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A REPARAMETERIZED BIRNBAUM–SAUNDERS DISTRIBUTION AND ITS MOMENTS, ESTIMATION AND APPLICATIONS

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Abstract:

- The Birnbaum–Saunders (BS) distribution is a model that is receiving considerable attention due to its good properties. We provide some results on moments of a reparameterized version of the BS distribution and a generation method of random numbers from this distribution. In addition, we propose estimation and inference for the mentioned parameterization based on maximum likelihood, moment, modified moment and generalized moment methods. By means of a Monte Carlo simulation study, we evaluate the performance of the proposed estimators. We discuss applications of the reparameterized BS distribution from different scientific fields and analyze two real-world data sets to illustrate our results. The simulated and real data are analyzed by using the R software.

Key-Words:

- *data analysis; maximum likelihood and moment estimation; Monte Carlo method; random number generation; statistical software.*

AMS Subject Classification:

- 62F86, 60E05.

1. INTRODUCTION

The Birnbaum–Saunders (BS) distribution is being widely considered. This distribution is unimodal and positively skewed, has positive support and two parameters corresponding to its shape and scale; see Birnbaum & Saunders (1969a), Johnson *et al.* (1995) and Athayde *et al.* (2012). Interest in the BS distribution is due to its physical theoretical arguments, its attractive properties and its relationship with the normal model. Although the BS distribution has its genesis from material fatigue, it has been used for applications in: agriculture, business, contamination, engineering, finance, food, forest and textile industries, informatics, insurance, medicine, microbiology, mortality, nutrition, pharmacology, psychology, quality control, queue theory, toxicology, water quality and wind energy; see Leiva *et al.* (2007, 2008c, 2010a,b, 2011, 2012, 2014a,b,d), Ahmed *et al.* (2008), Barros *et al.* (2008), Balakrishnan *et al.* (2009a,b, 2011), Bhatti (2010), Kotz *et al.* (2010), Vilca *et al.* (2010), Sanhueza *et al.* (2011), Santana *et al.* (2011), Villegas *et al.* (2011), Azevedo *et al.* (2012), Ferreira *et al.* (2012), Paula *et al.* (2012), Fierro *et al.* (2013), Marchant *et al.* (2013a,b) and Saulo *et al.* (2013).

One of the most studied topics in the BS distribution is its estimation and inference. Several types of estimators for its original parameterization have been proposed. Birnbaum & Saunders (1969b) found its maximum likelihood (ML) estimators. Bhattacharyya & Fries (1982) mentioned that the lack of an exponential family structure for the BS distribution complicates the statistical inference of its parameters. Engelhardt *et al.* (1981), Achcar (1993), Chang & Tang (1994) and Dupuis & Mills (1998) proposed other types of estimators of the original parameters. However, in all of these cases, it is not possible to find explicit expressions for its estimators, so that numerical procedures must be used. Ng *et al.* (2003) introduced a modified moment (MM) method for estimating the BS model parameters, which provides simple analytical expressions to compute them. From & Li (2006) presented and summarized several estimation methods for the BS distribution. Results about improved inference for this distribution are attributed to Lemonte *et al.* (2007) and Cysneiros *et al.* (2008). Thus, different estimation aspects related to the BS distribution have been considered by a number of authors. Nevertheless, not much attention has been paid to parameterizations that are different from that originally proposed by Birnbaum & Saunders (1969a), which was based on the physics of materials. Some works on reparameterizations of the BS distribution were proposed by Volodin & Dzhungurova (2000), Ahmed *et al.* (2008), Lio *et al.* (2010) and Santos-Neto *et al.* (2012). The present work is focused on Santos-Neto *et al.* (2012)'s reparameterization.

Our main motivation for studying this reparameterization of the BS distribution is based on the search of estimators with good statistical properties. Such a reparameterization is useful, because, first, moment estimates for the original parameterization of the BS distribution do not have a closed-form, but this is possible with Santos-Neto *et al.* (2012)'s reparameterization and, second, it allows a response variable to be modeled in its original scale (see Leiva *et al.*, 2014c), which is not possible with the parameterizations proposed until now.

The objectives of this paper are:

- **(i)** to provide some results on moments of a reparameterized version of the BS distribution and a generator of random numbers;
- **(ii)** to propose estimators for this reparameterization;
- **(iii)** to study the performance of these estimators;
- **(iv)** to apply the results to real-world data.

The proposed estimators are based on generalized moment (GM), ML, MM and moment methods.

The article is organized as follows. In Section 2, we present some results of the reparameterized version of the BS distribution that include a shape analysis, a generator of random numbers, its characteristic function (CF) and its moments. In Section 3, we develop estimation and inference for this reparameterization based on the GM, ML, MM and moment methods. In Section 4, we evaluate the performance of the proposed estimators through Monte Carlo (MC) simulations. In Section 5, we conduct an application with two real-world data sets, one from engineering and another from economics, which is a new application of the BS distribution. In Sections 4 and 5, computational aspects based on packages in the `R` software are discussed. In Section 6, we sketch some conclusions of this study.

2. BS DISTRIBUTIONS

In this section, we present some results of a reparameterized version of the BS distribution, including a shape analysis, a generator of random numbers and its moments.

2.1. The original parameterization

The first parameterization of the BS distribution was proposed by Birnbaum & Saunders (1969a) based on the physics of materials in terms of shape (α) and scale (β) parameters. Thus, if a random variable (RV) Y follows the BS distribution with parameters $\alpha > 0$ and $\beta > 0$, the notation $Y \sim \mathrm{BS}(\alpha, \beta)$ is used and the corresponding probability density function (PDF) is given by

$$f(y;\alpha,\beta) = \frac{1}{\sqrt{2\pi}}\,\exp\!\left(-\frac{1}{2\alpha^2}\left[\frac{y}{\beta}+\frac{\beta}{y}-2\right]\right)\frac{[y+\beta]}{2\,\alpha\sqrt{\beta y^3}}\,, \qquad y>0\,. \tag{2.1}$$

2.2. A reparameterized version of the BS distribution

Recently, Santos-Neto *et al.* (2012) proposed a reparameterized version of the BS distribution, given, with respect to the original parameterization, by $\alpha = \sqrt{2/\delta}$ and $\beta = \delta\mu/[\delta{+}1]$, such that $\delta = 2/\alpha^2$ and $\mu = \beta[1+\alpha^2/2]$, where $\delta > 0$ and $\mu > 0$ are shape and mean parameters, respectively. For details about motivations and justifications for this reparameterized version, see Santos-Neto *et al.* (2012) and Leiva *et al.* (2014c).

Thus, the PDF of $Y \sim \mathrm{BS}(\mu,\delta)$ is given by

$$\begin{aligned} f(y;\mu,\delta) &= \frac{\exp(\delta/2)\sqrt{\delta{+}1}}{4\sqrt{\pi\mu}\,y^{3/2}}\left[y + \frac{\delta\mu}{\delta{+}1}\right] \\ &\quad\times \exp\!\left(-\frac{\delta}{4}\left[\frac{y\{\delta{+}1\}}{\delta\mu} + \frac{\delta\mu}{y\{\delta{+}1\}}\right]\right), \qquad y > 0\,. \end{aligned} \tag{2.2}$$

From (2.1) and considering the indicated reparameterization, one can note that BS and standard normal RVs are related by

$$\begin{aligned} Y &= \frac{\delta\mu}{\delta{+}1}\left[\frac{Z}{\sqrt{2\delta}} + \sqrt{\left\{\frac{Z}{\sqrt{2\delta}}\right\}^2 + 1}\,\right]^2 \quad \text{and} \\ Z &= \sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{\{\delta{+}1\}Y}{\mu\delta}} - \sqrt{\frac{\mu\delta}{\{\delta{+}1\}Y}}\,\right]. \end{aligned} \tag{2.3}$$

Hence, from (2.3), the cumulative distribution function (CDF) and the quantile function (QF) of $Y \sim \mathrm{BS}(\mu,\delta)$ are, respectively, given by

$$F(y;\mu,\delta) = \Phi\left(\sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{\{\delta{+}1\}y}{\mu\delta}} - \sqrt{\frac{\mu\delta}{\{\delta{+}1\}y}}\,\right]\right), \qquad y > 0\,,$$

and

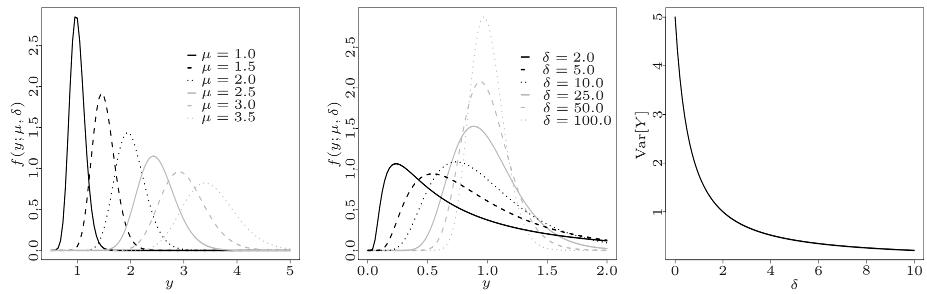
$$y(q;\mu,\delta) = F^{-1}(q) = \frac{\delta\mu}{\delta{+}1}\left[\frac{z(q)}{\sqrt{2\delta}} + \sqrt{\left\{\frac{z(q)}{\sqrt{2\delta}}\right\}^2 + 1}\,\right]^2, \qquad 0 < q < 1\,,$$

where $z(q)$ is the qth quantile of the standard normal distribution and F^{-1} is the inverse CDF of Y. The hazard rate function of Y is defined by

$$\begin{aligned} h(y;\mu,\delta) = \frac{f(y;\mu,\delta)}{1 - F(y;\mu,\delta)} &= \frac{\exp(\delta/2)\sqrt{\delta{+}1}}{4\sqrt{\pi\mu y^3}}\left[y + \frac{\delta\mu}{\delta{+}1}\right] \\ &\quad\times \frac{\exp\left(-\frac{\delta}{4}\left[\frac{y\{\delta+1\}}{\delta\mu} + \frac{\delta\mu}{y\{\delta+1\}}\right]\right)}{\Phi\left(-\sqrt{\frac{\delta}{2}}\left[\sqrt{\frac{\{\delta+1\}y}{\mu\delta}} - \sqrt{\frac{\mu\delta}{\{\delta+1\}y}}\right]\right)}, \qquad y > 0\,. \end{aligned}$$

2.3. Shape analysis

Figures 1(a)–1(b) show shapes for the PDF of $Y \sim \mathrm{BS}(\mu, \delta)$ considering different values of μ, when δ is fixed, and different values of δ, when μ is fixed. From Figure 1(a), note that the parameter μ controls the scale of the PDF, so that it is a scale parameter and also the mean of the distribution. This aspect can be formally verified because $b\,Y \sim \mathrm{BS}(b\,\mu, \delta)$, with $b > 0$. From Figure 1(b), notice that the parameter δ controls the shape of the PDF, making it more platykurtic as δ increases. Figure 1(c) shows a graphical plot of δ versus $\mathrm{Var}[Y]$, for $\mu = 1.0$. This figure allows the effect exerted by δ on the variance of the distribution to be detected. Note that such a variance decreases as δ increases, and it converges to 5.0, when δ goes to zero. Then, by means of this graphical analysis, we note that δ is a precision parameter.



