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FILTERS FOR SHORT NONSTATIONARY SEQUENCES: THE ANALYSIS OF THE BUSINESS CYCLE

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Abstract:

- This paper gives an account of some techniques of linear filtering which can be used for extracting the business cycle from economic data sequences of limited duration. It is argued that there can be no definitive definition of the business cycle. Both the definition of the business cycle and the methods that are used to extract it must be adapted to the purposes of the analysis; and different definitions may be appropriate to different eras.

Key-Words:

- *linear filters; spectral analysis; business cycles.*

AMS Subject Classification:

- 62M15, 93E11, 91B84.

1. INTRODUCTION

In recent years, there has been a renewed interest amongst economists in the business cycle. However, compared with the economic fluctuations of the nineteenth century, the business cycle in modern western economies has been a tenuous affair. For many years, minor fluctuations have been carried on the backs of strongly rising trends in national income. Their amplitudes have been so small in relative terms that they have rarely resulted in absolute reductions in the levels of aggregate income. Usually, they have succeeded only in slowing its upward progress.

Faced with this tenuous phenomenon, modern analysts have also had difficulties in reaching a consensus on how to define the business cycle and in agreeing on which methods should be used to extract it from macroeconomic data sequences. Thus, the difficulties have been both methodological and technical. This paper will deal with both of these aspects, albeit that the emphasis will be on technical matters.

It seems that many of the methodological difficulties are rooted in the tendency of economists to objectify the business cycle. If there is no doubt concerning the objective reality of a phenomenon, then it seems that it must be capable of a precise and an unequivocal definition.

However, the opinion that is offered in this paper is that it is fruitless to seek a definitive definition of the business cycle. The definition needs to be adapted to the purposes of the analysis in question; and it is arguable that it should also be influenced by the behaviour of the economy in the era that is studied.

It is also argued that a clear understanding of the business cycle can be achieved only in the light of its spectral analysis. However, the spectral approach entails considerable technical difficulties. The classical theory of statistical Fourier analysis deals with stationary stochastic sequences of unlimited duration. This accords well with the nature of the trigonometrical functions on which spectral analysis is based. In business cycle analysis, one is faced, by contrast, with macroeconomic sequences that are of strictly limited durations and that are liable to be strongly trended.

In order to apply the methods of spectral analysis to the macroeconomic data, two problems must be addressed. First, the data must be reduced to stationarity by an appropriate method of detrending. There are various ways of proceeding; and a judicious choice must be made. Then, there is the short duration of the data, which poses the problem acutely of how one should treat the ends of the sample.

One way of dealing with the end-of-sample problem is to create a circular sequence from the detrended data. By travelling around the circle indefinitely, the infinite periodic extension of the data sequence is generated, which is the essential object of an analysis that employs the discrete Fourier transform.

Such an analysis is liable to be undermined whenever there are radical disjunctions in the periodic extension at the points where the end of one replication joins the beginning of the next. Therefore, a successful Fourier analysis depends upon a careful detrending of the data. It seems that it was the neglect of this fact that led one renowned analyst to declare that spectral analysis was inappropriate to economic data. (See Granger 1966.)

2. INTERACTION OF TREND AND BUSINESS CYCLE

The business cycle has no fixed duration. In a Fourier analysis, it can be represented as a composite of sinusoidal motions of various frequencies that fall within some bandwidth. We shall consider one modern convention that defines the exact extent of this bandwidth; but it seems more appropriate that it should be determined in the light of the data.

If they are not allowed to overlap, it may be crucial to know where the low frequency range of the trend is deemed to end and where the higher range of the business cycle should begin. However, in this section, we shall avoid the issue by assuming that the business cycle is of a fixed frequency and that the trend is a simple exponential function.

In that case, the trend can be described by the function $T(t) = \exp\{rt\}$, where $r > 0$ is constant rate of growth. The business cycle, which serves to modulate the trend, is described by an exponentiated cosine function $C(t) = \exp\{\gamma \cos(\omega t)\}$. The product of the two functions, which can regarded as a model of the trajectory of aggregate income, is

$$Y(t) = \beta \exp\big\{rt + \gamma \cos(\omega t)\big\} . \tag{2.1}$$

The resulting business cycles, which are depicted in Figure 1, have an asymmetric appearance. Their contractions are of lesser duration than their expansions; and they become shorter as the growth rate r increases.

Eventually, when the rate exceeds a certain value, the periods of contraction will disappear and, in place of the local minima, there will be only points of inflection. In fact, the condition for the existence of local minima is that $\omega\gamma > r$, which is to say that the product of the amplitude of the cycles and their angular velocity must exceed the growth rate of the trend.

