2	4.8	3.1	5.5	4.6	3.0	10.831	4.089	1.745

Figure 3 — 3D Plot of the ratio $\overline{F}_{RL_g(5,3;q)}(m) / \overline{F}_{RL_0(5,3;q)}(m)$ for $q \in [0,2]$



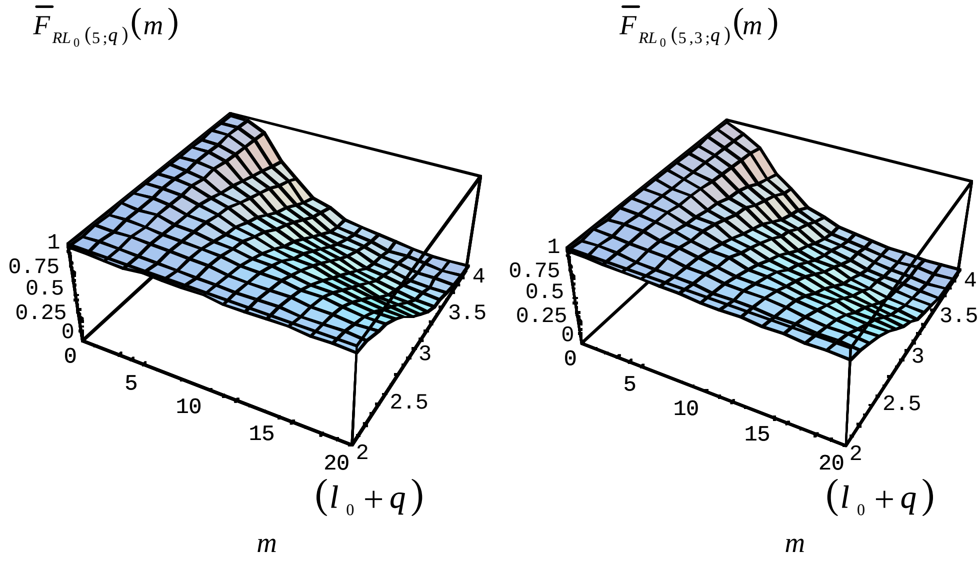
Property 2 — $RL_k(x;q)$ and $RL_k(x,y;q)$ stochastically decrease with the process parameter q, that is, for a fixed initial state $k \in \{0,1,\ldots,x\}$ —

$$RL_k(x;q') \leq_{st} RL_k(x;q), \quad 0 \leq q \leq q' < +\infty, \tag{31}$$

$$RL_k(x,y;q') \leq_{st} RL_k(x,y;q), \quad 0 \leq q \leq q' < +\infty \tag{32}$$

— as a consequence of $\widetilde{\mathbf{P}}(q) \in \mathrm{M}$ and (23) (result (31)), and of $\overline{\mathbf{P}}(q) \in \mathrm{M}$ and (28) (result (32)).

Figure 4 — 3D Plots of $RL_0(5;q)$ and $RL_0(5,3;q)$ survival functions for $q \in [0,2]$



These results mean that, for both decision rules, the control chart stochastically increases its ability to detect an increase in the expected defects count as this increase becomes more severe — as suggested by the two 3D plots of Figure 4, and the values of the several survival functions in Table 2 and of the average run lengths in Table 3.

It is worth mentioning here that results (29) and (31) can be thought as one of the possible extensions mentioned in the last section of Morais and Pacheco (1998a).

4. STOCHASTIC COMPARISON OF DECISION RULES

The stochastic improvement on the performance of the upper one-sided *Poisson CUSUM* chart, derived from the adoption of the increment rule, is discussed in detail in this section.

