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PEAKS OVER RANDOM THRESHOLD METHODOLOGY FOR TAIL INDEX AND HIGH QUANTILE ESTIMATION

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Abstract:

- In this paper we present a class of semi-parametric *high quantile* estimators which enjoy a desirable property in the presence of linear transformations of the data. Such a feature is in accordance with the empirical counterpart of the theoretical linearity of a quantile χ_p: $\chi_p(\delta X + \lambda) = \delta\chi_p(X) + \lambda$, for any real λ and positive δ. This class of estimators is based on the sample of excesses over a random threshold, originating what we denominate *PORT* (*Peaks Over Random Threshold*) methodology. We prove consistency and asymptotic normality of two high quantile estimators in this class, associated with the *PORT*-estimators for the tail index. The exact performance of the new tail index and quantile *PORT*-estimators is compared with the original semi-parametric estimators, through a simulation study.

Key-Words:

- *heavy tails; high quantiles; semi-parametric estimation; linear property; sample of excesses.*

AMS Subject Classification:

- 62G32, 62E20.

1. INTRODUCTION

In this paper we deal with semi-parametric estimators of the *tail index* γ and *high quantiles* χ_p, which enjoy desirable properties in the presence of linear transformations of the available data. We recall that a *high quantile* is a value exceeded with a small probability. Formally, we denote by F the heavy-tailed distribution function (d.f.) of a random variable (r.v.) X, the common d.f. of the i.i.d. sample $\underline{X} := \{X_i\}_{i=1}^n$, for which the high quantile

$$\chi_p(X) := F^{\leftarrow}(1-p)\,, \qquad p = p_n \to 0, \quad \text{as } n\to\infty\,, \qquad n\,p_n \to c \geq 0\;, \tag{1.1}$$

has to be estimated. Here $F^{\leftarrow}(t) := \inf\{x\colon F(x) \geq t\}$ denotes the generalized inverse function of F.

We consider estimators based on the $k+1$ top order statistics (o.s.), $X_{n:n} \geq \cdots \geq X_{n-k:n}$, where $X_{n-k:n}$ is an intermediate o.s., i.e., k is an intermediate sequence of integers such that

$$k = k_n \to \infty\,, \quad k_n/n \to 0, \qquad \text{as } n \to \infty\;. \tag{1.2}$$

We assume that we are working in a context of heavy tails, i.e., $\gamma > 0$ in the extreme value distribution

$$G_\gamma(x) = \begin{cases} \exp\bigl\{-(1+\gamma\,x)^{-1/\gamma}\bigr\}, & 1+\gamma\,x > 0, \quad \gamma \neq 0 \\ \exp\bigl(-e^{-x}\bigr), & x \in \mathbb{R}, \quad \gamma = 0\;, \end{cases} \tag{1.3}$$

the non-degenerate d.f. to which the maximum $X_{n:n}$ is attracted, after a suitable linear normalization. When this happens we say that the d.f. F is in the Fréchet domain of attraction and we write $F \in D(G_\gamma)_{\gamma>0}$.

The paper is developed under the first order regular variation condition, which allows the extension of the empirical d.f. beyond the range of the available data, assuming a polynomial decay of the tail. This condition can be expressed by

$$F \in D(G_\gamma)_{\gamma>0} \quad \text{iff} \quad \overline{F} := 1-F \in RV_{-1/\gamma} \quad \text{iff} \quad U \in RV_\gamma\;, \tag{1.4}$$

where U is the quantile function defined as $U(t) := F^{\leftarrow}(1-1/t)$, $t\geq 1$; the notation RV_α stands for the class of regularly functions at infinity with index of regular variation α, i.e., positive measurable functions h such that $\lim_{t\to\infty} h(tx)/h(t) = x^\alpha$, for all $x > 0$.

It is interesting to note that the p-quantile can be expressed as $\chi_{p_n} = U(1/p_n)$.

To get asymptotic normality of estimators of parameters of extreme events, it is usual to assume the following extra second regular variation condition, that involves a non-positive parameter ρ:

$$\lim_{t\to\infty} \frac{U(tx)/U(t) - x^\gamma}{A(t)} = x^\gamma\, \frac{x^\rho - 1}{\rho}\ , \tag{1.5}$$

for all $x > 0$, where A is a suitably chosen function of constant sign near infinity. Then, $|A| \in RV_\rho$ and ρ is called the second order parameter (Geluk and de Haan, 1987). For the strict Pareto model, with tail function $\overline{F}(x) = (x/C)^{-1/\gamma}$ and quantile function $U(t) = Ct^\gamma$, $U(tx)/U(t) - x^\gamma \equiv 0$. We then consider that (1.5) holds with $A(t) \equiv 0$.