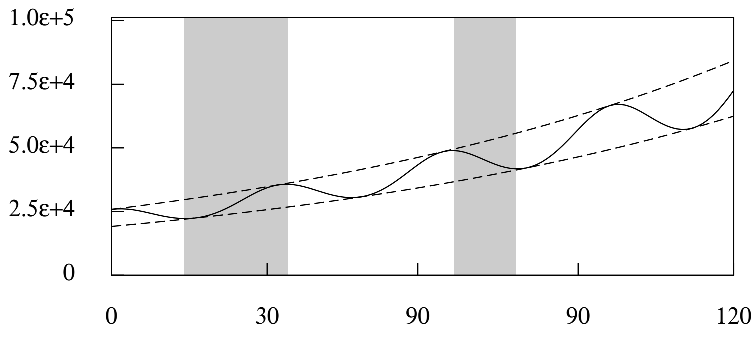

Figure 1: The function $Y(t) = \beta \exp\{rt + \gamma\cos(\omega t)\}$ as a model of the business cycle. Observe that, when $r > 0$, the duration of an expansion exceeds the duration of a contraction.

Next, we take logarithms of the data to obtain a model, represented in Figure 2, that has additive trend and cyclical components. This gives

$$\ln\{Y(t)\} = y(t) = \mu + rt + \gamma\cos(\omega t) \,, \tag{2.2}$$

where $\mu = \ln\{\beta\}$. Since logs effect a monotonic transformation, there is no displacement of the local maxima and minima. However, the amplitude of the fluctuations around the trend, which has become linear in the logs, is now constant.

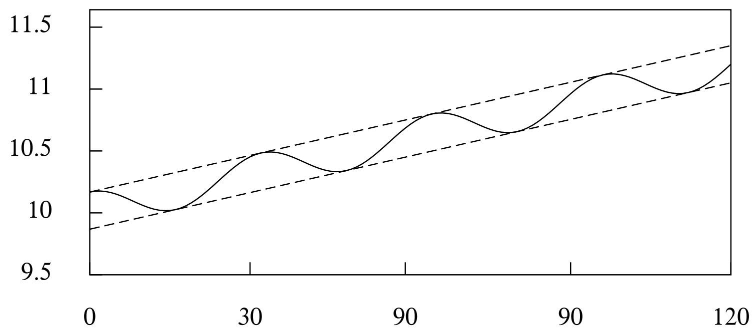

Figure 2: The function $\ln\{Y(t)\} = \ln\{\beta\} + rt + \gamma\cos(\omega t)$ representing the logarithmic business cycle data. The duration of the expansions and the contractions are not affected by the transformation.

The final step is to create a stationary function by eliminating the trend. There are two equivalent ways of doing this in the context of the schematic model. On the one hand, the linear trend $\xi(t) = \mu + rt$ can be subtracted from $y(t)$ to create the pure business cycle $\gamma\cos(\omega t)$.

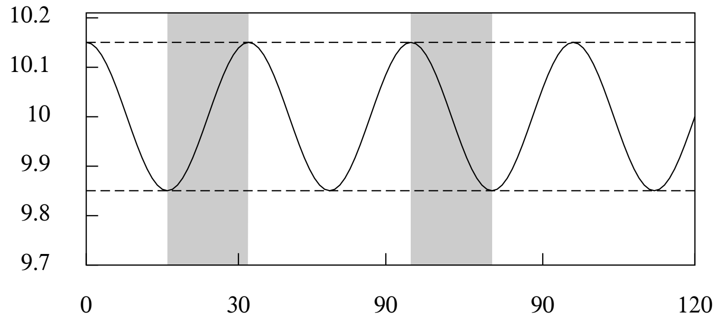

Figure 3: The function $\mu + \gamma\cos(\omega t)$ representing the detrended business cycle. The duration of the expansions and the contractions are equal.

Alternatively, the function $y(t)$ can be differentiated to give $dy(t)/dt = r - \gamma\,\omega\,\sin(\omega t)$. When the latter is adjusted by subtracting the growth rate r, by dividing by ω and by displacing its phase by $-\pi/2$ radians — which entails replacing the argument t by $t - \pi/2$ — we obtain the function $\gamma\cos(\omega t)$ again. Through the process of detrending, the phases of expansion and contraction acquire equal duration, and the asymmetry of the business cycle vanishes.

There is an enduring division of opinion, in the literature of economics, on whether we should be looking at the turning points and phase durations of the original data or at those of the detrended data. The task of finding the turning points is often a concern of analysts who wish to make international comparisons of the timing of the business cycle.

However, since the business cycle is a low-frequency component of the data, it is difficult to find the turning points with great accuracy. In fact, the pinnacles and pits that are declared to be the turning points often seem to be the products of whatever high-frequency components happen to remain in the data after they have been subjected to a process of seasonal adjustment.

If the objective is to compare the turning points of the cycles, then the trends should be eliminated from the data. The countries that might be compared are liable to be growing at differing rates. From the trended data, it will appear that those with higher rates of growth have shorter recessions with delayed onsets, and this can be misleading.