(a) (b) (c)

Figure 1: PDF plots of a reparameterized BS distribution for different values of μ with $\delta = 100.0$ (a) and of δ with $\mu = 1.0$ (b), and plot of δ versus $\mathrm{Var}[Y]$ (c).

2.4. Number generation

Random numbers from the reparameterized BS distribution can be obtained by using the generator described in Algorithm 1.

Algorithm 1 – Generator of BS random numbers
1: Generate a random number z from a RV $Z \sim \mathrm{N}(0, 1)$;
2: Set values for μ and δ of $Y \sim \mathrm{BS}(\mu, \delta)$;
3: Compute a random number y from $Y \sim \mathrm{BS}(\mu, \delta)$, using formula given in (2.3);
4: Repeat steps 1 to 3 until the required amount of numbers to be completed.

2.5. Moments

Another way to characterize a distribution is by using its CF, which allows us to obtain its moments. Here, we provide some results on the CF and moments of the reparameterized BS distribution. Moments for the original parameterization of the BS distribution can be found in Leiva *et al.* (2008a) and Balakrishnan *et al.* (2009a). In the literature on the BS distribution, the CF is practically not studied. From the PDF given in (2.2), we obtain the CF of $Y \sim \mathrm{BS}(\mu, \delta)$ in the following theorem.

Theorem 2.1. *Let $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the CF $\varphi\colon \mathbb{R} \to \mathbb{C}$ of Y is*

$$\begin{aligned}\varphi(t) &= \mathrm{E}\Big[\exp(itY)\Big] \\ &= \frac{1}{2}\left[\left\{1 + \frac{\sqrt{\delta+1}}{\sqrt{1+\delta-4ti\mu}}\right\} \exp\left(\frac{\delta\left\{\sqrt{\delta+1} - \sqrt{1+\delta-4ti\mu}\right\}}{2\sqrt{\delta+1}}\right)\right], \qquad t \in \mathbb{R}\,,\end{aligned}$$

where $i = \sqrt{-1}$ is the imaginary unit.

Proof: The result is obtained using algebraic and integration methods. □

Corollary 2.1. *Let $Y \sim \mathrm{BS}(\mu, \delta)$ with CF φ as given in Theorem 2.1. Then, the rth derivative of φ with respect to t, evaluated at the point $t = 0$, is*

$$\begin{aligned}\varphi(0)^{(r)} &= \frac{\mathrm{d}^r\varphi(t)}{\mathrm{d}t^r}\bigg|_{t=0} \\ &= i^r\,\mathrm{E}\Big[Y^r\exp(itY)\Big]\Big|_{t=0} \\ &= \frac{1}{2\sqrt{\pi}\,[\delta+1]^{\frac{3}{2}}}\left[i^r\mu^r\delta^2\exp\left(\frac{\delta}{2}\right)\right. \\ &\quad\left.\times\left\{\left(\delta^{r-\frac{1}{2}} + \delta^{r-\frac{3}{2}}\right)(\delta+1)^{\frac{1}{2}-r}K_{r+\frac{1}{2}}\left(\frac{\delta}{2}\right) + \delta^{r-\frac{3}{2}}(\delta+1)^{\frac{3}{2}-r}K_{r-\frac{1}{2}}\left(\frac{\delta}{2}\right)\right\}\right],\end{aligned}$$

where K_v is the modified Bessel function of second type.

Table 1 displays the values of the function K_v (see Abramowitz & Stegun, 1972) for some values of v, which are useful for calculating the moments around zero of the BS distribution.

Table 1: Values of $K_v(\delta/2)$ for the indicated values of v.

v	$K_v(\delta/2)$
$\frac{1}{2}$	$\dfrac{\sqrt{\pi}\,\exp(-\frac{1}{2}\,\delta)}{\sqrt{\delta}}$
$\frac{3}{2}$	$K_{\frac{1}{2}}\!\left(\frac{\delta}{2}\right)\left[1+\frac{2}{\delta}\right]$
$\frac{5}{2}$	$K_{\frac{1}{2}}\!\left(\frac{\delta}{2}\right)\left[1+\frac{6}{\delta}+\frac{12}{\delta^2}\right]$
$\frac{7}{2}$	$K_{\frac{1}{2}}\!\left(\frac{\delta}{2}\right)\left[1+\frac{12}{\delta}+\frac{60}{\delta^2}+\frac{120}{\delta^3}\right]$
$\frac{9}{2}$	$K_{\frac{1}{2}}\!\left(\frac{\delta}{2}\right)\left[1+\frac{20}{\delta}+\frac{180}{\delta^2}+\frac{840}{\delta^3}+\frac{1680}{\delta^4}\right]$

By means of Theorem 2.1 and Corollary 2.1, it is possible to obtain the moments around zero of $Y \sim \mathrm{BS}(\mu,\delta)$. By using the fact that $\varphi(0)^{(r)} = i^r \mathrm{E}[Y^r]$, we can easily find, for example, the four first moments of Y as

$$\begin{aligned}
\mathrm{E}\big[Y\big] &= \mu\,, \qquad \mathrm{E}\big[Y^2\big] = \mu^2\,\frac{[\delta^2+4\,\delta+6]}{[\delta+1]^2}\,,\\
\mathrm{E}\big[Y^3\big] &= \mu^3\,\frac{[\delta^3+9\,\delta^2+36\,\delta+60]}{[\delta+1]^3} \qquad \text{and} \\
\mathrm{E}\big[Y^4\big] &= \mu^4\,\frac{[\delta^4+16\delta^3+120\delta^2+460\delta+840]}{[\delta+1]^4}\,.
\end{aligned} \tag{2.4}$$

The rth central moment of $Y \sim \mathrm{BS}(\mu,\delta)$, which we denote by μ_r, is given by

$$\mu_r = \mathrm{E}\big[Y-\mu\big]^r = \sum_{j=0}^{r}\binom{r}{j}\,(-1)^{r-j}\,\mathrm{E}\big[Y^j\big]\,\mu^{r-j}\,, \qquad r=2,3,... \tag{2.5}$$

From (2.4) and (2.5), we have that the variance of Y is $\mathrm{Var}[Y] = \mu^2[2\,\delta+5]/[\delta+1]^2$, which allows the parameter δ to be interpreted as a precision parameter because, for μ fixed, the variance of Y decreases when δ increases. In addition, we can rewrite this variance as $\mathrm{Var}[Y] = V(\mu)/\phi$, where $\phi = [\delta+1]^2/[2\,\delta+5]$ and $V(\mu) = \mu^2$, with $V(\mu)$ acting as a "variance function", such as in generalized linear models.

Another interesting result is that the reparameterized BS distribution preserves the reciprocation property of the original BS distribution, that is, $1/Y$ is in the same family of distributions of Y. Thus, if $Y \sim \mathrm{BS}(\mu,\delta)$, then $1/Y \sim \mathrm{BS}([\delta+1]^2/\mu\,\delta^2,\,\delta)$ and, consequently,

$$\mathrm{E}\big[1/Y\big] = \frac{[\delta+1]^2}{\mu\,\delta^2} \qquad \text{and} \qquad \mathrm{Var}\big[1/Y\big] = \frac{[2\,\delta+5]\,[\delta+1]^2}{\mu^2\,\delta^4}\,.$$

3. ESTIMATION

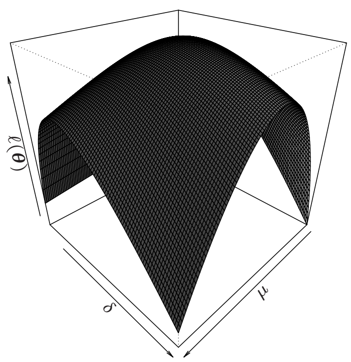
In this section, we derive estimation and inference for the parameters, in the sequel denoted by $\boldsymbol{\theta} = [\mu, \delta]^\top$, of the reparameterized BS distribution based on the GM, ML, MM and moment methods.

3.1. Maximum likelihood estimation

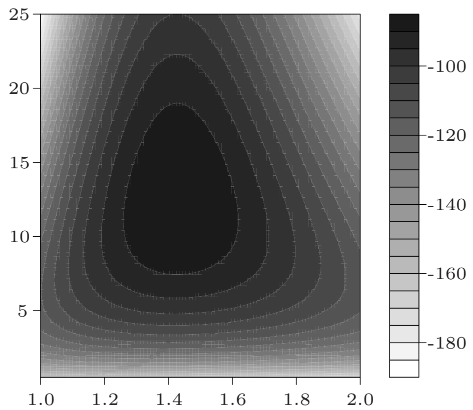
Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the log-likelihood function for $\boldsymbol{\theta}$ is

$$\ell(\boldsymbol{\theta}) = \sum_{j=1}^{n} \ell_j(\boldsymbol{\theta}) \ , \tag{3.1}$$

where $\ell_j(\boldsymbol{\theta})$ is the logarithm of the PDF given in (2.2) replacing y by y_j. Figure 2 displays graphical plots of the log-likelihood function and its respective contours, considering, as illustration, a sample from $Y \sim \mathrm{BS}(\mu{=}1.5, \delta{=}10)$. In this figure, note that the shape of the log-likelihood function is well behaved and, through its contours, it is easy to see the region where the values that maximize the function $\ell(\boldsymbol{\theta})$ given in (3.1) are located.