More restrictively, we might consider that F belonged to the wide class of Hall [11], that is, the associated quantile function U satisfies

$$U(t) = Ct^\gamma\big(1 + Dt^\rho + o(t^\rho)\big)\ , \qquad \rho<0, \quad C>0, \quad D\in\mathbb{R}, \quad \text{as } t\to\infty\ , \tag{1.6}$$

or equivalently, (1.5) holds, with $A(t) = D\rho t^\rho$. The strict Pareto model appears when both D and the remainder term $o(t^\rho)$ are null.

Returning to the problem of high quantile estimation, we recall the classical semi-parametric Weissman-type estimator of χ_{p_n} (Weissman, 1978),

$$\widehat{\chi}_{p_n} = \widehat{\chi}_{p_n}(\underline{X}) = X_{n-k_n:n} \left(\frac{k_n}{n p_n}\right)^{\hat{\gamma}_n}, \tag{1.7}$$

with $\hat{\gamma}_n = \hat{\gamma}_n(\underline{X})$ some consistent estimator of the tail parameter γ.

In the classical approach one considers for $\hat{\gamma}_n$ the well known Hill estimator (Hill, 1975),

$$\hat{\gamma}_n^H = \hat{\gamma}_n^H(\underline{X}) = \frac{1}{k_n} \sum_{j=1}^{k_n} \log \frac{X_{n-j+1:n}}{X_{n-k_n:n}}\ , \tag{1.8}$$

or the Moment estimator (Dekkers et al., 1989),

$$\hat{\gamma}_n^M = \hat{\gamma}_n^M(\underline{X}) = M_n^{(1)} + 1 - \frac{1}{2} \left\{1 - \frac{\big(M_n^{(1)}\big)^2}{M_n^{(2)}}\right\}^{-1}, \tag{1.9}$$

with $M_n^{(r)}$, the r-Moment of the log-excesses, defined by

$$M_n^{(r)} = M_n^{(r)}(\underline{X}) = \frac{1}{k_n} \sum_{j=1}^{k_n} \left(\log \frac{X_{n-j+1:n}}{X_{n-k_n:n}}\right)^r, \qquad r = 1, 2\ . \tag{1.10}$$

We use the following notation:

$$\widehat{\chi}_{p_n}^H = X_{n-k_n:n} \left(\frac{k_n}{n p_n}\right)^{\hat{\gamma}_n^H}, \qquad \widehat{\chi}_{p_n}^M = X_{n-k_n:n} \left(\frac{k_n}{n p_n}\right)^{\hat{\gamma}_n^M}. \tag{1.11}$$

Finally, we explain the question that motivated this paper. It is well known that scale transformations to the data do not interfere with the stochastic behaviour of the tail index estimators (1.8) and (1.9), i.e., we can say that they enjoy scale invariance. The incorporation of (1.8) or (1.9) in the Weissman-type estimator in (1.7), allows us to obtain the following desirable exact property for quantile estimators: for any real positive δ,

$$\widehat{\chi}_{p_n}(\delta\underline{X}) \;=\; \delta X_{n-k_n:n}\left(\frac{k_n}{n p_n}\right)^{\hat{\gamma}_n} \;=\; \delta\,\widehat{\chi}_{p_n}(\underline{X})\ . \tag{1.12}$$

But we want a similar linear property in the case of location transformations to the data, $Z_j := X_j + \lambda$, $j = 1, ..., n$, for any real λ. That is, our main goal is that, for the transformed data $\underline{Z} := \{Z_j\}_{j=1}^n$, the quantile estimator satisfies

$$\widehat{\chi}_{p_n}(\underline{Z}) \;=\; \widehat{\chi}_{p_n}(\underline{X}) + \lambda\ . \tag{1.13}$$

Altogether, this represents the empirical counterpart of the following theoretical linear property for quantiles,

$$\chi_p(\delta X + \lambda) \;=\; \delta\chi_p(X) + \lambda\,, \qquad \text{for any real } \lambda \text{ and real positive } \delta\,. \tag{1.14}$$

Here we present a class of high quantile-estimators for which (1.12) and (1.13) hold exactly, pursuing the empirical counterpart of the theoretical linear property (1.14). For a simple modification of (1.7) that enjoys (1.13) approximately, see Fraga Alves and Araújo Santos (2004). For the use of reduced bias tail index estimation in high quantile estimation for heavy tails, see Gomes and Figueiredo (2003), Matthys and Beirlant (2003) and Gomes and Pestana (2005), where the second order reduced bias tail index estimator in Caeiro *et al.* (2005) is used for the estimation of the *Value at Risk*.