The various indices of an expanding economy will also grow at diverse rates. Unless they are reduced to a common basis by eliminating their trends, their fluctuations cannot be compared easily. Amongst such indices will be the percentage rate of unemployment, which constitutes a trend-stationary sequence. It would be difficult to collate the turning points in this index with those within a rapidly growing series of aggregate income, which might not exhibit any absolute reductions in its level. A trenchant opinion to the contrary, which opposes the practice of detrending the data for the purposes of describing the business cycle, has been offered by Harding and Pagan (2003).

3. BANDPASS DEFINITION OF THE BUSINESS CYCLE

The modern definition of the business cycle that has been alluded to in the previous section is that of a quasi cyclical motion comprising sinusoidal elements that have durations of no less than one-and-a-half years and not exceeding eight years.

This definition has been proposed by Baxter and King (1999) who have declared that it was the definition adopted by Burns and Mitchell (1947) in their study of the economic fluctuations in the U.S. in the late nineteenth century and in the early twentieth century. However, it is doubtful whether Burns and Mitchell were so firm in their definition of what constitutes the business cycle. It seems, instead, that they were merely speaking of what they had discerned in their data.

The definition in question suggests that the data should be filtered in order to extract the components that fall within the stated range, which is described as the pass band. Given a doubly infinite data sequence, this objective would be fulfilled, in theory, by an ideal bandpass filter comprising a doubly infinite sequence of coefficients.

The ideal bandpass filter that transmits all elements within the frequency range $[\alpha, \beta]$ and blocks all others has the following frequency response:

$$\psi(\omega) = \begin{cases} 1, & \text{if } |\omega| \in (\alpha, \beta)\,, \\ 0, & \text{otherwise}\,. \end{cases} \tag{3.1}$$

The coefficients of the corresponding time-domain filter are obtained by applying an inverse Fourier transform to this response to give

$$\psi_k = \int_{\alpha}^{\beta} e^{\mathrm{i}k\omega}\, d\omega = \frac{1}{\pi k}\left\{\sin(\beta k) - \sin(\alpha k)\right\}\,. \tag{3.2}$$

In practice, all data sequences are finite, and it is impossible to apply a filter that has an infinite number of coefficients. However, a practical filter may be obtained by selecting a limited number of the central coefficients of an ideal infinite-sample filter. In the case of a truncated filter based on $2q + 1$ central coefficients, the elements of the filtered sequence are given by

$$\begin{aligned} x_t &= \psi_q\, y_{t-q} + \psi_{q-1}\, y_{t-q+1} + \cdots + \psi_1\, y_{t-1} + \psi_0\, y_t + \psi_1\, y_{t+1} \\ &\quad + \cdots + \psi_{q-1}\, y_{t+q-1} + \psi_q\, y_{t+q}\,. \end{aligned} \tag{3.3}$$

Given a sample $y_0, y_1, ..., y_{T-1}$ of T data points, only $T - 2q$ processed values $x_q, x_{q+1} ..., x_{T-q-1}$ are available, since the filter cannot reach the ends of the sample, unless it is extrapolated.

If the coefficients of the truncated bandpass or highpass filter are adjusted so that they sum to zero, then the z-transform polynomial $\psi(z)$ of the coefficient sequence will contain two roots of unit value. The adjustments may be made by subtracting $\sum_k \psi_k/(2q+1)$ from each coefficient. The sum of the adjusted coefficients is $\psi(1) = 0$, from which it follows that $1 - z$ is a factor of $\psi(z)$. The condition of symmetry, which is that $\psi(z) = \psi(z^{-1})$, implies that $1 - z^{-1}$ is also a factor. Thus the polynomial contains the factor

$$(3.4) \qquad (1-z)(1-z^{-1}) = -z^{-1}(1-z)^2 \ ,$$

within which $\nabla^2(z) = (1-z)^2$ corresponds to a twofold differencing operator.

Since it incorporates the factor $\nabla^2(z)$, the effect of applying the filter to a data sequence with a linear trend will be to produce an untrended sequence with a zero mean. The effect of applying it to a sequence with a quadratic trend will be to produce an untrended sequence with a nonzero mean.

The usual effect of the truncation will be to cause a considerable spectral leakage. Thus, if the filter is applied to trended data, then it is liable to transmit some powerful low-frequency elements that will give rise to cycles of high amplitudes within the filtered output. The divergence of the frequency response function from the ideal specification of (3.1) is illustrated in Figure 4.