One starts by establishing a stochastic order relation, in the Kalmikov sense (Kulkarni (1995), pp. 148-149), between the transition probability matrices $\tilde{\mathbf{P}}(q)$ and $\overline{\mathbf{P}}(q)$:

$$\tilde{\mathbf{P}}(q) \leq_K \overline{\mathbf{P}}(q), \tag{33}$$

i.e.,

$$\sum_{l=k}^{x+1} \tilde{p}_{il}(q) \leq \sum_{l=k}^{x+1} \overline{p}_{ml}(q), 0 \leq i \leq m \leq x+1, 0 \leq k \leq x+1, \tag{34}$$

This is due to the facts that

$$\tilde{a}_{ij}(q) \geq \overline{a}_{ij}(q), i, j \in \{0,1,\ldots,x\} \tag{35}$$

(which is rather obvious after checking (9)) and both $\tilde{a}_{ij}(q)$ and $\overline{a}_{ij}(q)$ decrease with i. Result (33) can also be written as

$$\left(\tilde{X}_{n+1}(q)\middle|\tilde{X}_n(q)=i\right)\leq_{st}\left(\overline{X}_{n+1}(q)\middle|\overline{X}_n(q)=m\right), n\in \mathrm{N}_0, 0\leq i\leq m\leq x, \quad (36)$$

which implies, for example, that $\tilde{X}_{n+1}(q)$ tends to be smaller than $\overline{X}_{n+1}(q)$ if the two Markov chains, $\tilde{X}(q)$ and $\overline{X}(q)$, have the same initial state. In fact, using theorem 3.31 of Kulkarni (1995, p.149), one has

$$\tilde{X}(q)\leq_{st}\overline{X}(q) \quad (37)$$

provided that

$$\tilde{X}_0(q)\leq_{st}\overline{X}_0(q) \text{ and } \tilde{\mathbf{P}}(q)\leq_K \overline{\mathbf{P}}(q). \quad (38)$$

Then the two Markov chains referred above can be compared stochastically.

The stochastic relation order (37) is equivalent to

$$P\left[\left(\tilde{X}_{n_1}(q),\ldots,\tilde{X}_{n_k}(q)\right)\in A\right]\leq P\left[\left(\overline{X}_{n_1}(q),\ldots,\overline{X}_{n_k}(q)\right)\in A\right] \quad (39)$$

for all $k\geq 1$, $0\leq n_1\leq\ldots\leq n_k$ and increasing sets $A\subset \mathrm{R}^n$,[3] it is also equivalent to a far more convenient result (Shaked and Shanthikumar (1994), p.124):

$$E\left\{g\left[\tilde{X}(q)\right]\right\}\leq E\left\{g\left[\overline{X}(q)\right]\right\} \quad (40)$$

for every increasing functional g for which the expectations in (40) exist.[4]

Let

$$g(x_0,\ldots,x_m)=\min\left\{1,\sum_{n=0}^m I_{[x+1,+\infty)}(x_n)\right\}, m=0,1,\ldots. \quad (41)$$

Noting that g is an increasing functional and the survival function of $RL_k(x;q)$ and $RL_k(x,y;q)$ can be written in terms of the number of visits to the absorbing state as

$$P[RL_k(x;q)>m]=1-E\left\{g\left[\tilde{X}(q)\right]\right\}=1-E\left(\min\left\{1,\sum_{n=0}^m I_{\{x+1\}}\left[\tilde{X}_n(q)\right]\right\}\right) \quad (42)$$

$$P[RL_k(x,y;q)>m]=1-E\left\{g\left[\overline{X}(q)\right]\right\}=1-E\left(\min\left\{1,\sum_{n=0}^m I_{\{x+1\}}\left[\overline{X}_n(q)\right]\right\}\right), \quad (43)$$

one immediately concludes from (37) and (40) that

$$P[RL_k(x,y;q)>m]\leq P[RL_k(x;q)>m], -\infty<m<+\infty, q\geq 0. \quad (44)$$

---

3 Recall that a set $A\subset\mathrm{R}^n$ is said to be an increasing set if $\underline{x}\in A\Rightarrow \underline{y}\in A$ for all $\underline{x},\underline{y}\in\mathrm{R}^n$ such that $x_i\leq y_i$, $i=1,\ldots,n$. See Kulkarni (1995), p. 588, for the definition of stochastic ordering in a multivariate setting.