1.1. The class of high quantile estimators under study

The class of estimators suggested here is function of a sample of excesses over a random threshold $X_{n_q:n}$,

$$\underline{X}^{(q)} := \left(X_{n:n} - X_{n_q:n},\ X_{n-1:n} - X_{n_q:n},\ ...,\ X_{n_q+1:n} - X_{n_q:n}\right)\ , \tag{1.15}$$

where $n_q := [nq] + 1$, with:

- $0 < q < 1$, for d.f.'s with finite or infinite left endpoint $x_F := \inf\{x\colon F(x) > 0\}$ (*the random threshold is an empirical quantile*);
- $q = 0$, for d.f.'s with finite left endpoint x_F (*the random threshold is the minimum*).

A statistical inference method based on the sample of excesses $\underline{X}^{(q)}$ defined in (1.15) will be called a *PORT*-methodology, with *PORT* standing for*Peaks Over Random Threshold*. We propose the following *PORT*-*Weissman* estimators:

$$\hat{\chi}_{p_n}^{(q)} = (X_{n-k_n:n} - X_{n_q:n}) \left(\frac{k_n}{n p_n}\right)^{\hat{\gamma}_n^{(q)}} + X_{n_q:n} \ , \tag{1.16}$$

where $\hat{\gamma}_n^{(q)}$ is any consistent estimator of the tail parameter γ, made location/scale invariant by using the transformed sample $\underline{X}^{(q)}$. Indeed, the incorporation in the Adapted-Weissman estimator in (1.16), of tail index estimators, as function of the sample of excesses, allows us to obtain exactly the linear property (1.13).

1.2. Shifts in a Pareto model

To illustrate the behaviour of the new quantile estimators in (1.16), we shall first consider a parent X from a *Pareto*$(\gamma, \lambda, \delta)$,

$$F_{\gamma,\lambda,\delta}(z) = 1 - \left(\frac{z-\lambda}{\delta}\right)^{-1/\gamma}, \qquad z > \lambda + \delta, \quad \delta > 0 \ , \tag{1.17}$$

with $\lambda = 0$ and $\gamma = \delta = 1$. Let us assume that we want to estimate an upper $p = p_n = \frac{1}{n}$-quantile in a sample of size $n = 500$. Then, we want to estimate the parameter $\chi_p(X) = 500$. If we induce a shift $\lambda = 100$ to our data, we would obviously like our estimates to approach $\chi_p(X+100) = 600$.

In Figure 1 we plot, for the *Pareto*$(\lambda, 1, 1)$ parents, with $\lambda = 0$ and $\lambda = 100$ and for $q = 0$ in (1.15), the simulated mean values of the *Weissman* and *PORT*-*Weissman* quantile estimators based on the Hill, denoted $\hat{\chi}_p^H$ and $\hat{\chi}_p^{H(q)}$, respectively. These mean values are based on $N = 500$ replications, for each value k, $5 \leq k \leq 500$, from the above mentioned models.

Similarly to the Hill horror plots (Resnick, 1997), associated to slowly varying functions $L_U(t) = t^{-\gamma}\, U(t)$, we also obtain here Weissman–Hill horror plots whenever we induce a shift in the simple standard Pareto model. Indeed, for a standard Pareto model ($\lambda = 0$ in (1.17)), Weissman type estimators in (1.7) perform reasonably well, with $\hat{\gamma}_n = \hat{\gamma}_n^H$. However, a small shift in the data may lead to disastrous results, even in this simple and specific case. For the *PORT*-*Weissman* estimates, the shift in the quantile estimates is equal to the shift induced in the data, a sensible property of quantile estimates. Figure 1 also illustrates how serious can be the consequences to the sample paths of the classical high quantile estimators, when we induce a shift in the data, as suggested in Drees (2003). We may indeed be led to dangerous misleading conclusions, like a systematic underestimation, for instance, mainly due to "stable zones" far away of the target quantile to be estimated.