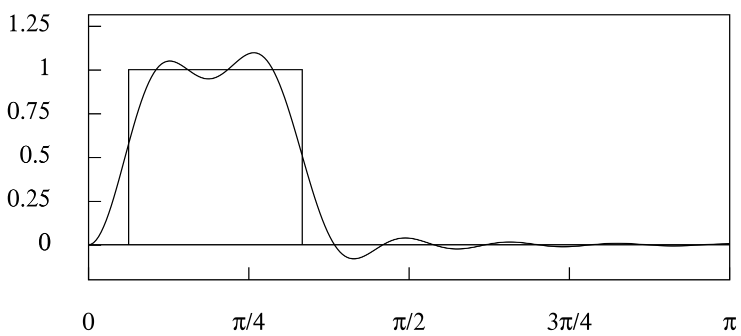

Figure 4: The frequency response of the truncated bandpass filter of 25 coefficients superimposed upon the ideal frequency response. The lower cut-off point is at $\pi/15$ radians (11.25°), corresponding to a period of 6 quarters, and the upper cut-off point is at $\pi/3$ radians (60°), corresponding to a period of the 32 quarters.

An indication of the effect of the truncated filter is provided by its application to a quarterly sequence of the logarithms of consumption in the U.K. that is illustrated in Figure 5. The filtered sequence is in Figure 6, where the loss of the data from the ends is indicated by the vertical lines.

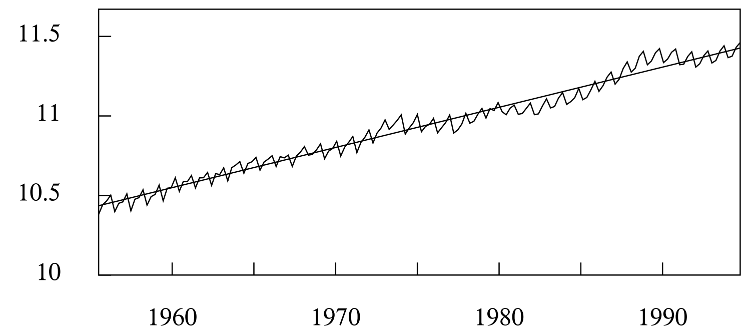

Figure 5: The quarterly sequence of the logarithms of consumption in the U.K., for the years 1955 to 1994, together with a linear trend interpolated by least-squares regression.

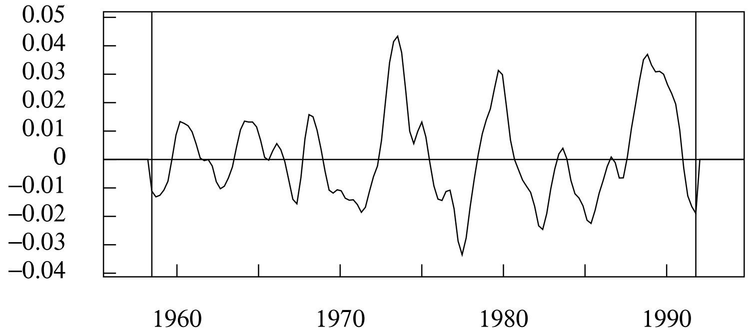

Figure 6: The sequence derived by applying the truncated bandpass filter of 25 coefficients to the quarterly logarithmic data on U.K. Consumption.

An alternative filter that is designed to reach the ends of the sample has been proposed by Christiano and Fitzgerald, (2003). The filter is described by the equation

$$
\begin{aligned}
(3.5) \qquad x_t \;=\; & Ay_0 + \psi_t\, y_0 + \cdots + \psi_1\, y_{t-1} + \psi_0\, y_t + \psi_1 y_{t+1} \\
& + \cdots + \psi_{T-1-t}\, y_{T-1} + B\, y_{T-1}\ .
\end{aligned}
$$

This equation comprises the entire data sequence $y_0, ..., y_{T-1}$; and the value of t determines which of the coefficients of the infinite-sample filter are entailed in producing the current output. Thus, the value of x_0 is generated by looking forwards to the end of the sample, whereas the value of x_{T-1} is generated by looking backwards to the beginning of the sample.

If the process generating the data is stationary and of zero mean, then it is appropriate to set $A = B = 0$, which is tantamount to approximating the extra-sample elements by zeros. In the case of a data sequence that appears to follow a first-order random walk, it has been proposed to set A and B to the values of the sums of the coefficients that lie beyond the span of the data on either side.

Since the filter coefficients must sum to zero, it follows that

$$(3.6) \qquad A = -\left(\frac{1}{2}\psi_0 + \psi_1 + \cdots + \psi_t\right) \quad \text{and} \quad B = -\left(\frac{1}{2}\psi_0 + \psi_1 + \cdots + \psi_{T-t-1}\right).$$

The effect is tantamount to extending the sample at either end by constant sequences comprising the first and the last sample values respectively.

For data that have the appearance of having been generated by a first-order random walk with a constant drift, it is appropriate to extract a linear trend before filtering the residual sequence. In fact, this has proved to be the usual practice in most circumstances.