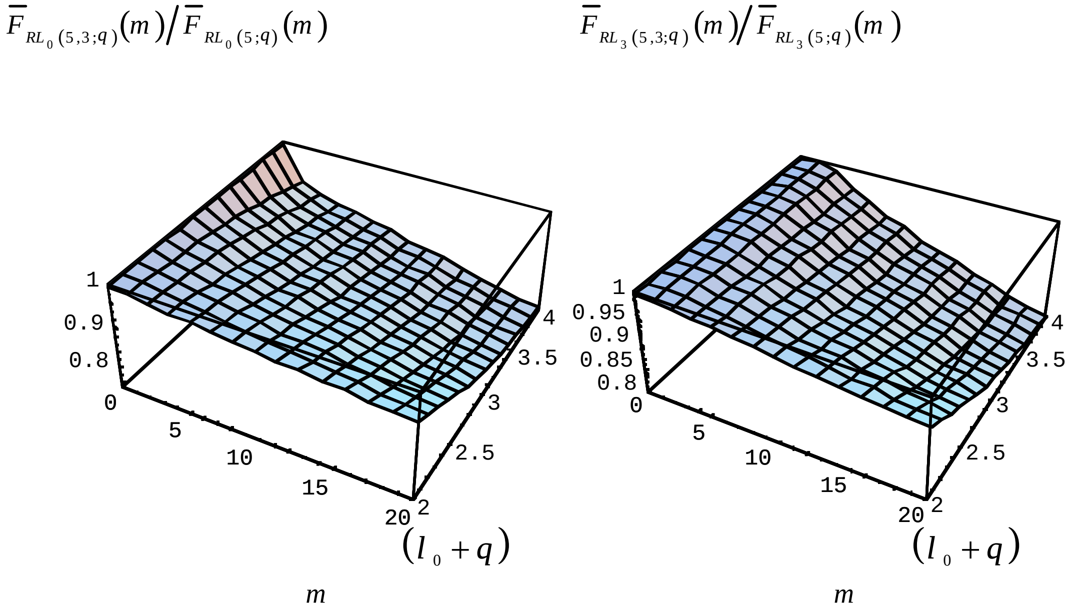
4 A functional g is called increasing if $g(\{x_n, n\geq 0\})\leq g(\{y_n, n\geq 0\})$ whenever $x_n\leq y_n, n\geq 0$ (Shaked and Shanthikumar (1994), p.124).

Property 3 — The stochastic result (44) can be phrased more vividly by saying that the adoption of the increment rule stochastically reduces the run length of the upper one-sided *Poisson CUSUM* control chart, i.e.,

$$RL_k(x,y;q) \leq_{st} RL_k(x;q), k \in \{0,1,\ldots,x\}, q \geq 0 \,. \tag{45}$$

Thus, this chart has a larger ability of discriminating increases in the expected value of the defects count when the increment decision rule is adopted, leading to a smaller average run length. These results are illustrated by columns 2-7 and 8-13 of Table 2 (with $q \neq 0$) and by columns 8-10 of Table 3 (also considering $q \neq 0$) and by the comparison of the two 3D plots from Figure 4.

Figure 5 — 3D Plots of the ratio $\overline{F}_{RL_0(5,3;q)}(m)/\overline{F}_{RL_0(5;q)}(m)$ and $\overline{F}_{RL_3(5,3;q)}(m)/\overline{F}_{RL_3(5;q)}(m)$, for $q \in [0,2]$



For example, the adoption of the increment decision rule yields a benefit of $[1-ARL_0(5,3;0.2)/ARL_0(5;0.2)]\times 100\% = 44.665\%$ in the average run length of the chart. This benefit in the average run length seems to dependent weakly on the initial value of the summary statistic, for small and moderate values of q. But it is clearly dependent on q (for fixed k).

However, the adoption of the increment decision rule has an unpleasant disadvantage: it leads to stochastically smaller run lengths when the production process is in-control, therefore false alarms occur more frequently, as suggested by columns 2, 5, 8 and 11 of Table 2.