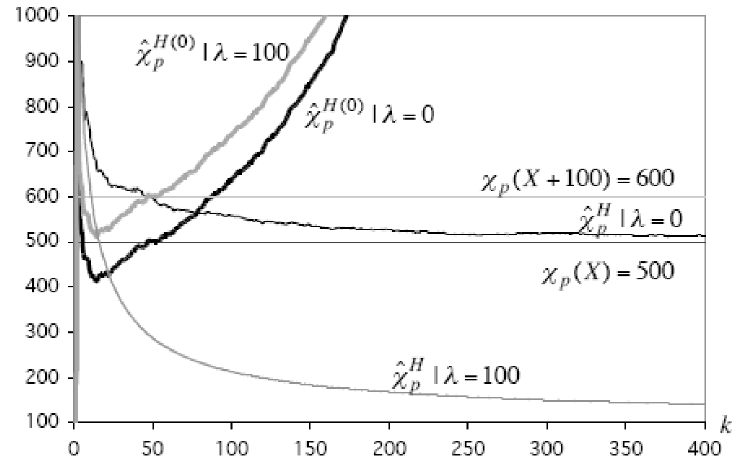

Figure 1: Mean values of $\widehat{\chi}^H_{p_n}$ and $\widehat{\chi}^{H(0)}_{p_n}$, $p_n = 0.002$ for samples of size $n=500$ from a $Pareto(1,0,1)$ parent (target quantile $\chi_{p_n} = 500$) and from the $Pareto(1,100,1)$ (target quantile $\chi_{p_n} = 600$).

1.3. Scope of the paper

As far as we know, no systematic study has been done concerning asymptotic and exact properties of semi-parametric methodologies for tail index and high quantile estimation, using the transformed sample in (1.15). Somehow related with this subject, Gomes and Oliveira (2003), in a context of regularly varying tails, suggested a simple generalization of the classical Hill estimator associated to artificially shifted data. The shift imposed to the data is deterministic, with the aim of reducing the main component of the bias of Hill's estimator, getting thus estimates with stable sample paths around the target value. A preliminary study has also been carried out, by the same authors, replacing the artificial deterministic shift by a random shift, which in practice represents a transformation of the original data through the subtraction of the smallest observation, added by one, whenever we are aware that the underlying heavy-tailed model has a finite left endpoint.

With the purpose of tail index and high quantile estimation there is, in our opinion, a gap in the literature regarding classical semi-parametric estimation methodologies adapted for shifted data, the main topic of this paper.

In Section 2, we derive asymptotic properties for the adapted Hill and Moment estimators, as functions of the sample of excesses (1.15). In Section 3, we propose two estimators for χ_p that belong to the class (1.16) and prove their asymptotic normality. In Section 4, and through simulation experiments, we compare the performance of the new estimators with the classical ones. Finally, in Section 5, we draw some concluding remarks.

2. TAIL INDEX PORT-ESTIMATORS

For the classical Hill and Moment estimators, we know that for any intermediate sequence k as in (1.2) and under the validity of the second order condition in (1.5),

$$\hat{\gamma}_n^H \stackrel{d}{=} \gamma + \frac{\gamma}{\sqrt{k}} P_k^H + \frac{A(n/k)}{1-\rho} \big(1 + o_p(1)\big) \tag{2.1}$$

and

$$\hat{\gamma}_n^M \stackrel{d}{=} \gamma + \frac{\sqrt{\gamma^2+1}}{\sqrt{k}} P_k^M + \frac{\big(\gamma(1-\rho)+\rho\big)\, A(n/k)}{\gamma(1-\rho)^2} \big(1 + o_p(1)\big) \;, \tag{2.2}$$

where P_k^H and P_k^M are asymptotically standard normal r.v.'s.

In this section we present asymptotic results for the classical Hill estimator in (1.8) and the Moment estimator in (1.9), both based on the sample of excesses $\underline{X}^{(q)}$ in (1.15), which will be denoted respectively, by

$$\hat{\gamma}_n^{H(q)} := \hat{\gamma}_n^H\big(\underline{X}^{(q)}\big) \quad \text{and} \quad \hat{\gamma}_n^{M(q)} := \hat{\gamma}_n^M\big(\underline{X}^{(q)}\big)\,, \qquad 0 \le q < 1 \;. \tag{2.3}$$

In the following, χ_q^* denotes the q-quantile of F: $F(\chi_q^*) = q$ (by convention $\chi_0^* := x_F$), so that

$$X_{n_q:n} \stackrel{p}{\longrightarrow} \chi_q^*\,, \quad \text{as} \;\; n \to \infty\,, \qquad \text{for} \;\; 0 \le q < 1 \;.$$

For the estimators in (2.3) we have the asymptotic distributional representations expressed in Theorem 2.1.