It has been proposed to subtract from the data a linear function $f(t) = \alpha + \beta t$ interpolated through the first and the final data points, such that $\alpha = y_0$ and $\beta = (y_{T-1} - y_0)/T$. In that case, there should be $A = B = 0$. This procedure is appropriate to seasonally adjusted data. For data that manifest strong seasonal fluctuations, such as the U.K. consumption data, a line can be fitted by least squares through the data points of the first and the final years. Figure 7, shows the effect of the application of the filter to the U.K. data adjusted in this manner.

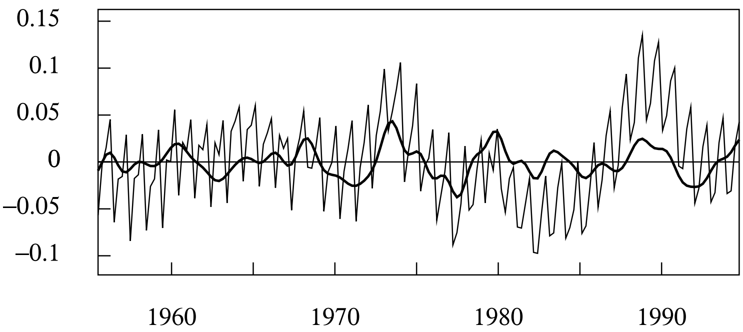

Figure 7: The sequence derived by applying the bandpass filter of Christiano and Fitzgerald to the quarterly logarithmic data on U.K. Consumption.

The filtered sequence of Figure 7 has much the same profile in its middle section as does the sequence of Figure 6, which is derived by applying truncated bandpass filter. (The difference in the scale of the two diagrams tends to conceal this similarity.) However, in comparing filtered sequence to the adjusted data, it seems fair to say that it fails adequately to represent the prominent low-frequency fluctuations. It is also beset by some noisy high-frequency fluctuations that would not normally be regarded as part of the business cycle.

4. POLYNOMIAL DETRENDING

The problems besetting the filtered sequence can be highlighted with reference to the periodogram of the residuals that are obtained by interpolating a polynomial trend line thorough the logarithmic data. Therefore, it is appropriate, at this juncture, to derive a formula for polynomial regression.

Therefore, let $L_T = [e_1, e_2, ..., e_{T-1}, 0]$ be the matrix version of the lag operator, which is formed from the identity matrix $I_T = [e_0, e_1, e_2, ..., e_{T-1}]$ of order T by deleting the leading column and by appending a column of zeros to the end of the array. The matrix that takes the p-th difference of a vector of order T is

$$\nabla_T^p = (I - L_T)^p \ . \tag{4.1}$$

We may partition this matrix so that $\nabla_T^p = [Q_*, Q]'$, where Q_*' has p rows. If y is a vector of T elements, then

$$\nabla_T^p y = \begin{bmatrix} Q_*' \\ Q' \end{bmatrix} y = \begin{bmatrix} g_* \\ g \end{bmatrix}; \tag{4.2}$$

and g_* is liable to be discarded, whereas g will be regarded as the vector of the p-th differences of the data.

The inverse matrix, which corresponds to the summation operator, is partitioned conformably to give $\nabla_T^{-p} = [S_*, S]$. It follows that

$$\begin{bmatrix} S_* & S \end{bmatrix} \begin{bmatrix} Q_*' \\ Q' \end{bmatrix} = S_* Q_*' + S Q' = I_T \ , \tag{4.3}$$

and that

$$\begin{bmatrix} Q_*' \\ Q' \end{bmatrix} \begin{bmatrix} S_* & S \end{bmatrix} = \begin{bmatrix} Q_*' S_* & Q_*' S \\ Q' S_* & Q' S \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_{T-p} \end{bmatrix}. \tag{4.4}$$

If g_* is available, then y can be recovered from g via $y = S_* g_* + S g$.

The lower-triangular Toeplitz matrix $\nabla_T^{-p} = [S_*, S]$ is completely characterised by its leading column. The elements of that column are the ordinates of a polynomial of degree $p-1$, of which the argument is the row index $t = 0, 1, ..., T-1$. Moreover, the leading p columns of the matrix ∇_T^{-p}, which constitute the submatrix S_*, provide a basis for all polynomials of degree $p-1$ that are defined on the integer points $t = 0, 1, ..., T-1$.