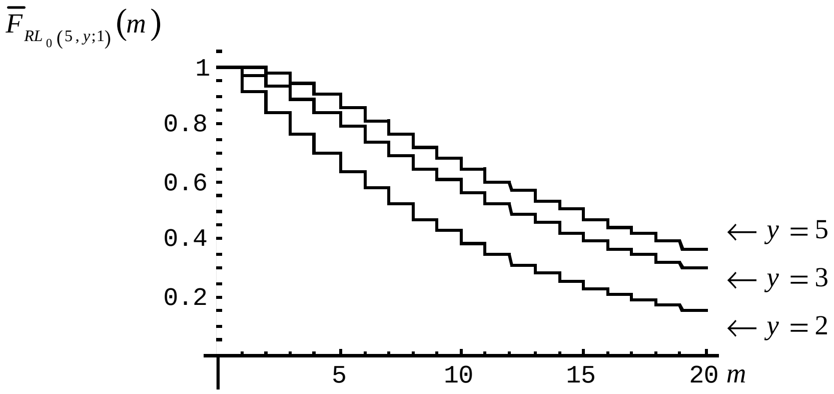
It is also possible to state a stochastic monotonicity result in y: if the critical increment for the summary statistic increases from y to y' $(0 \leq y \leq y' \leq x)$, one is bound to stochastically increase the run length of the control chart. In fact, by inspection of the block matrix (9), one has the following stochastic order relation, in the Kalmikov sense, between the transition probability matrices $\overline{\mathbf{P}}(q)$ and $\overline{\mathbf{P}}'(q)$ (with this latter matrix obviously associated to the critical increment y'):

$$\overline{\mathbf{P}}'(q) \leq_K \overline{\mathbf{P}}(q). \quad (46)$$

Property 4 — By similar arguments to the used before, one concludes that, for $k \in \{0,1,\ldots,x\}$ and $q \geq 0$,

$$RL_k(x,y;q) \leq_{st} RL_k(x,y';q), 0 \leq y \leq y' \leq x. \quad (47)$$

Figure 6 — Graphs of $RL_0(5,y;1)$ survival function for several values of y



Increasing y means being less demanding with the increment decision rule. This property is fully suggested by Figures 6 and 7, by columns 2-7 and 8-13 of Table 4, and by Table 5.

For instance, a false alarm occurs within the first 100 samples with probability $1-\overline{F}_{RL_0(5,3;0)}(100)=0.433$, for the chart with the increment decision rule, no head start and $y=3$. This probability decreases to $1-\overline{F}_{RL_0(5,4;0)}(100)=0.253$ if one adopts $y=4$.

Table 4 — Values of $\overline{F}_{RL_k(5,y;q)}(m)$ for two values of y, several *abcissae* and some values of q