Theorem 2.1 (PORT-Hill and PORT-Moment)**.** *For any intermediate sequence k as in* (1.2), *under the validity of the second order condition in* (1.5), *for any real q, $0 \le q < 1$, and with T generally denoting either H or M, the asymptotic distributional representation*

$$\hat{\gamma}_n^{T(q)} \stackrel{d}{=} \gamma + \frac{\sigma_T}{\sqrt{k}} P_k^T + \left(c_T\, A(n/k) + d_T\, \frac{\chi_q^*}{U(n/k)}\right) \big(1 + o_p(1)\big) \tag{2.4}$$

holds, where P_k^T is an asymptotically standard normal r.v.,

$$\sigma_H^2 := \gamma^2 \;, \qquad c_H := \frac{1}{1-\rho} \;, \qquad d_H := \frac{\gamma}{\gamma+1} \;, \tag{2.5}$$

$$\sigma_M^2 := \gamma^2 + 1\,, \quad c_M := \frac{\gamma(1-\rho)+\rho}{\gamma(1-\rho)^2} \quad \text{and} \quad d_M := \left(\frac{\gamma}{\gamma+1}\right)^2 . \tag{2.6}$$

Remark 2.1. Notice that $\sigma^2_{_M} = \sigma^2_{_H}+1$, $c_{_M} = c_{_H} + \frac{\rho}{\gamma(1-\rho)^2}$ and $d_{_M} = (d_{_H})^2$. Consequently, $\sigma_{_M} > \sigma_{_H}$, $c_{_M} \le c_{_H}$ and $d_{_M} < d_{_H}$.

The proof of Theorem 2.1 relies on the the following Lemmas 2.1 and 2.2.

Lemma 2.1. *Let F be the d.f. of X, and assume that the associated U-quantile function satisfies the second order condition* (1.5). *Consider a deterministic shift transformation to X, defining the r.v. $X_q := X - \chi^*_q$ with d.f. $F_q(x) = F(x) + \chi^*_q$ and associated U_q-quantile function given by $U_q(t) := F_q^{\leftarrow}(1-1/t) = U(t) - \chi^*_q$.*

Then U_q satisfies a second order condition similar to (1.5), *that is*

$$\lim_{t\to\infty} \frac{U_q(tx)/U_q(t) \,-\, x^\gamma}{A_q(t)} \;=\; x^\gamma \left(\frac{x^{\rho_q}-1}{\rho_q}\right), \qquad \textit{for} \quad x>0, \quad \rho_{_q} \le 0 \;, \tag{2.7}$$

with

$$\big(A_q(t), \rho_{_q}\big) := \begin{cases} \big(A(t),\, \rho\big) & \textit{if} \quad \rho > -\gamma\,; \\ \left(A(t) + \dfrac{\gamma\chi^*_q}{U(t)},\, -\gamma\right) & \textit{if} \quad \rho = -\gamma\,; \\ \left(\dfrac{\gamma\chi^*_q}{U(t)},\, -\gamma\right) & \textit{if} \quad \rho < -\gamma\,. \end{cases} \tag{2.8}$$

Proof: Under (1.5), for $x > 0$,

$$\begin{aligned} \frac{U_q(tx)}{U_q(t)} &= \frac{U(tx) - \chi^*_q}{U(t) - \chi^*_q} \\ &= \frac{U(tx)}{U(t)} \left\{\frac{1 - \chi^*_q/U(tx)}{1 - \chi^*_q/U(t)}\right\} \\ &= \frac{U(tx)}{U(t)} \left\{1 + \chi^*_q\, \frac{1/U(t) - 1/U(tx)}{1 - \chi^*_q/U(t)}\right\} \\ &= \frac{U(tx)}{U(t)} \left\{1 + \frac{\chi^*_q}{U(t)} \left[1 - \frac{U(t)}{U(tx)}\right] \big(1 + o(1)\big)\right\} \\ &= x^\gamma \left\{1 + \frac{x^\rho - 1}{\rho}\, A(t)\, \big(1 + o(1)\big)\right\} \left\{1 + \frac{\gamma\chi^*_q}{U(t)}\, \frac{x^{-\gamma} - 1}{-\gamma}\, \big(1 + o(1)\big)\right\} \\ &= x^\gamma \left\{1 + \frac{x^\rho - 1}{\rho}\, A(t) + \frac{\gamma\chi^*_q}{U(t)}\, \frac{x^{-\gamma} - 1}{-\gamma} + o\big(A(t)\big) + o\big(1/U(t)\big)\right\}. \end{aligned}$$