(a)



(b)

Figure 2: Plots of the log-likelihood function (a) and its respective contours (b), for the $\mathrm{BS}(\mu{=}1.5, \delta{=}10)$ distribution.

As is well-known, to obtain the ML estimates of the parameters, we must equal the score functions to zero. In the case of the reparameterized BS distribution, the score vector for $\boldsymbol{\theta}$ is given by $U(\boldsymbol{\theta}) = [U_\mu, U_\delta]^\top$, where

$$U_\mu = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \mu} = \sum_{j=1}^{n} \left[\frac{\delta}{\delta y_j + y_j + \delta\mu} + \frac{y_j\{\delta+1\}}{4\mu^2} - \frac{\delta^2}{4y_j\{\delta+1\}} - \frac{1}{2\mu} \right]$$

and

$$U_\delta = \frac{\partial \ell(\boldsymbol{\theta})}{\partial \delta} = \sum_{j=1}^{n} \left[\frac{y_j + \mu}{\delta y_j + y_j + \delta\mu} - \frac{y_j}{4\mu} - \frac{\delta\{\delta+2\}\mu}{4\{\delta+1\}^2 y_j} + \frac{\delta}{2\{\delta+1\}} \right].$$

Such as in the case of the original BS parameterization, for the reparameterized version, it is not possible to find closed-form estimators for its parameters. Then, we must use an iterative numerical method to optimize the function $\ell(\boldsymbol{\theta})$ given in (3.1). For example, a Newton–Raphson type algorithm can be used in this case.

The corresponding expected Fisher information matrix, denoted by $\mathcal{K}(\boldsymbol{\theta}) = [\mathcal{K}_{\theta_j\theta_k}]$, has elements

$$\begin{aligned}
\mathcal{K}_{\mu\mu} &= -\mathrm{E}\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \mu^2}\right] = n\left[\frac{\delta}{2\mu^2} + \frac{\delta^2}{\{\delta+1\}^2}\, I(\boldsymbol{\theta})\right], \\
\mathcal{K}_{\delta\mu} &= -\mathrm{E}\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \mu\, \partial \delta}\right] = n\left[\frac{1}{2\mu\{\delta+1\}} + \frac{\delta\mu}{\{\delta+1\}^3}\, I(\boldsymbol{\theta})\right] \quad \text{and} \\
\mathcal{K}_{\delta\delta} &= -\mathrm{E}\left[\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial \delta^2}\right] = n\left[\frac{\delta_j^2 + 3\delta_j + 1}{2\delta_j^2\{\delta_j+1\}^2} + \frac{\mu_j^2}{\{\delta_j+1\}^4}\, I(\boldsymbol{\theta})\right],
\end{aligned} \tag{3.2}$$

where $\mathcal{K}_{\delta\mu} = \mathcal{K}_{\mu\delta}$ and

$$I(\boldsymbol{\theta}) = \int_0^\infty \left[y + \frac{\mu\delta}{\delta+1}\right]^{-2} f(y;\boldsymbol{\theta})\, \mathrm{d}y\,.$$

Under regularity conditions (see Cox & Hinkley, 1974), we have that the corresponding variance-covariance matrix is $\mathrm{Cov}[\widehat{\mu}, \widehat{\delta}] = \mathcal{K}(\boldsymbol{\theta})^{-1}$, whose elements of $\mathcal{K}(\boldsymbol{\theta})$ are given in (3.2). In addition, in general, as is well-known, ML estimators have an asymptotic bivariate normal joint distribution. Thus, in our case, $[\hat{\mu}, \hat{\delta}]^\top$ approximately follows the distribution

$$\mathrm{N}_2\left(\begin{bmatrix}\mu \\ \delta\end{bmatrix}, \mathcal{K}(\boldsymbol{\theta})^{-1}\right).$$

3.2. Moment estimation

Moment conditions are needed to estimate parameters by using the moment method; see Mátyás (1999). Next, we define these conditions.

Definition 3.1. Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from any distribution. We want to estimate an unknown $p\times 1$ parameter vector $\boldsymbol{\theta}$, with true value $\boldsymbol{\theta}_0$. Let $g(Y_j, \boldsymbol{\theta})$ be a $q \times 1$ vector, which is a continuous function of $\boldsymbol{\theta}$, and assume that $\mathrm{E}[g(Y_j, \boldsymbol{\theta})]$ exists and it is finite for all j and $\boldsymbol{\theta}$. Then, the moment conditions to estimate $\boldsymbol{\theta}$ are that $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$.

We want to estimate the vector $\boldsymbol{\theta}$ using the moment conditions given in Definition 3.1. First, we consider the case when $p = q$, that is, when $\boldsymbol{\theta}$ is exactly identified by the moment conditions. Thus, these conditions represent a set of p equations, with p unknown parameters. Solving these equations, we find the true value of $\boldsymbol{\theta}$, $\boldsymbol{\theta}_0$ say, which satisfies the mentioned moment conditions. However, it is not possible to observe $\mathrm{E}[g(Y_j, \boldsymbol{\theta})]$, but only $g(y_j, \boldsymbol{\theta})$. In this way, a natural procedure is to define the sample moments of $g(Y_j, \boldsymbol{\theta})$, given by

$$g_n(\boldsymbol{\theta}) = \frac{1}{n} \sum_{j=1}^{n} g(Y_j, \boldsymbol{\theta}) \; . \tag{3.3}$$

If the sample moments are estimators of the population moments with good properties, we then hope that the estimator $\widetilde{\boldsymbol{\theta}}$ holding the sample moment conditions $g_n(\boldsymbol{\theta}) = \mathbf{0}$ is a good estimator of the true value $\boldsymbol{\theta}_0$, which holds the population moment conditions $\mathrm{E}[g(Y_j, \boldsymbol{\theta})] = \mathbf{0}$. Hence, $\widetilde{\boldsymbol{\theta}}$ is a moment estimator of $\boldsymbol{\theta}$.

Theorem 3.1. *Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the moment estimators of μ and δ are, respectively,*

$$\widetilde{\mu} = \bar{Y} \qquad \text{and} \qquad \widetilde{\delta} = \frac{\bar{Y}^2 - S^2 + \sqrt{\bar{Y}^4 + 3\bar{Y}^2 S^2}}{S^2} \; ,$$

where $\bar{Y} = [1/n] \sum_{j=1}^{n} Y_j$ and $S^2 = [1/n] \sum_{j=1}^{n} [Y_j - \bar{Y}]^2$.

Proof: Recall from (2.4) and (2.5) that $\mathrm{E}[Y - \mu]^2 = \mu^2 [2\delta + 5]/[\delta + 1]^2$ and $\mathrm{E}[Y] = \mu$. Also, recall $\boldsymbol{\theta} = [\mu, \delta]^\top$ and define the vector of functions

$$g(Y_j, \boldsymbol{\theta}) = \left[Y_j - \mu, \;\; \{Y_j - \mu\}^2 - \frac{\mu^2 \{2\delta + 5\}}{\{\delta + 1\}^2} \right]^\top .$$

Then, the moment conditions are $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$. We have that $g_n(\widetilde{\boldsymbol{\theta}}) = \mathbf{0}$, with g_n defined in (3.3), implies that

$$\frac{1}{n} \sum_{j=1}^{n} Y_j - \widetilde{\mu} = 0 \qquad \text{and} \qquad \frac{1}{n} \sum_{j=1}^{n} [Y_j - \widetilde{\mu}]^2 - \frac{\widetilde{\mu}^2 [2\widetilde{\delta} + 5]}{[\widetilde{\delta} + 1]^2} = 0 \; ,$$

which, after some algebraic manipulations, result to be

$$\widetilde{\mu} = \bar{Y} \qquad \text{and} \qquad \widetilde{\delta} = \frac{1 - \widetilde{\kappa}^2 + \sqrt{3\widetilde{\kappa}^2 + 1}}{\widetilde{\kappa}^2} \; , \tag{3.4}$$

where $\widetilde{\kappa} = \sqrt{S^2}/\bar{Y}$ is the sample coefficient of variation (CV), with $0 < \widetilde{\kappa} < \sqrt{5}$. Therefore, we have that (3.4) can be rewritten as

$$\widetilde{\mu} = \bar{Y} \qquad \text{and} \qquad \widetilde{\delta} = \frac{\bar{Y}^2 - S^2 + \sqrt{\bar{Y}^4 + 3\bar{Y}^2 S^2}}{S^2} \; . \qquad \square$$

Theorem 3.2. *Let* $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ *be a random sample of size* n *from* $Y \sim \mathrm{BS}(\mu, \delta)$. *Then,* $\widetilde{\mu}$ *and* $\widetilde{\delta}$ *have an asymptotic bivariate normal joint distribution, that is,* $[\widetilde{\mu}, \widetilde{\delta}]^\top$ *approximately follows the distribution*

$$\mathrm{N}_2\left(\begin{bmatrix}\mu\\ \delta\end{bmatrix}, \frac{1}{n}\begin{bmatrix}\frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} & -\frac{\mu\{2\delta^2+8\delta-3\}}{\{\delta+1\}\{\delta+4\}}\\ -\frac{\mu\{2\delta^2+8\delta-3\}}{\{\delta+1\}\{\delta+4\}} & \frac{2\delta^4+28\delta^3+122\delta^2+126\delta+57}{\{\delta+4\}^2}\end{bmatrix}\right).$$