A polynomial of degree $p-1$, represented by its ordinates in the vector f, can be interpolated through the data by minimising the criterion

$$(y - f)'(y - f) = (y - S_* f_*)'(y - S_* f_*) \tag{4.5}$$

with respect to f_*. The resulting values are

$$f_* = (S_*'S_*)^{-1}S_*'y \quad \text{and} \quad f = S_*(S_*'S_*)^{-1}S_*'y\ . \tag{4.6}$$

An alternative representation of the estimated polynomial is available, which is provided by the identity

$$S_*(S_*'S_*)^{-1}S_*' = I - Q(Q'Q)^{-1}Q'\ . \tag{4.7}$$

It follows that the polynomial fitted to the data by least-squares regression can be written as

$$f = y - Q(Q'Q)^{-1}Q'y\ . \tag{4.8}$$

A more general method of curve fitting, which embeds polynomial regression as a special case, is one that involves the minimisation of a combination of two sums of squares. Let f denote the vector of fitted values. Then, the criterion for finding the vector is to minimise

$$L = (y-f)'(y-f) + f'Q\Lambda Q'f\ . \tag{4.9}$$

The first term penalises departures of the resulting curve from the data, whereas the second term imposes a penalty for a lack of smoothness in the curve. The second term comprises $d = Q'f$, which is the vector of p-th-order differences of f. The matrix Λ serves to generalise the overall measure of the curvature of the function that has the elements of f as its sampled ordinates, and it serves to regulate the penalty for roughness, which may vary over the sample.

Differentiating L with respect to f and setting the result to zero, in accordance with the first-order conditions for a minimum, gives

$$y - f = Q\Lambda Q'f = Q\Lambda d\ . \tag{4.10}$$

Multiplying the equation by Q' gives $Q'(y-f) = Q'y - d = Q'Q\Lambda d$, whence $\Lambda d = (\Lambda^{-1} + Q'Q)^{-1}Q'y$. Putting this into the equation $f = y - Q\Lambda d$ gives

$$f = y - Q(\Lambda^{-1} + Q'Q)^{-1}Q'y\ . \tag{4.11}$$

If $\Lambda^{-1} = 0$ in (4.11), and if Q' is the matrix version of the twofold difference operator, then the least-squares interpolator of a linear function is derived in the form equation (4.8). The sequence of regression residuals will be given by the vector $r = Q(Q'Q)^{-1}Q'y$; and it is notable that these residuals contain exactly the same information as the vector $g = Q'y$ of the twofold differences of the data. However, whereas the low-frequency structure would be barely visible in the periodogram of the differenced data, it will be fully evident in the periodogram of the residuals of a polynomial regression.

The periodogram of the residual sequence obtained from a linear detrending of the logarithmic consumption data is presented in Figure 8. Superimposed upon the figure is a highlighted band that spans the interval $[\pi/16, \pi/3]$, which corresponds to the nominal pass band of the filters applied in the previous section.

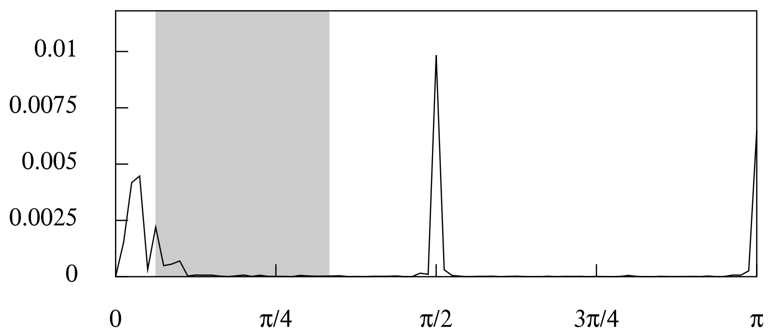

Figure 8: The periodogram of the residual sequence obtained from the linear detrending of the logarithmic consumption data. A band, with a lower bound of $\pi/16$ radians and an upper bound of $\pi/3$ radians, is masking the periodogram.

Within this periodogram, the spectral structure extending from zero frequency up to $\pi/8$ belongs to the business cycle. The prominent spikes located at the frequency $\pi/2$ and at the limiting Nyquist frequency of π are property of the seasonal fluctuations. Elsewhere in the periodogram, there are wide dead spaces, which are punctuated by the spectral traces of minor elements of noise. The highlighted pass band omits much of the information that might be used in synthesising the business cycle.

5. SYNTHESIS OF THE BUSINESS CYCLE

To many economists, it seems implausible that the trend of a macroeconomic index, which is the product of events within the social realm, should be modelled by polynomial, which may be described as a deterministic function. A contrary opinion is represented in this paper. We deny the objective reality of the trend. Instead, we consider it to be the product of our subjective perception of the data. From this point of view, a polynomial function can often serve as a firm benchmark against which to measure the fluctuations of the index. Thus, the linear trend that we have interpolated through the logarithms of the consumption data provides the benchmark of constant exponential growth.

It is from the residuals of a log-linear detrending of the consumption data that we wish to extract the business cycle. The appropriate method is to extract the Fourier components of the residual sequence that lie within the relevant frequency band. Reference to Figure 8 suggests that this band should stretch from zero up to the frequency of $\pi/8$ radians per quarter, which corresponds to a cycle with a duration of 4 years. In Figure 9, the sequence that is synthesised from these Fourier ordinates has been superimposed upon the sequence of the residuals of the linear detrending.