Then U_q satisfies (2.7), for A_q and ρ_q defined in (2.8) and the result follows. □

Lemma 2.2. *Denote by* $M_n^{(r,q)}$ *the* $M_n^{(r)}$ *statistics in* (1.10), *as functions of the transformed sample* $\underline{X}^{(q)}$, $0 \le q < 1$ *in* (1.15); *that is,*

$$M_n^{(r,q)} := M_n^{(r)}\big(\underline{X}^{(q)}\big) = \frac{1}{k}\sum_{j=1}^{k}\left(\log\frac{X_{n-j+1:n}-X_{n_q:n}}{X_{n-k:n}-X_{n_q:n}}\right)^r, \qquad r=1,2 \ .$$

Then, for any intermediate sequence k *as in* (1.2), *under the validity of the second order condition in* (1.5) *and for any real* q, $0 \le q < 1$,

$$M_n^{(r,q)} - \frac{1}{k}\sum_{j=1}^{k}\left(\log\frac{X_{n-j+1:n}-\chi_q^*}{X_{n-k:n}-\chi_q^*}\right)^r = o_p\!\left(\frac{1}{U(n/k)}\right) , \qquad r=1,2 \ .$$

Proof: We will consider $r=1$. Using the first order approximation $\ln(1+x)\sim x$, as $x\to 0$, together with the fact that $X_{n_q:n} = \chi_q^*\big(1+o_p(1)\big)$, we will have successively

$$\begin{aligned}
M_n^{(1,q)} - \frac{1}{k}\sum_{j=1}^{k}\log\frac{X_{n-j+1:n}-\chi_q^*}{X_{n-k:n}-\chi_q^*} &= \\
&= \frac{1}{k}\sum_{j=1}^{k}\log\frac{X_{n-j+1:n}-X_{n_q:n}}{X_{n-k:n}-X_{n_q:n}} - \log\frac{X_{n-j+1:n}-\chi_q^*}{X_{n-k:n}-\chi_q^*} \\
&= \frac{1}{k}\sum_{j=1}^{k}\log\frac{1-X_{n_q:n}/X_{n-j+1:n}}{1-X_{n_q:n}/X_{n-k:n}} - \log\frac{1-\chi_q^*/X_{n-j+1:n}}{1-\chi_q^*/X_{n-k:n}} \\
&= \frac{1}{k}\sum_{j=1}^{k}\left(\frac{X_{n_q:n}}{X_{n-k:n}} - \frac{X_{n_q:n}}{X_{n-j+1:n}} + \frac{\chi_q^*}{X_{n-j+1:n}} - \frac{\chi_q^*}{X_{n-k:n}}\right)\big(1+o_p(1)\big) \\
&= \frac{X_{n_q:n}-\chi_q^*}{X_{n-k:n}}\,\frac{1}{k}\sum_{j=1}^{k}\left(1-\frac{X_{n-k:n}}{X_{n-j+1:n}}\right)\big(1+o_p(1)\big) \\
&= \frac{o_p(1)}{X_{n-k:n}}\,\frac{1}{k}\sum_{j=1}^{k}\left(1-\frac{X_{n-k:n}}{X_{n-j+1:n}}\right)\big(1+o_p(1)\big) .
\end{aligned}$$

Denote by $\{Y_j\}_{j=1}^k$ i.i.d. Y standard Pareto r.v.'s, with d.f. $F_Y(y) = 1-y^{-1}$, for $y>1$ and $\{Y_{j:k}\}_{j=1}^k$ the associated o.s.'s.