Proof: Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be independent identically distributed (IID) RVs according to $Y \sim \mathrm{BS}(\mu, \delta)$ and $\mathrm{E}[Y^4]$ given in (2.4) be finite. In addition, let $\widetilde{\mu} = f_1(\bar{Y}, S^2)$ and $\widetilde{\delta} = f_2(\bar{Y}, S^2)$ be the moment estimators of the parameters μ and δ, respectively. Assume that the random vector

$$\sqrt{n}\begin{bmatrix}\bar{Y} - \mathrm{E}[Y]\\ S^2 - \mathrm{E}[Y-\mu]^2\end{bmatrix}$$

approximately follows the distribution

$$\mathrm{N}_2\left(\begin{bmatrix}0\\ 0\end{bmatrix}, \boldsymbol{\Sigma}\right), \qquad \text{where } \boldsymbol{\Sigma} = \begin{bmatrix}\nu & \mu_3\\ \mu_3 & \mu_4 - \nu^2\end{bmatrix},$$

with

$$\nu = \mathrm{Var}[Y] = \frac{\mu^2[2\delta+5]}{[\delta+1]^2}, \quad \mu_3 = \frac{4[3\delta+11]\mu^3}{[\delta+1]^3} \quad \text{and} \quad \mu_4 - \nu^2 = \frac{8\mu^4[\delta^2+20\delta+76]}{[\delta+1]^4}.$$

We want to determine the asymptotic joint distribution of the estimators $\widetilde{\mu} = f_1(\bar{Y}, S^2)$ and $\widetilde{\delta} = f_2(\bar{Y}, S^2)$. These estimators can be expressed as

$$f_1(x, y) = x \qquad \text{and} \qquad f_2(x, y) = \frac{x^2 - y + \sqrt{x^4 + 3x^2 y}}{y}.$$

By using the delta method (see Rao, 1965), we obtain that the random vector

$$\sqrt{n}\begin{bmatrix}\widetilde{\mu} - \mu\\ \widetilde{\delta} - \delta\end{bmatrix}$$

approximately follows the distribution

$$\mathrm{N}_2\left(\begin{bmatrix}0\\ 0\end{bmatrix}, \boldsymbol{\Sigma}\right),$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix}\frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} & -\frac{\mu\{2\delta^2+8\delta-3\}}{\{\delta+1\}\{\delta+4\}}\\ -\frac{\mu\{2\delta^2+8\delta-3\}}{\{\delta+1\}\{\delta+4\}} & \frac{\{2\delta^4+28\delta^3+122\delta^2+126\delta+57\}}{\{\delta+4\}^2}\end{bmatrix}. \qquad \square$$

3.3. Modified moment estimation

Ng *et al.* (2003) used the fact that the BS distribution satisfies the reciprocation property to propose MM estimates for its parameters. The MM estimation method is a variation of the moment estimation method, substituting the expression that equates the second population and sample moments by equating the expected value of $1/Y$ with $[1/n]\sum_{j=1}^{n} 1/Y_j$. Because the reparameterized BS distribution preserves the reciprocation property, once again, the MM estimates of its parameters μ and δ can be easily obtained.

Theorem 3.3. *Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, the MM estimators of μ and δ are, respectively,*

$$\breve{\mu} = \bar{Y} \qquad \textit{and} \qquad \breve{\delta} = \left[\sqrt{\frac{\bar{Y}}{\bar{Y}_h}} - 1\right]^{-1},$$

where $\bar{Y}_h = \left[\{1/n\}\sum_{j=1}^{n}\{1/Y_j\}\right]^{-1}$.

Proof: Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, $\mathrm{E}[Y] = \mu$ and $\mathrm{E}\,[1/Y] = [\delta+1]^2/[\mu\delta^2]$. Thus,

$$g(Y_j, \boldsymbol{\theta}) = \left[Y_j - \mu\,,\; \frac{1}{Y_j} - \frac{\{\delta+1\}^2}{\mu\,\delta^2}\right]^\top.$$

Recall the moment conditions are $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$. We have that $g_n(\breve{\boldsymbol{\theta}}) = \mathbf{0}$, with g_n defined in (3.3), implies that

$$\text{(3.5)} \qquad \frac{1}{n}\sum_{j=1}^{n} Y_j - \breve{\mu} = 0 \qquad \text{and} \qquad \frac{1}{n}\sum_{j=1}^{n}\frac{1}{Y_j} - \frac{[\breve{\delta}+1]^2}{\breve{\mu}\,\breve{\delta}^2} = 0\,.$$

Hence, solving (3.5), we obtain the MM estimators

$$\breve{\mu} = \bar{Y} \qquad \text{and} \qquad \breve{\delta} = \left[\sqrt{\frac{\bar{Y}}{\bar{Y}_h}} - 1\right]^{-1},$$

where $\bar{Y}_h$ is defined in Theorem 3.3. In addition, we have that $\breve{\delta}$ is well-defined for $\bar{Y}_h \neq \bar{Y}$, when $\bar{Y}_h < \bar{Y}$. $\square$

Theorem 3.4. *Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from $Y \sim \mathrm{BS}(\mu, \delta)$. Then, $\breve{\mu}$ and $\breve{\delta}$ have an asymptotic bivariate normal joint distribution, that is, $[\breve{\mu}, \breve{\delta}]^\top$ approximately follows the distribution*

$$\mathrm{N}_2\left(\begin{bmatrix}\mu \\ \delta\end{bmatrix}, \frac{1}{n}\begin{bmatrix}\frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} & -\frac{2\mu\delta}{\delta+1} \\ -\frac{2\mu\delta}{\delta+1} & 2\,\delta^2\end{bmatrix}\right).$$

Proof: Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be IID RVs according to $Y \sim \mathrm{BS}(\mu, \delta)$ and $\mathrm{E}[Y_j^4] < \infty$. Then, the vector $[\bar{Y}, \bar{Y}_h^{-1}]^\top$ follows an asymptotic bivariate normal distribution, which implies that

$$\sqrt{n} \begin{bmatrix} \bar{Y} - \mathrm{E}[Y] \\ \bar{Y}_h^{-1} - \mathrm{E}[Y^{-1}] \end{bmatrix} \;\dot{\sim}\; \mathrm{N}_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{\Sigma} \right),$$

where "$\dot{\sim}$" means "approximately follows the distribution" and

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathrm{Var}[Y] & \mathrm{Cov}[Y, Y^{-1}] \\ \mathrm{Cov}[Y, Y^{-1}] & \mathrm{Var}[Y^{-1}] \end{bmatrix},$$

with

$$\mathrm{Var}[Y] = \frac{\mu^2 [2\delta+5]}{[\delta+1]^2}, \quad \mathrm{Cov}[Y, Y^{-1}] = 1 - \frac{[\delta+1]^2}{\delta^2} \quad \text{and} \quad \mathrm{Var}[Y^{-1}] = \frac{[2\delta+5]\,[\delta+1]^2}{\mu^2\,\delta^4}.$$

However, our interest is to find the asymptotic joint distribution of $\breve{\mu} = f_1(\bar{Y}, \bar{Y}_h^{-1})$ and $\breve{\delta} = f_2(\bar{Y}, \bar{Y}_h^{-1})$. For these estimators, consider $f_1(x,y) = x$, $f_2(x,y) = [\sqrt{xy} - 1]^{-1}$ and the delta method. Then,

$$\sqrt{n} \begin{bmatrix} \breve{\mu} - \mu \\ \breve{\delta} - \delta \end{bmatrix} \;\dot{\sim}\; \mathrm{N}_2\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{\Sigma} \right),$$

where

$$\boldsymbol{\Sigma} = \begin{bmatrix} \frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} & -\frac{2\mu\delta}{\delta+1} \\ -\frac{2\mu\delta}{\delta+1} & 2\,\delta^2 \end{bmatrix}. \qquad \square$$

3.4. Generalized moment estimation

The GM method provides estimators that are in general consistent, but in general not efficient. The GM method is an extension of the usual moment estimation method; see details in Mátyás (1999) and in the following definition.

Definition 3.2. Let $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ be a random sample of size n from any distribution. We want to estimate an unknown $p \times 1$ parameter vector $\boldsymbol{\theta}$, with true value $\boldsymbol{\theta}_0$. Let $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$ be a set of q moment conditions and $g_n(\boldsymbol{\theta})$ be the corresponding sample moments given in (3.3). Define the criterion function

$$Q_n(\boldsymbol{\theta}) = g_n(\boldsymbol{\theta})^\top \boldsymbol{A}_n^{-1} g_n(\boldsymbol{\theta}),$$

where $\boldsymbol{A}_n$ is a $O_p(1)$ stochastic positive definite matrix. Then, the GM estimator of $\boldsymbol{\theta}$ is

$$\breve{\boldsymbol{\theta}} = \underset{\boldsymbol{\theta}}{\operatorname{argmin}}\; Q_n(\boldsymbol{\theta}). \tag{3.6}$$

As mentioned, in general, the GM method provides consistent estimators, but $\boldsymbol{\theta}$ must be the unique solution of $\mathrm{E}[g(Y_j, \boldsymbol{\theta})]$ and an element of a compact space. Some assumptions on high order moments of $g(Y_j, \boldsymbol{\theta})$ also are needed. However, there are no restrictions on the model that generates the data, except for the case of dependent data.

Considering $q > p$ in Definition 3.2, we can perform the $\mathcal{J}$ test (see Hansen, 1982) to assess the moment conditions and/or the specification of model, because it acts as an omnibus test for model misspecification. In this case, the null hypothesis H_0: $\mathrm{E}[g(Y_j, \boldsymbol{\theta}_0)] = \mathbf{0}$ can be tested by using the statistic $n g_n(\breve{\boldsymbol{\theta}})^\top \breve{\boldsymbol{A}}_n^{-1} g_n(\breve{\boldsymbol{\theta}})$, which approximately follows the χ^2_{q-p} distribution under H_0; see Mátyás (1999). If the model is misspecified and/or some of the moment conditions do not hold, then the $\mathcal{J}$ statistic will have a small p-value.