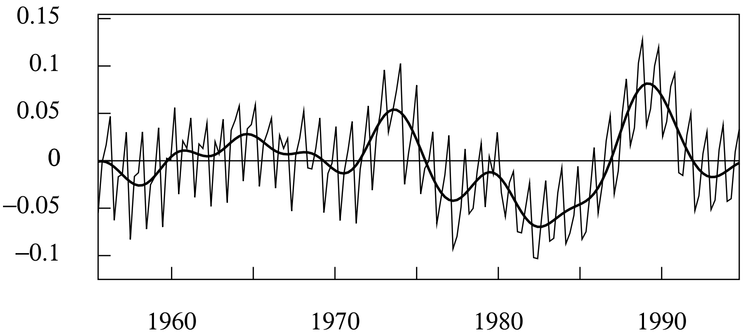

Figure 9: The residual sequence from fitting a quadratic trend to the logarithmic consumption data. The interpolated line, which represents the business cycle, has been synthesised from the Fourier ordinates in the frequency interval $[0, \pi/8]$.

To provide a symbolic representation of the method, we may denote the matrix of the discrete Fourier transform and its inverse by

$$\begin{aligned} U &= T^{-1/2}\big[\exp\{-\mathrm{i}\,2\pi tj/T\};\ t,j=0,...,T-1\big]\ , \\ \bar{U} &= T^{-1/2}\big[\exp\{\mathrm{i}\,2\pi tj/T\};\ t,j=0,...,T-1\big]\ . \end{aligned} \tag{5.1}$$

Then, the residual vector $r = Q(Q'Q)^{-1}Q'y$ and its Fourier transform ρ are represented by

$$r = T^{1/2}\bar{U}\rho \quad \longleftrightarrow \quad \rho = T^{-1/2}Ur\ . \tag{5.2}$$

Let J be a matrix of which the elements are zeros apart from a string of units on the diagonal, which serve to select from ρ the requisite Fourier ordinates within the band $[0, \pi/8]$. Then, the filtered vector that represents the business cycle is given by

$$x = T^{1/2}\bar{U}J\rho = \{\bar{U}JU\}r = \Psi r\ . \tag{5.3}$$

Here, $\bar{U}JU = \Psi = \big[\psi^{\circ}_{|i-j|};\ i,j=0,...,T-1\big]$ is a circulant matrix of the filter coefficients that would result from wrapping the infinite sequence of the ideal bandpass coefficients around a circle of circumference T and adding the overlying elements. Thus

$$\psi^{\circ}_{k} = \sum_{q=-\infty}^{\infty}\psi_{qT+k}\ . \tag{5.4}$$

Applying the wrapped filter to the finite data sequence via a circular convolution is equivalent to applying the original filter to an infinite periodic extension of the data sequence. In practice, the wrapped coefficients would be obtained from the Fourier transform of the vector of the diagonal elements of the matrix J.

The Fourier method can also be exploited to create a sequence that represents a combination of the trend and the business cycle. There are various ways of proceeding. One of them is to add the vector x to that of the linear or polynomial trend that has generated the sequence of residuals. An alternative method is to obtain the trend/cycle component by subtracting its complement from the data vector.

The complement of the trend/cycle component is a stationary component. Since a Fourier method can be applied only to a stationary vector, we are constrained to work with the vector $g = Q'y$, obtained by taking the twofold differences of the data.

Since the twofold differencing entails the loss of two points, the vector g may be supplemented by a point at the beginning and a point at the end. The resulting vector may be denoted by q. The relevant Fourier ordinates are extracted by applying the selection matrix $I-J$ to the transformed vector $\gamma = Uq$. Thereafter, they need to be reinflated to compensate for the differencing operation.

The frequency response of the twofold difference operator, which is obtained be setting $z = \exp\{-i\omega\}$ in equation (3.4), is

$$f(\omega) = 2 - 2\cos(\omega) \ , \tag{5.5}$$

and that of the anti-differencing operation is the inverse $1/f(\omega)$. The Fourier ordinates of a differenced vector will be reinflated by pre-multiplying their vector by the diagonal matrix $V = \mathrm{diag}\{v_0, v_1, ..., v_{T-1}\}$, which comprises the values $v_j = 1/f(\omega_j);\ j = 0, ..., T-1$, where $\omega_j = 2\pi j/T$.

The matrix that is to be applied to the Fourier ordinates of the differenced data is therefore $H = V(I - J)$. The resulting vector is transformed back to the time domain via the matrix $\bar{U}$ to produce the vector that is to be subtracted from the data vector y. The resulting estimate of the trend/cycle component is

$$z = y - \bar{U}HUq \ . \tag{5.6}$$

This is represented in Figure 10.