Since $X_{n-k:n} \overset{d}{=} U(Y_{n-k:n})$, with $Y_{n-k:n}$ the $(n-k)$-th o.s. associated to an i.i.d. standard Pareto sample of size n and $\big(\frac{k}{n}\big)Y_{n-k:n} \overset{p}{\longrightarrow} 1$, for any intermediate sequence k, then $\frac{X_{n-k:n}}{U(n/k)} \overset{p}{\longrightarrow} 1$; this together with the fact that

$$\left\{\frac{Y_{n-j+1:n}}{Y_{n-k:n}}\right\}_{j=1}^{k} \overset{d}{=} \Big\{Y_{k-j+1:k}\Big\}_{j=1}^{k}$$

allow us to write

$$
\begin{aligned}
M_n^{(1,q)} - \frac{1}{k}\sum_{j=1}^{k} \log \frac{X_{n-j+1:n}-\chi_q^*}{X_{n-k:n}-\chi_q^*} \;&= \\
&= \frac{o_p(1)}{U(Y_{n-k:n})}\,\frac{1}{k}\sum_{j=1}^{k}\left(1-\frac{U(Y_{n-k:n})}{U\Big(\frac{Y_{n-j+1:n}}{Y_{n-k:n}}\,Y_{n-k:n}\Big)}\right)\big(1+o_p(1)\big) \\
&= \frac{1}{k}\sum_{j=1}^{k}\Big(1-Y_{k-j+1:k}^{-\gamma}\Big)\,o_p\!\left(\frac{1}{U(n/k)}\right)\big(1+o_p(1)\big) \\
&= \frac{1}{k}\sum_{j=1}^{k}\Big(1-Y_j^{-\gamma}\Big)\,o_p\!\left(\frac{1}{U(n/k)}\right)\big(1+o_p(1)\big)\,.
\end{aligned}
$$

Now $E\big[Y^{-\gamma}\big]=\frac{1}{\gamma+1}$ and by the weak law of large numbers we obtain

$$
\begin{aligned}
M_n^{(1,q)} - \frac{1}{k}\sum_{j=1}^{k} \log \frac{X_{n-j+1:n}-\chi_q^*}{X_{n-k:n}-\chi_q^*} \;&= \\
&= \frac{\gamma}{\gamma+1}\Big(1+o_p\big(1/\sqrt{k}\big)\Big)\,o_p\!\left(\frac{1}{U(n/k)}\right) \\
&= o_p\!\left(\frac{1}{U(n/k)}\right)\,.
\end{aligned}
$$

For $r=2$ steps similar to the previous ones lead us to the result. □

Remark 2.2. Note that if $q\in(0,1)$, $X_{n_q:n}-\chi_q^* = O_p(1/\sqrt{n})$ and for $r=1,2$, $\sqrt{k}\left[M_n^{(r,q)}-\frac{1}{k}\sum_{j=1}^{k}\left\{\log\frac{X_{n-j+1:n}-\chi_q^*}{X_{n-k:n}-\chi_q^*}\right\}^r\right] = O_p\Big(\sqrt{k/n}\;\frac{1}{U(n/k)}\Big) = o_p(1)$ holds.

Proof of Theorem 2.1: Taking into account Lemma 2.2

$$
\hat\gamma_n^{H(q)} = \frac{1}{k}\sum_{j=1}^{k}\log\frac{X_{n-j+1:n}-\chi_q^*}{X_{n-k:n}-\chi_q^*} + o_p\!\left(\frac{1}{U(n/k)}\right)\,.
$$

Now, considering the result in Lemma 2.1 and representation (2.1) adapted for the deterministic shift data from $X_q := X-\chi_q^*$ model, we obtain the following representation for *PORT*-Hill estimator

$$
\hat\gamma_n^{H(q)} \overset{d}{=} \gamma + \frac{\gamma}{\sqrt{k}}\,P_k^H + \frac{A_q(n/k)}{1-\rho_q}\big(1+o_p(1)\big) + o_p\!\left(\frac{1}{U(n/k)}\right)\,,
$$

with $A_q(t)$ provided in (2.8), and the result (2.4) follows with $T=H$.