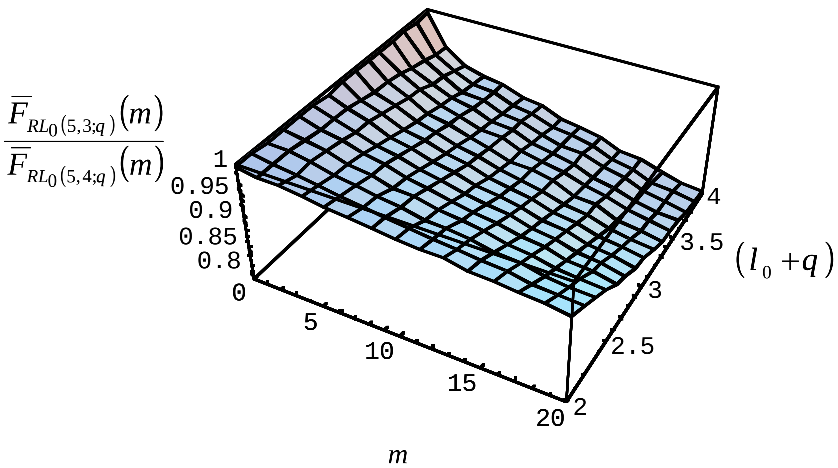
m	$\overline{F}_{RL_k(5,3;q)}(m)$						$\overline{F}_{RL_k(5,4;q)}(m)$					
	$k=0$			$k=3$			$k=0$			$k=3$		
	q			q			q			q		
	0.0	0.2	0.5	0.0	0.2	0.5	0.0	0.2	0.5	0.0	0.2	0.5
1	0.995	0.994	0.986	0.983	0.975	0.958	0.999	0.998	0.996	0.983	0.975	0.957
2	0.991	0.988	0.971	0.967	0.949	0.911	0.997	0.995	0.989	0.969	0.951	0.913
3	0.986	0.982	0.954	0.956	0.929	0.872	0.995	0.991	0.978	0.959	0.934	0.879
4	0.981	0.975	0.935	0.947	0.914	0.841	0.993	0.985	0.965	0.953	0.921	0.852
5	0.975	0.967	0.916	0.940	0.901	0.815	0.990	0.980	0.950	0.948	0.912	0.830
10	0.948	0.931	0.817	0.911	0.851	0.716	0.976	0.949	0.872	0.932	0.878	0.751
20	0.895	0.861	0.645	0.861	0.766	0.565	0.947	0.889	0.729	0.905	0.822	0.627
50	0.755	0.682	0.318	0.725	0.560	0.279	0.867	0.731	0.426	0.828	0.675	0.366
100	0.567	0.463	0.098	0.545	0.332	0.086	0.747	0.527	0.174	0.714	0.487	0.149
200	0.321	0.213	0.009	0.308	0.117	0.008	0.556	0.274	0.029	0.531	0.253	0.025
500	0.058	0.021	0.000	0.056	0.005	0.000	0.229	0.038	0.000	0.218	0.036	0.000

Table 5 — $ARL_k(5,y;q)$ for several head starts, values of y and different q

q	$ARL_k(5,2;q)$			$ARL_k(5,3;q)$			$ARL_k(5,5;q)=ARL_k(5;q)$		
	k			k			k		
	0	1	3	0	1	3	0	1	3
0.0	59.4	59.3	58.3	176.5	175.9	169.7	412.5	410.5	393.3
0.1	47.8	47.7	46.7	130.1	129.5	123.6	264.5	262.7	248.5
0.2	39.0	38.9	37.8	97.2	96.6	91.0	175.6	173.9	162.2
0.3	32.1	32.0	30.8	73.7	73.0	67.8	120.6	119.0	109.1
0.4	26.8	26.6	25.4	56.7	56.0	51.2	85.5	84.1	75.6
0.5	22.5	22.4	21.1	44.3	43.6	39.2	62.6	61.2	54.0
1.0	9.6	9.4	8.3	16.4	15.8	12.9	19.5	18.5	14.7
1.5	6.3	6.2	5.3	8.6	8.1	6.2	9.7	8.9	6.5
2.0	4.3	4.14	3.5	5.5	5.1	3.8	6.2	5.6	4.0

An increase of $q=0.2$ is not detected by none of the 100 samples with probability $\overline{F}_{RL_0(5,3;0.2)}(100)=0.463$, if one has adopted the chart with the increment decision rule, no head start and $y=3$. This probability increases to $\overline{F}_{RL_0(5,4;0.2)}(100)=0.527$ if the critical value of the increment is increased to $y=4$.

Figure 7 — 3D plot of $\overline{F}_{RL_0(5,3;q)}(m)/\overline{F}_{RL_0(5,4;q)}(m)$, for $q\in[0,2]$



4. CONCLUDING REMARKS

A new decision rule was proposed in this paper, the increment decision rule. It was shown that it gives tighter process control than the standard decision rule, in the sense that it stochastically increases the ability to detect increases in the expected value of the defects count by the upper-one sided *Poisson CUSUM* control chart. However, it also increases stochastically the time between two consecutive false alarms. Hence, the choice of the critical value for the increment — i.e., the design of an upper-one sided *Poisson CUSUM* chart with the increment decision rule — should be done having in mind that a compromise between short run lengths when the production is out-of-control and large run lengths while it is in-control should always be made.