Theorem 3.5. *Let* $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ *be a random sample of size* n *from* $Y \sim \mathrm{BS}(\mu, \delta)$*. Then, the GM estimators of* μ *and* δ*,* $\breve{\mu}$ *and* $\breve{\delta}$ *say, can be obtained in a general setting from* (3.6)*.*

Proof: The result is direct from (3.6). □

Theorem 3.6. *Let* $\boldsymbol{Y} = [Y_1, ..., Y_n]^\top$ *be a random sample of size* n *from* $Y \sim \mathrm{BS}(\mu, \delta)$*. Then,* $\breve{\mu}$ *and* $\breve{\delta}$ *have an asymptotic bivariate normal joint distribution, that is,* $[\breve{\mu}, \breve{\delta}]^\top$ *approximately follows the distribution*

$$\mathrm{N}_2\left(\begin{bmatrix}\mu \\ \delta\end{bmatrix}, \frac{1}{n}\boldsymbol{V}\right),$$

where

$$\boldsymbol{V} = \mathrm{E}\left[\frac{\partial g(Y_j, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]^\top \boldsymbol{A}_n^{-1} \mathrm{E}\left[\frac{\partial g(Y_j, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right].$$

Proof: Given some regularity conditions (see Mátyás, 1999, Section 1.3.2), as n goes to infinity, the GM estimator converges to a bivariate normal distribution and so the random vector $\sqrt{n}[\breve{\boldsymbol{\theta}} - \boldsymbol{\theta}] \dot{\sim} \mathrm{N}_2(\mathbf{0}, \boldsymbol{V})$, where

$$\boldsymbol{V} = \mathrm{E}\left[\frac{\partial g(Y_j, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]^\top \boldsymbol{A}_n^{-1} \mathrm{E}\left[\frac{\partial g(Y_j, \boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\right]. \qquad \square$$

To obtain point and interval estimates of the parameters of the BS distribution, we can use the `gmm` package (see Chaussé, 2010) of the `R` software (`www.R-project.org`). The matrix $\boldsymbol{A}_n$, which produces efficient estimators for $\boldsymbol{\theta}$, can be estimated by an heteroskedasticity and autocorrelation consistent covariance matrix; see Newey & West (1987) and Chaussé (2010). To obtain the corresponding estimates, we run the `gmm` function using as starting values $\mu_0 = \breve{\mu}$ and $\delta_0 = \breve{\delta}$. To test the specification of estimated model, we use the $\mathcal{J}$ test through of the `specTest()` function also available in the `gmm` package.

4. SIMULATION

In this section, we conduct a study based on MC simulations to evaluate the performance of the GM, ML, MM and moment estimators for the reparameterized BS distribution.

MC replications are based on Algorithm 1.For each replication generated by this algorithm, we calculate GM, ML, MM and moment estimates. The algorithm and estimation methods are implemented in the `R` software by using the `gamlss` (see Stasinopoulos & Rigby, 2007) and `gmm` packages. For details about generation of numbers from the BS distribution, see Leiva *et al.* (2008b) and Barros *et al.* (2009). Then, the mean, bias, standard error (SE) and squared root of the mean squared error ($\sqrt{\text{MSE}}$) of these estimators are empirically computed. We obtain point estimates, confidence intervals (CIs) and their coverage probabilities (CPs) of 95% level, based on the asymptotic results associated with each estimator given in Section 3. The ML estimates are obtained from the `gamlss()` function and the GM estimates from the `gmm()` function. The CIs based on the GM estimates are obtained by using the `R` function `confint()`, where the main argument is an object of the `gmm` class. The scenario of this simulation study considers 10 000 MC replications in each case, sample sizes $n \in \{30, 50, 75, 100, 200\}$ and values for $\delta \in \{0.5, 2.0, 8.0, 32.0, 200\}$ (according to different levels of skewness) and $\mu = 1.0$ (without loss of generality). The obtained results are presented in Tables 2, 3, 4 and 5.

To perform the GM estimation of the parameters μ and δ of the BS distribution, we consider the following vector of moment conditions:

$$\mathrm{E}\left[g(Y_j, \boldsymbol{\theta})\right] = \mathrm{E}\begin{bmatrix} \mu - Y_j \\ \frac{\mu^2\{2\delta+5\}}{\{\delta+1\}^2} - \{Y_j - \mu\}^2 \\ \frac{\{\delta+1\}^2}{\mu\delta^2} - \frac{1}{Y_j} \end{bmatrix} = \mathbf{0}\,,$$

where the gradient function of $g_n(\boldsymbol{\theta})$ is given by

$$G = \frac{\partial g_n(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathrm{E}\begin{bmatrix} 1 & 0 \\ \frac{2\mu\{2\delta+5\}}{\{\delta+1\}^2} - 2\mu + 2\bar{Y} & -\frac{2\mu^2\{\delta+4\}}{\{\delta+1\}^3} \\ -\frac{\{\delta+1\}^2}{\{\mu\delta\}^2} & -\frac{2\{\delta+1\}}{\mu\delta^3} \end{bmatrix}.$$

From Tables 2 through 5, note that the ML, MM and moment estimators of the parameter μ present similar statistical properties in relation to the empirical bias and $\sqrt{\text{MSE}}$. However, the GM estimator presents similar properties to the other estimators only when the sample size is large. In the case of the parameter δ, its ML and MM estimators present similar properties for the different sample sizes and true values assumed for this parameter. Table 3 shows that, in general, the GM method underestimates the true value of μ. From Tables 4 and 5, note that the values of the empirical SE and $\sqrt{\text{MSE}}$ increase as δ increases, for all the considered methods, in the case of the parameter δ. Nevertheless, in the case of the parameter μ, we have a reverse behavior, that is, the values of the empirical SE and $\sqrt{\text{MSE}}$ decrease as δ increases, for all the considered methods. In addition, the GM estimator presents the worse behavior in terms of statistical properties, but, as the sample size increases, the estimators obtained by this method turn to be more competitive, with respect to the other estimators considered.

Table 6 provides empirical CPs of 95% CIs for the parameters of the BS(μ, δ) distribution. Note that the CIs based on the GM estimates have CPs smaller than those from the other methods. However, as the sample size increases, the distance between CPs for the fixed confidence levels decreases. Also, when the true value of δ increases, the distance between the confidence level (0.95) and the empirical CP decreases. Thus, such as in the study based on point estimation, for interval estimation, ML and MM estimators present similar statistical properties and better than the other estimators considered.

Table 2: Empirical mean of the estimator of the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	1.004	1.002	1.002	0.869	0.561	0.772	0.561	0.633
	2.0	1.001	1.001	1.001	0.929	2.232	2.526	2.232	2.508
	8.0	1.000	1.000	1.000	0.978	8.886	9.285	8.886	9.949
	32.0	1.000	1.000	1.000	1.005	35.477	35.920	35.477	40.352
	200.0	1.000	1.000	1.000	1.003	221.734	222.150	221.734	245.663
50	0.5	0.999	0.998	0.998	0.896	0.536	0.668	0.536	0.578
	2.0	0.999	0.999	0.999	0.946	2.137	2.321	2.137	2.319
	8.0	1.000	1.000	1.000	0.981	8.522	8.775	8.522	9.174
	32.0	1.000	1.000	1.000	1.002	34.058	34.339	34.058	37.064
	200.0	1.000	1.000	1.000	1.002	212.782	213.040	212.782	227.794
75	0.5	0.998	0.996	0.996	0.916	0.524	0.610	0.524	0.552
	2.0	0.999	0.998	0.998	0.958	2.092	2.210	2.092	2.220
	8.0	0.999	0.999	0.999	0.985	8.355	8.518	8.355	8.835
	32.0	1.000	1.000	1.000	1.000	33.385	33.559	33.385	35.463
	200.0	1.000	1.000	1.000	1.001	208.676	208.810	208.676	219.441
100	0.5	0.999	0.998	0.998	0.933	0.518	0.581	0.518	0.539
	2.0	0.999	0.999	0.999	0.967	2.068	2.150	2.068	2.163
	8.0	1.000	1.000	1.000	0.988	8.261	8.377	8.261	8.634
	32.0	1.000	1.000	1.000	0.999	33.022	33.148	33.022	34.590
	200.0	1.000	1.000	1.000	1.001	206.366	206.453	206.366	214.828
200	0.5	0.998	0.997	0.997	0.960	0.509	0.541	0.509	0.521
	2.0	0.999	0.999	0.999	0.980	2.036	2.077	2.036	2.085
	8.0	1.000	0.999	0.999	0.993	8.137	8.195	8.137	8.338
	32.0	1.000	1.000	1.000	0.998	32.529	32.600	32.529	33.313
	200.0	1.000	1.000	1.000	1.001	203.274	203.362	203.274	207.927

Table 3: Empirical bias of the estimator of the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	0.004	0.002	0.002	−0.131	0.061	0.272	0.061	0.133
	2.0	0.001	0.001	0.001	−0.071	0.232	0.526	0.232	0.508
	8.0	0.000	0.000	0.000	−0.022	0.886	1.285	0.886	1.949
	32.0	0.000	0.000	0.000	0.005	3.477	3.920	3.477	8.352
	200.0	0.000	0.000	0.000	0.003	21.734	22.150	21.734	45.663
50	0.5	−0.001	−0.002	−0.002	−0.104	0.036	0.168	0.036	0.078
	2.0	−0.001	−0.001	−0.001	−0.054	0.137	0.321	0.137	0.319
	8.0	0.000	0.000	0.000	−0.019	0.522	0.775	0.522	1.174
	32.0	0.000	0.000	0.000	0.002	2.058	2.339	2.058	5.064
	200.0	0.000	0.000	0.000	0.002	12.782	13.040	12.782	27.794
75	0.5	−0.002	−0.004	−0.004	−0.084	0.024	0.110	0.024	0.052
	2.0	−0.001	−0.002	−0.002	−0.042	0.092	0.210	0.092	0.220
	8.0	−0.001	−0.001	−0.001	−0.015	0.355	0.518	0.355	0.835
	32.0	0.000	0.000	0.000	0.000	1.385	1.559	1.385	3.463
	200.0	0.000	0.000	0.000	0.001	8.676	8.810	8.676	19.441
100	0.5	−0.001	−0.002	−0.002	−0.067	0.018	0.081	0.018	0.039
	2.0	−0.001	−0.001	−0.001	−0.033	0.068	0.150	0.068	0.163
	8.0	0.000	0.000	0.000	−0.012	0.261	0.377	0.261	0.634
	32.0	0.000	0.000	0.000	−0.001	1.022	1.148	1.022	2.590
	200.0	0.000	0.000	0.000	0.001	6.366	6.453	6.366	14.828
200	0.5	−0.002	−0.003	−0.003	−0.040	0.009	0.041	0.009	0.021
	2.0	−0.001	−0.001	−0.001	−0.020	0.036	0.077	0.036	0.085
	8.0	0.000	−0.001	−0.001	−0.007	0.137	0.195	0.137	0.338
	32.0	0.000	0.000	0.000	−0.002	0.529	0.600	0.529	1.313
	200.0	0.000	0.000	0.000	0.001	3.274	3.362	3.274	7.927