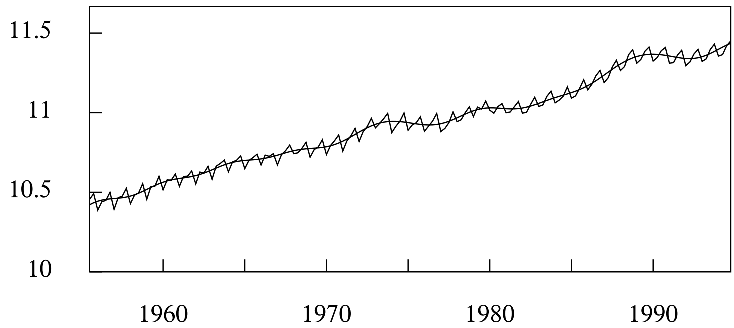

Figure 10: The trend/cycle component of U.K. Consumption determined by the Fourier method, superimposed on the logarithmic data.

6. MORE FLEXIBLE METHODS OF DETRENDING

Methods of detrending may be required that are more flexible than the polynomial interpolations that we have we considered so far. For a start, there is a need to minimise the disjunctions that occur in the periodic extension of the data sequence where the end of one replication joins the beginning of the next. This purpose can be served by a weighted version of a least-squares polynomial regression. If extra weight is given to the data points at the beginning and the end of the sample, then the interpolated line can be constrained pass through their midst; and, thereby, a major disjunction can be avoided.

The more general method of trend estimation that is represented by equation (4.11) can also be deployed. By setting $\Lambda^{-1} = \lambda^{-1}I$, a familiar filtering device is obtained that has been attributed by economists to Hodrick and Prescott (1980, 1997). In fact, an earlier exposition this filter was provided by Leser (1961), and its essential details can be found in a paper of Whittaker (1923).

The effect of the Hodrick–Prescott (H–P) filter depends upon the value of the smoothing parameter λ. As the value of the parameter increases, the vector f converges upon that of a linear trend. As the value of λ tends to zero, f converges to the data vector y. The effect of using the more flexible H–P trend in place of a linear trend is to generate estimates of the business cycle fluctuations that have lesser amplitudes and a greater regularity.

The enhanced regularity of the fluctuations is a consequence of the removal from the residual sequence of a substantial proportion of the fluctuations of lowest frequency, which can cause wide deviations from the line. This enhancement might be regarded as a spurious. However, it can be argued that such low-frequency fluctuations are liable to escape the attention of many economic agents, which is a reason for excluding them from a representation of the business cycle.

Whereas the H–P filter employs a globally constant value for the λ, it is possible to vary this parameter over the course of the sample. This will allow the trend to absorb the structural breaks or disturbances that might occasionally interrupt the steady progress of the economy. If it can be made to absorb the structural breaks, then the trend will not be thrown off course for long; and, therefore, it should serve as a benchmark against which to measure the cyclical variations when the economy resumes its normal progress. At best, the residual sequence will serve to indicate how the economy might have behaved in the absence of the break.

Figure 11 shows a trend function that has been fitted, using a variable smoothing parameter, to the logarithms of a sequence of annual data on real U.K. gross domestic product that runs from 1873 to 2001. Only the breaks after the ends of the first and second world wars have been accommodated, leaving the disruptions of the 1929 recession to be expressed in the residual sequence. The effect has been achieved by attributing a greatly reduced value to the smoothing parameter in the vicinity of the post-war breaks. In the regions that are marked by shaded bands, the smoothing parameter has been given a value of 5. Elsewhere, it has been given a high value of 100,000, which results in trend segments that are virtually linear.

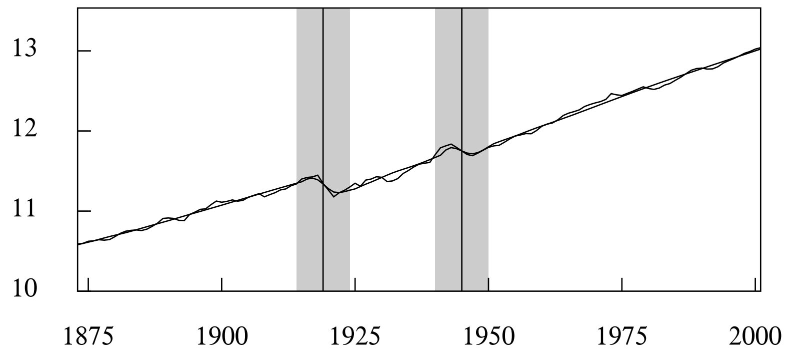

Figure 11: The logarithms of annual U.K. real GDP from 1873 to 2001 with an interpolated trend. The trend is estimated via a filter with a variable smoothing parameter.

This example serves to illustrate the contention that the trend and the accompanying business cycle are best regarded as subjective concepts. The intention of the example is to remove from the residual sequence — and, therefore, from the representation of business cycle — the effects of two major economic disruptions. For the purpose of emphasising the extent of these disruptions, the contrary approach of fitting a stiff polynomial trend line through the data should be followed.

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