The increment decision rule can be implemented with any upper one-sided control chart. If the summary statistic takes non-negative integer values then:

- $RL_k(x,y;q)$ stochastically decreases with the initial value k given to the summary statistic.
- If in addition the partial sums $\bar{a}_{ij}(q)$ of $\bar{\mathbf{P}}(q)$ decrease with q, an increase in q implies a stochastic reduction of $RL_k(x,y;q)$.
- $RL_k(x,y;q)$ stochastically increases with y since an increase in y is followed by a decrease of $\bar{\mathbf{P}}(q)$ in the Kalmikov sense.
- The adoption of the increment decision rule stochastically reduces the run length of the chart.

Further extensions are possible. Take for instance, the operation of an upper one-sided control scheme with a summary statistic that can be regarded as forming a Markov chain with continuous state space. In such cases, the run length is approximated by the first passage time of a Markov chain with N_0 as state space (see Lucas and Saccucci (1990)). Under the conditions mentioned above, the run length of the approximating chain have the four stochastic properties referred here. Moreover, as noted by Morais and Pacheco (1998b), those properties are still valid for the run length associated to the Markov chain with continuous state space.

Analogous results can be derived for lower one-sided control charts with what could be called the decrement decision rule.

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REFERENCES

BROOK, D. and EVANS, D.A. (1972) "An approach to the probability distribution of cusum run length", *Biometrika*, Vol. 59, pp. 539-549.

COLLANI, E. (1989) *The Economic Design of Control Charts*, B.G. Teubner, Stuttgart.

DEVOR, R.E., CHANG, T. and SUTHERLAND, J.W. (1992) *Statistical Quality Design and Control: Contemporary Concepts and Methods*, Macmillan Publishing Company.

KEILSON, J. and KESTER, A. (1977) "Monotone matrices and monotone Markov processes", *Stochastic Processes and their Applications*, Vol. 5, pp. 231-241.

KULKARNI, V.G. (1995) *Modeling and Analysis of Stochastic Systems*, Chapman and Hall.

LI, H. and SHAKED, M. (1997) "Ageing first-passage times of Markov processes: a matrix approach", *Journal of Applied Probability*, Vol. 34, pp. 1-13.

LUCAS, J.M. (1985) "Counted data CUSUM's", *Technometrics*, Vol. 32, pp. 129-144.

LUCAS, J.M. and SACCUCCI, M.S. (1990) "Exponentially Weighted Moving Average control schemes:

properties and enhancements", *Technometrics*, Vol. 32, pp. 1-12.

MONTGOMERY, D.C. (1985) *Introduction to Statistical Quality Control*, John Wiley & Sons.

MORAIS, M.C. and NATÁRIO, I. (1998) "Improving an upper one-sided $c-$chart ", *Communications in Statistics — Theory and Methods*, Vol. 27, pp. 353-364.

MORAIS, M.C. and PACHECO, A. (1998a) "Two stochastic properties of one-sided Exponentially Weighted Moving Average controls charts", *Communications in Statistics — Simulation and Computation*, Vol. 27, pp. 937-952.

MORAIS, M. C. and PACHECO, A. (1998b) "Comparação estocástica de tempos de primeira passagem: uma aplicação a controlo de qualidade, fiabilidade e filas de espera", (in portuguese). Accepted for publication in the *Proceedings of the VI Annual Meeting of the Portuguese Statistical Society* — Tomar, 9-12 June 1998.

SHAKED, M. and SHANTHIKUMAR, J.G. (1994). *Stochastic Orders and Their Applications*, Academic Press.

WOLFRAM, S. (1996). *The Mathematica Book — 3rd edition (Mathematica Version 3.0)*, Wolfram Media, Cambridge University Press.