Table 4: Empirical SE of the estimator of the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	0.296	0.298	0.298	0.308	0.162	0.440	0.162	0.257
	2.0	0.182	0.182	0.182	0.195	0.638	0.986	0.638	0.925
	8.0	0.092	0.092	0.092	0.102	2.532	2.993	2.532	3.533
	32.0	0.046	0.046	0.046	0.051	10.100	10.627	10.100	13.359
	200.0	0.018	0.018	0.018	0.020	63.122	63.636	63.122	78.065
50	0.5	0.226	0.228	0.228	0.237	0.113	0.340	0.113	0.153
	2.0	0.139	0.139	0.139	0.148	0.448	0.733	0.448	0.582
	8.0	0.071	0.071	0.071	0.078	1.786	2.186	1.786	2.212
	32.0	0.035	0.035	0.035	0.040	7.134	7.624	7.134	8.809
	200.0	0.014	0.014	0.014	0.015	44.580	45.136	44.580	51.508
75	0.5	0.185	0.187	0.187	0.193	0.089	0.276	0.089	0.104
	2.0	0.114	0.114	0.114	0.121	0.353	0.591	0.353	0.432
	8.0	0.058	0.058	0.058	0.063	1.404	1.744	1.404	1.663
	32.0	0.029	0.029	0.029	0.032	5.609	6.025	5.609	6.693
	200.0	0.012	0.012	0.012	0.013	35.043	35.502	35.043	39.398
100	0.5	0.159	0.160	0.160	0.166	0.075	0.240	0.075	0.084
	2.0	0.099	0.099	0.099	0.104	0.299	0.504	0.299	0.347
	8.0	0.051	0.051	0.051	0.055	1.191	1.484	1.191	1.372
	32.0	0.025	0.025	0.025	0.028	4.764	5.128	4.764	5.535
	200.0	0.010	0.010	0.010	0.011	29.733	30.126	29.733	32.884
200	0.5	0.114	0.115	0.115	0.118	0.051	0.172	0.051	0.055
	2.0	0.070	0.070	0.070	0.073	0.206	0.354	0.206	0.221
	8.0	0.036	0.036	0.036	0.037	0.820	1.028	0.820	0.884
	32.0	0.018	0.018	0.018	0.019	3.283	3.538	3.283	3.563
	200.0	0.007	0.007	0.007	0.008	20.510	20.790	20.510	21.865

Table 5: Empirical $\sqrt{\text{MSE}}$ of the estimator of the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	0.296	0.298	0.298	0.334	0.173	0.517	0.173	0.290
	2.0	0.182	0.182	0.182	0.208	0.679	1.117	0.679	1.055
	8.0	0.092	0.092	0.092	0.104	2.683	3.257	2.683	4.035
	32.0	0.046	0.046	0.046	0.052	10.682	11.327	10.682	15.755
	200.0	0.018	0.018	0.018	0.020	66.759	67.380	66.759	90.440
50	0.5	0.226	0.228	0.228	0.259	0.119	0.379	0.119	0.172
	2.0	0.139	0.139	0.139	0.158	0.469	0.800	0.469	0.663
	8.0	0.071	0.071	0.071	0.080	1.861	2.320	1.861	2.505
	32.0	0.035	0.035	0.035	0.040	7.425	7.975	7.425	10.161
	200.0	0.014	0.014	0.014	0.016	46.376	46.981	46.376	58.528
75	0.5	0.185	0.187	0.187	0.210	0.092	0.297	0.092	0.116
	2.0	0.114	0.114	0.114	0.128	0.365	0.627	0.365	0.485
	8.0	0.058	0.058	0.058	0.065	1.448	1.819	1.448	1.861
	32.0	0.029	0.029	0.029	0.032	5.777	6.223	5.777	7.536
	200.0	0.012	0.012	0.012	0.013	36.101	36.578	36.101	43.933
100	0.5	0.159	0.161	0.161	0.179	0.077	0.253	0.077	0.093
	2.0	0.099	0.099	0.099	0.109	0.307	0.526	0.307	0.383
	8.0	0.051	0.051	0.051	0.056	1.219	1.531	1.219	1.511
	32.0	0.025	0.025	0.025	0.028	4.873	5.255	4.873	6.111
	200.0	0.010	0.010	0.010	0.011	30.407	30.809	30.407	36.072
200	0.5	0.114	0.115	0.115	0.125	0.052	0.177	0.052	0.059
	2.0	0.070	0.070	0.070	0.075	0.209	0.362	0.209	0.237
	8.0	0.036	0.036	0.036	0.038	0.832	1.046	0.832	0.947
	32.0	0.018	0.018	0.018	0.019	3.325	3.589	3.325	3.797
	200.0	0.007	0.007	0.007	0.008	20.770	21.060	20.770	23.258

Table 6: CP of 95% CIs for the indicated parameter, method, n and δ, with $\mu = 1.0$.

n	δ	μ				δ			
		ML	Moment	MM	GM	ML	Moment	MM	GM
30	0.5	0.899	0.884	0.896	0.622	0.956	0.993	0.957	0.864
	2.0	0.917	0.906	0.916	0.707	0.956	0.983	0.956	0.861
	8.0	0.930	0.924	0.930	0.785	0.956	0.970	0.956	0.858
	32.0	0.937	0.935	0.937	0.826	0.956	0.961	0.956	0.836
	200.0	0.942	0.940	0.942	0.815	0.956	0.958	0.956	0.880
50	0.5	0.999	0.903	0.914	0.703	0.955	0.984	0.955	0.886
	2.0	0.929	0.921	0.930	0.779	0.954	0.978	0.954	0.878
	8.0	0.939	0.934	0.938	0.826	0.954	0.967	0.954	0.886
	32.0	0.943	0.941	0.942	0.857	0.954	0.960	0.953	0.864
	200.0	0.943	0.943	0.943	0.843	0.953	0.954	0.953	0.896
75	0.5	0.928	0.920	0.926	0.757	0.954	0.982	0.953	0.904
	2.0	0.936	0.930	0.936	0.820	0.954	0.973	0.953	0.899
	8.0	0.941	0.938	0.940	0.862	0.954	0.964	0.954	0.901
	32.0	0.943	0.942	0.942	0.880	0.954	0.957	0.953	0.887
	200.0	0.944	0.944	0.944	0.862	0.954	0.953	0.954	0.906
100	0.5	0.935	0.929	0.933	0.794	0.952	0.978	0.952	0.913
	2.0	0.942	0.939	0.942	0.848	0.952	0.972	0.952	0.910
	8.0	0.944	0.942	0.944	0.879	0.953	0.961	0.953	0.911
	32.0	0.944	0.940	0.944	0.888	0.952	0.955	0.952	0.897
	200.0	0.944	0.944	0.943	0.869	0.952	0.953	0.952	0.912
200	0.5	0.940	0.935	0.938	0.851	0.952	0.978	0.952	0.926
	2.0	0.944	0.942	0.943	0.888	0.951	0.969	0.951	0.926
	8.0	0.949	0.947	0.948	0.916	0.950	0.958	0.950	0.926
	32.0	0.948	0.949	0.948	0.916	0.950	0.952	0.950	0.925
	200.0	0.947	0.947	0.947	0.894	0.950	0.950	0.950	0.927

5. APPLICATIONS

In this section, we provide a practical illustration of the proposed estimation methods based on two real-world data sets, with moderate and large sample sizes and from two fields: economics and engineering.

5.1. Data set I (S1): Griffiths *et al.* (1993)

The first data set (S1) is presented in Griffiths *et al.* (1993) and corresponds to household expenditures for food in the United States (US) expressed in thousands of US dollars (M$). These data are provided in Table 7.

Table 7: Household expenditures for food (in M$) (Griffiths *et al.*, 1993).

15.998	16.652	21.741	7.431	10.481	13.548	23.256	17.976	14.161	8.825
14.184	19.604	13.728	21.141	17.446	9.629	14.005	9.160	18.831	7.641
13.882	9.670	21.604	10.866	28.980	10.882	18.561	11.629	18.067	14.539
19.192	25.918	28.833	15.869	14.910	9.550	23.066	14.751		

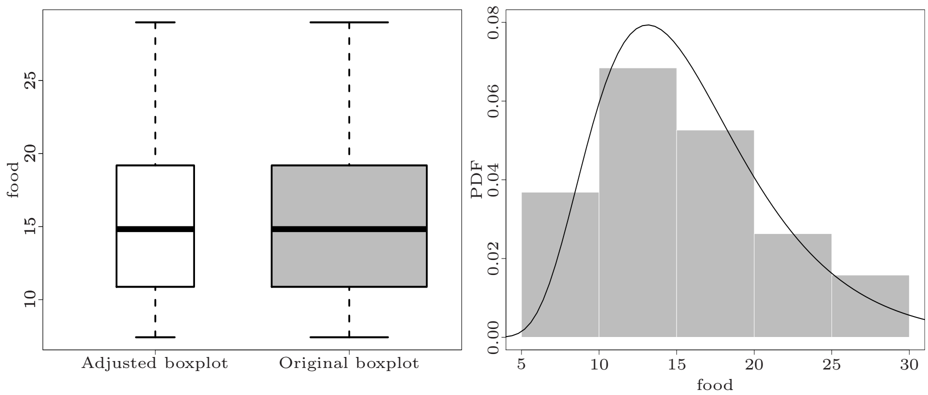
Table 8 presents a descriptive summary of S1 that includes sample mean ($\bar{y}$), median ($\tilde{y}$), standard deviation (SD), CV, coefficients of skewness (CS) and of kurtosis (CK), and minimum ($y_{(1)}$) and maximum ($y_{(n)}$) values. Note that the empirical distribution of the studied RV is slightly positive skewed. Figure 3 presents the boxplots and histogram for S1. From Figure 3(a), note that the adjusted and usual boxplots exhibit the same behavior, which makes sense because the data have little asymmetry. From Figure 3(b), note that the BS distribution fits the data well, whose PDF is estimated with $\widehat{\mu} = 15.95$ and $\widehat{\delta} = 15.57$. Point estimates of the μ and δ parameters of the BS distribution for the proposed methods, and 90% and 95% CIs for these parameters, are displayed in Table 9.

Table 8: Descriptive statistics for S1 (in M$).

$y_{(1)}$	$\tilde{y}$	$\bar{y}$	$y_{(n)}$	SD	CV	CS	CK
7.431	14.831	15.953	28.980	5.624	0.353	0.525	2.556

Table 9: Estimates and CIs for indicated parameter and method with S1.

Method	μ			δ		
	Estimate	CI(90%)	CI(95%)	Estimate	CI(90%)	CI(95%)
ML	15.95	[14.41;17.50]	[14.11;17.79]	15.57	[9.70;21.45]	[8.57;22.57]
Moment	15.95	[14.47;17.43]	[14.19;17.72]	16.91	[9.51;24.31]	[8.10;25.72]
MM	15.95	[14.41;17.50]	[14.11;17.79]	15.57	[9.70;21.45]	[8.57;22.57]
GM	15.30	[14.31;16.30]	[14.12;16.49]	15.94	[10.96;20.92]	[10.00;21.87]



(a) (b)

Figure 3: Boxplots (a) and histogram with estimated PDF (b) for S1.

Next, we evaluate the fitting of the BS distribution to S1 with goodness-of-fit tests. Consider the null hypothesis H_0: "the data come from a RV $Y \sim \text{BS}(\mu, \delta)$" versus the alternative hypothesis H_1: "the data do not come from this RV". We use the Cramér–von Mises (CM) and Anderson–Darling (AD) statistics to test these hypotheses; see Barros *et al.* (2014). The corresponding p-values of the CM and AD tests obtained for S1, with the BS distribution under H_0, are 0.656 and 0.608, respectively. Thus, we do not have evidence to indicate than the BS distribution does not fit these data well. We check moment conditions of the GM method for S1 with the $\mathcal{J}$ test, by using the `R` function `specTest()`, whose p-value is 0.430. Thus, once again the null hypothesis is not rejected for any usual significance level. Therefore, we do not have evidence to conclude that the moment conditions are incorrect or that the BS distribution does not fit S1 well.

5.2. Data set II (S2): Birnbaum & Saunders (1969b)

The second data set (S2) is a classical one used in the literature on the topic. These data were introduced by Birnbaum & Saunders (1969b) and correspond to lifetimes of 6061-T6 aluminum coupons expressed in cycles ($\times 10^{-3}$) at a maximum stress level of 3.1 psi ($\times 10^4$), until the failure to occur. These coupons were cut parallel to the direction of rolling and oscillating at 18 cycles per seconds. The data are displayed in Table 10.

Table 10: Lifetimes (in cycles $\times 10^{-3}$) (Birnbaum & Saunders, 1969b).

70	90	96	97	99	100	103	104	104	105	107	108	108	108	109
109	112	112	113	114	114	114	116	119	120	120	120	121	121	123
124	124	124	124	124	128	128	129	129	130	130	130	131	131	131
131	131	132	132	132	133	134	134	134	134	134	136	136	137	138
138	138	139	139	141	141	142	142	142	142	142	142	144	144	145
146	148	148	149	151	151	152	155	156	157	157	157	157	158	159
162	163	163	164	166	166	168	170	174	196	212				

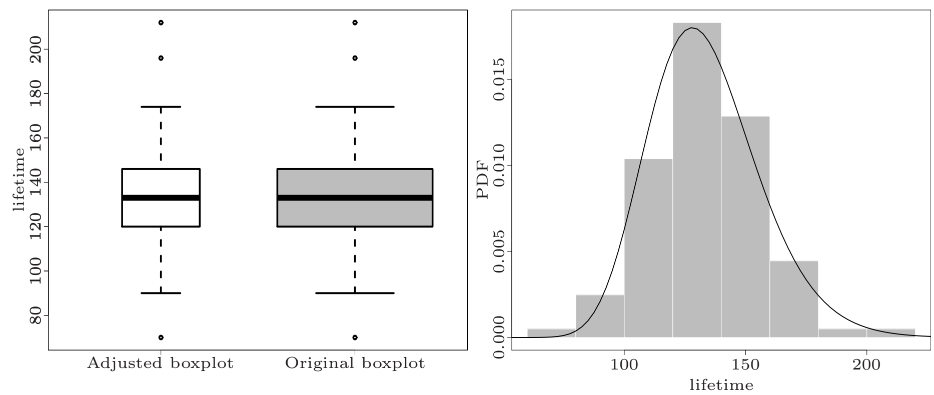
Table 11 presents a descriptive summary of S2 in a similar way to S1. Note that the empirical distribution of the studied RV is relatively symmetric and leptokurtic. Figure 3 presents the boxplots and histogram for S2. From Figure 4(a), note also that the adjusted and usual boxplots exhibit the same behavior, which makes sense because the data have little asymmetry. From Figure 4(b), note that the BS distribution fits the data well, whose PDF is estimated with $\widehat{\mu} = 133.73$ and $\widehat{\delta} = 68.89$. Point estimates of the μ and δ parameters of the BS distribution for the proposed methods, and 90% and 95% CIs for these parameters, for S2, are displayed in Table 12. From this table, we note that less accurate CIs are obtained by the GM method.

Table 11: Descriptive statistics for S2 (in cycles $\times 10^{-3}$).

$y_{(1)}$	$\tilde{y}$	$\bar{y}$	$y_{(n)}$	SD	CV	CS	CK
70.00	133.000	133.733	212.000	22.356	0.167	0.326	3.973

Table 12: Estimates and CIs for indicated parameter and method with S2.

Method	μ			δ		
	Estimate	CI(90%)	CI(95%)	Estimate	CI(90%)	CI(95%)
ML	133.73	[129.99;137.47]	[129.27;138.19]	68.89	[52.95; 84.84]	[49.89;87.89]
Moment	133.73	[130.09;137.37]	[129.39;138.07]	72.76	[55.24; 90.27]	[51.88;93.63]
MM	133.73	[129.99;137.47]	[129.27;138.19]	68.89	[52.95; 84.84]	[49.89;87.89]
GM	137.69	[129.62;145.76]	[128.08;147.31]	75.36	[33.88;116.85]	[25.93;124.80]



(a) (b)

Figure 4: Boxplots (a) and histogram with estimated PDF (b) for S2.

The corresponding p-values of the CM and AD tests obtained for S2 are 0.202 and 0.169, respectively. Thus, we do not have evidence to indicate than the BS distribution does not fit S2 well. The $\mathcal{J}$ test presented a p-value $= 0.720$, so that the null hypothesis is not rejected for any usual significance level. Therefore, we do not have evidence to conclude that the moment conditions are incorrect or that the BS distribution does not fit S2 well.

6. CONCLUSIONS

In this paper, we have provided some novel results on moments and generation of random numbers from a reparameterized version of the Birnbaum–Saunders distribution. In addition, we have studied several estimation methods for this parameterization. We have considered the generalized moment, maximum likelihood, modified moment and moment methods to estimate the corresponding parameters. Furthermore, we have conducted a Monte Carlo study to evaluate the performance of these estimators. From this study, we can conclude that the maximum likelihood and modified moment estimators present similar statistical properties and better that those of the other estimators considered. Therefore, due to the modified moment estimators are easier to compute, we recommend their use for the reparameterized Birnbaum–Saunders distribution. In addition, we have obtained moment estimators in a closed-form, which is not possible with the original parameterization of the Birnbaum–Saunders distribution. However, the parameter estimators obtained by the moment method, as well as those obtained by the generalized moment method, are underperformed with respect to their statistical properties. Nevertheless, for the case of large sample sizes, all the studied estimators have similar statistical properties. We have discussed applications of the BS distribution in different scientific fields and taken advantage of the computational implementation in the `R` software for carrying an application with two real-world data sets, which allowed us to illustrate the obtained results.

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REFERENCES

[1] Abramowitz, M. and Stegun, I. (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, NY.

[2] Achcar, J.A. (1993). Inference for the Birnbaum–Saunders fatigue life model using Bayesian methods, *Comp. Stat. Data Anal.*, **15**, 367–380.

[3] Ahmed, S.E.; Budsaba, K.; Lisawadi, S. and Volodin, A. (2008). Parametric estimation for the Birnbaum–Saunders lifetime distribution based on a new parametrization, *Thailand Stat.*, **6**, 213–240.

[4] Ahmed, S.E.; Castro, C.; Flores, E.; Leiva, V. and Sanhueza, A. (2010). A truncated version of the Birnbaum–Saunders distribution with an application in financial risk, *Pak. J. Stat.*, **26**, 293–311.

[5] Athayde, E.; Azevedo, C.; Leiva, V. and Sanhueza, A. (2012). About Birnbaum–Saunders distributions based on the Johnson system, *Comm. Stat. Theor. Meth.*, **41**, 2061–2079.

[6] Azevedo, C.; Leiva, V.; Athayde, E. and Balakrishnan, N. (2012). Shape and change point analyses of the Birnbaum–Saunders-t hazard rate and associated estimation, *Comp. Stat. Data Anal.*, **56**, 3887–3897.

[7] Balakrishnan, N.; Gupta, R.; Kundu, D.; Leiva, V. and Sanhueza, A. (2011). On some mixture models based on the Birnbaum–Saunders distribution and associated inference, *J. Stat. Plan. Infer.*, **141**, 2175–2190.

[8] Balakrishnan, N.; Leiva, V.; Sanhueza, A. and Cabrera, E. (2009a). Mixture inverse Gaussian distribution and its transformations, moments and applications, *Statistics*, **43**, 91–104.

[9] Balakrishnan, N.; Leiva, V.; Sanhueza, A. and Vilca, F. (2009b). Estimation in the Birnbaum–Saunders distribution based on scale-mixture of normals and the EM-algorithm, *Stat. Oper. Res. Trans.*, **33**, 171–192.

[10] Barros, M.; Leiva, V.; Ospina, R. and Tsuyuguchi, A. (2014). Goodness-of-fit tests for the Birnbaum–Saunders distribution with censored reliability data, *IEEE Trans. Rel.*, **63**, 543–554.

[11] Barros, M.; Paula, G.A. and Leiva, V. (2008). A new class of survival regression models with heavy-tailed errors: robustness and diagnostics, *Lifetime Data Anal.*, **14**, 316–332.

[12] Barros, M.; Paula, G.A. and Leiva, V. (2009). An R implementation for generalized Birnbaum–Saunders distributions, *Comp. Stat. Data Anal.*, **53**, 1511–1528.

[13] Bhatti, C.R. (2010). The Birnbaum–Saunders autoregressive conditional duration model, *Math. Comp. Simul.*, **80**, 2062–2078.

[14] Bhattacharyya, G.K. and Fries, A. (1982). Fatigue failure models: Birnbaum–Saunders versus inverse Gaussian, *IEEE Trans. Rel.*, **31**, 439–440.

[15] Birnbaum, Z.W. and Saunders, S.C. (1969a). A new family of life distributions, *J. Appl. Prob.*, **6**, 319–327.

[16] Birnbaum, Z.W. and Saunders, S.C. (1969b). Estimation for a family of life distributions with applications to fatigue, *J. Appl. Prob.*, **6**, 328–347.

[17] Chang, D.S. and Tang, L.C. (1994). Graphical analysis for Birnbaum–Saunders distribution, *Microelect. Rel.*, **34**, 17–22.

[18] Chaussé, P. (2010). Computing generalized method of moments and generalized empirical likelihood with R, *J. Stat. Soft.*, **34**, Issue 11, May 2010.

[19] Cox, D.R. and Hinkley, D.V.(1974). *Theoretical Statistics*, Chapman & Hall, UK.

[20] Cysneiros, A.H.M.A.; Cribari-Neto, F. and Araujo, C.G.J. (2008). On Birnbaum–Saunders inference, *Comp. Stat. Data Anal.*, **52**, 4939–4950.

[21] Dupuis, D.J. and Mills, J.E. (1998). Robust estimation of the Birnbaum–Saunders distribution, *IEEE Trans. Rel.*, **47**, 88–95.

[22] Engelhardt, M.; Bain, L.J. and Wright, F.T. (1981). Inferences on the parameters of the Birnbaum–Saunders fatigue life distribution based on maximum likelihood estimation, *Technometrics*, **23**, 251–256.

[23] Ferreira, M.; Gomes, M.I. and Leiva, V. (2012). On an extreme value version of the Birnbaum–Saunders distribution, *Revstat Stat. J.*, **10**, 181–210.

[24] Fierro, R.; Leiva, V.; Ruggeri, F. and Sanhueza, A. (2013). On a Birnbaum–Saunders distribution arising from a non-homogeneous Poisson process, *Stat. Prob. Let.*, **83**, 1233–1239.

[25] From, S.G. and Li, L.X. (2006). Estimation of the parameters of the Birnbaum–Saunders distribution, *Comm. Stat. Theor. Meth.*, **35**, 2157–2169.

[26] Griffiths, W.; Hill, R. and Judge, G. (1993). *Learning and Practicing Econometrics*, Wiley, NY.

[27] Hansen, L.P. (1982). Large sample properties of generalized method of moments estimators, *Econometrica*, **50**, 1029–1054.

[28] Johnson, N.L.; Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions*, Wiley, NY.

[29] Kotz, S.; Leiva, V. and Sanhueza, A. (2010). Two new mixture models related to the inverse Gaussian distribution, *Meth. Comp. App. Prob.*, **12**, 199–212.

[30] Leiva, V.; Athayde, E.; Azevedo, C. and Marchant, C. (2011). Modeling wind energy flux by a Birnbaum–Saunders distribution with unknown shift parameter, *J. Appl. Stat.*, **38**, 2819–2838.

[31] Leiva, V.; Barros, M.; Paula, G.A. and Galea, M. (2007). Influence diagnostics in log-Birnbaum–Saunders regression models with censored data, *Comp. Stat. Data Anal.*, **51**, 5694–5707.

[32] Leiva, V.; Barros, M.; Paula, G.A. and Sanhueza, A. (2008a). Generalized Birnbaum–Saunders distributions applied to air pollutant concentration, *Environmetrics*, **19**, 235–249.

[33] Leiva, V.; Marchant, C.; Saulo, H.; Aslam, M. and Rojas, F. (2014a). Capability indices for Birnbaum–Saunders processes applied to electronic and food industries, *J. Appl. Stat.*, **41**, 1881–1902.

[34] Leiva, V.; Ponce, M.G.; Marchant, C. and Bustos, O. (2012). Fatigue statistical distributions useful for modeling diameter and mortality of trees, *Col. J. Stat.*, **35**, 349–367.

[35] Leiva, V.; Rojas, E.; Galea, M. and Sanhueza, A. (2014b). Diagnostics in Birnbaum–Saunders accelerated life models with an application to fatigue data, *Appl. Stoch. Mod. Bus. Ind.*, **30**, 115–131.

[36] Leiva, V.; Sanhueza, A.; Kotz, S. and Araneda, N. (2010a). A unified mixture model based on the inverse gaussian distribution, *Pak. J. Stat.*, **26**, 445–460.

[37] Leiva, V.; Sanhueza, A.; Sen, P.K. and Paula, G.A. (2008c). Random number generators for the generalized Birnbaum–Saunders distribution, *J. Stat. Comp. Simul.*, **78**, 1105–1118.

[38] Leiva, V.; Sanhueza, A.; Silva, A. and Galea, M. (2008c). A new three-parameter extension of the inverse Gaussian distribution, *Stat. Prob. Let.*, **78**, 1266–1273.

[39] Leiva, V.; Soto, G.; Cabrera, E. and Cabrera, G. (2011). New control charts based on the Birnbaum–Saunders distribution and their implementation, *Col. J. Stat.*, **34**, 147–176.

[40] Leiva, V.; Santos-Neto, M.; Cysneiros, F.J.A. and Barros, M. (2014c). Birnbaum–Saunders statistical modelling: a new approach, *Stat. Mod.*, **14**, 21–48.

[41] Leiva, V.; Saulo, H.; Leao, J. and Marchant, C. (2014d). A family of autoregressive conditional duration models applied to financial data, *Comp. Stat. Data Anal.*, **79**, 175–191.

[42] Lemonte, A.; Cribari-Neto, F. and Vasconcellos, K.L.P. (2007). Improved statistical inference for the two-parameter Birnbaum–Saunders distribution, *Comp. Stat. Data Anal.*, **51**, 4656–4681.

[43] Lio, Y.L.; Tsai, T.-R. and Wu, S.-J. (2010). Acceptance sampling plans from truncated life tests based on the Birnbaum–Saunders distribution for percentiles, *Comm. Stat. Simul. Comp.*, **39**, 119–136.

[44] Marchant, C.; Bertin, K.; Leiva, V. and Saulo, H. (2013a). Generalized Birnbaum–Saunders kernel density estimators and an analysis of financial data, *Comp. Stat. Data Anal.*, **63**, 1–15.

[45] Marchant, C.; Leiva, V.; Cavieres, M.F. and Sanhueza, A. (2013b). Air contaminant statistical distributions with application to PM10 in Santiago, Chile, *Rev. Environ. Contam. Tox.*, **223**, 1–31.

[46] Mátyás, L. (1999). *Generalized Method of Moments Estimation*, Cambridge University Press, NY.

[47] Newey, W.K. and West, K.D. (1987). A simple, positive semi-definite, heteroskedasticity and autocorrelation consistent covariance matrix, *Econometrica*, **55**, 703–708.

[48] Ng, H.K.; Kundu, D. and Balakrishnan, N. (2003). Modified moment estimation for the two-parameter Birnbaum–Saunders distribution, *Comp. Stat. Data Anal.*, **43**, 283–298.

[49] Paula, G.A.; Leiva, V.; Barros, M. and Liu, S. (2012). Robust statistical modeling using the Birnbaum–Saunders-t distribution applied to insurance, *Appl. Stoch. Mod. Bus. Ind.*, **28**, 16–34.

[50] Rao, C.R. (1965). *Linear Statistical Inference and its Applications*, Wiley, NY.

[51] Sanhueza, A.; Leiva, V. and Lopez-Kleine, L. (2011). On the Student-t mixture inverse Gaussian model with an application to protein production, *Col. J. Stat.*, **34**, 177–195.

[52] Santana, L.; Vilca, F. and Leiva, V. (2011). Influence analysis in skew-Birnbaum–Saunders regression models and applications, *J. Appl. Stat.*, **38**, 1633–1649.

[53] Santos-Neto, M.; Cysneiros, F.J.A.; Leiva, V. and Ahmed, S.E. (2012). On new parameterizations of the Birnbaum–Saunders distribution, *Pak. J. Stat.*, **28**, 1–26.

[54] Saulo, H.; Leiva, V.; Ziegelmann, F.A. and Marchant, C. (2013). A nonparametric method for estimating asymmetric densities based on skewed Birnbaum–Saunders distributions applied to environmental data, *Stoch. Environ. Res. Risk. Assess.*, **27**, 1479–1491.

[55] Stasinopoulos, D.M. and Rigby, R.A. (2007). Generalized additive models for location scale and shape (GAMLSS), *J. Stat. Soft.*, **23**, Issue 7, December 2007.

[56] Vilca, F.; Sanhueza, A.; Leiva, V. and Christakos, G. (2010). An extended Birnbaum–Saunders model and its application in the study of environmental quality in Santiago, Chile, *Stoch. Environ. Res. Risk. Assess.*, **24**, 771–782.

[57] Villegas, C.; Paula, G.A. and Leiva, V. (2011). Birnbaum–Saunders mixed models for censored reliability data analysis, *IEEE Trans. Rel.*, **60**, 748–758.

[58] Volodin, I.N. and Dzhungurova, O.A. (2000). On limit distribution emerging in the generalized Birnbaum–Saunders model, *J. Math. Sc.*, **99**, 1348–1366.