Similarly, considering Lemmas 2.1 and 2.2 and the representation (2.2) adapted for the deterministic shift data from $X_q := X - \chi_q^*$ model, we obtain for the *PORT*-Moment estimator the representation

$$\hat{\gamma}_n^{M(q)} \stackrel{d}{=} \gamma + \frac{\sqrt{\gamma^2+1}}{\sqrt{k}} P_k^M + \frac{\Big(\gamma(1-\rho_q)+\rho_q\Big) A_q(n/k)}{\gamma(1-\rho_q)^2} \big(1+o_p(1)\big) + o_p\left(\frac{1}{U(n/k)}\right) ,$$

and result (2.4) follows with $T = M$. □

Remark 2.3. Note that if we induce a deterministic shift λ to data X from a model $F =: F_0$, i.e., if we work with the new model $F_\lambda(x) := F_0(x-\lambda)$, the associated U-quantile function changes to $U_\lambda(t) = \lambda + \delta\, U_0(t)$. Then, as expected, (2.4) holds whenever we replace $\hat{\gamma}_n^{H(q)}$ by $\hat{\gamma}_n^H|\lambda$ (the Hill estimator associated with the shifted population with shift λ) provided that we replace χ_q^* by $-\lambda$. This topic has been handled in Gomes and Oliveira (2003), where the shift λ is regarded as a *tuning parameter* of the statistical procedure that leads to the tail index estimates. The same comments apply to the classical Moment estimator.

Corollary 2.1. *For the strict Pareto model, i.e., the model in* (1.17) *with* $\lambda = 0$ *and* $\gamma = \delta = 1$, *the distributional representations* (2.4) *holds with* $A(t)$ *replaced by 0.*

Under the conditions of Theorems 2.1 and with the notations defined in (2.5) and (2.6), the following results hold:

Corollary 2.2. *Let* μ_1 *and* μ_2 *be finite constants and let* T *generically denote either* H *or* M.

i) *For* $\gamma > -\rho$,

$$\hat{\gamma}_n^{T(q)} \stackrel{d}{=} \gamma + \frac{\sigma_T}{\sqrt{k}} P_k^T + c_T\, A(n/k) \big(1 + o_p(1)\big) .$$

If $\sqrt{k}\, A(n/k) \to \mu_1$, *then*

$$\sqrt{k}\Big(\hat{\gamma}_n^{T(q)} - \gamma\Big) \underset{n\to\infty}{\stackrel{d}{\longrightarrow}} Normal\big(\mu_1\, c_T, \sigma_T^2\big) .$$

ii) *For* $\gamma < -\rho$,

$$\hat{\gamma}_n^{T(q)} \stackrel{d}{=} \gamma + \frac{\sigma_T}{\sqrt{k}} P_k^T + d_T \frac{\chi_q^*}{U(n/k)} \big(1 + o_p(1)\big) .$$

If $\sqrt{k}/U(n/k) \to \mu_2$, *then*

$$\sqrt{k}\Big(\hat{\gamma}_n^{T(q)} - \gamma\Big) \underset{n\to\infty}{\stackrel{d}{\longrightarrow}} Normal\big(\mu_2\, d_T \chi_q^*, \sigma_T^2\big) .$$

iii) *For* $\gamma = -\rho$,

$$\hat{\gamma}_n^{T(q)} \stackrel{d}{=} \gamma + \frac{\sigma_T}{\sqrt{k}} P_k^T + \left[c_T A(n/k) + d_T \frac{\chi_q^*}{U(n/k)}\right] \big(1 + o_p(1)\big) .$$

If $\sqrt{k}\, A(n/k) \to \mu_1$ *and* $\sqrt{k}/U(n/k) \to \mu_2$, *then*

$$\sqrt{k}\left(\hat{\gamma}_n^{T(q)} - \gamma\right) \underset{n\to\infty}{\stackrel{d}{\longrightarrow}} Normal\big(\mu_1 c_T + \mu_2 d_T \chi_q^*, \sigma_T^2\big) .$$

3. HIGH QUANTILE PORT-ESTIMATORS

On the basis of (1.16), we shall now consider the following estimators of χ_{p_n}, functions of the sample of excesses over $X_{n_q:n}$, i.e., of the sample $\underline{X}^{(q)}$ in (1.15):

(3.1) $$\widehat{\chi}_{p_n}^{H(q)} := \big(X_{n-k_n:n} - X_{n_q:n}\big)\left(\frac{k_n}{np_n}\right)^{\hat{\gamma}_n^{H(q)}} + X_{n_q:n} , \qquad 0 \le q < 1 ,$$

(3.2) $$\widehat{\chi}_{p_n}^{M(q)} := \big(X_{n-k_n:n} - X_{n_q:n}\big)\left(\frac{k_n}{np_n}\right)^{\hat{\gamma}_n^{M(q)}} + X_{n_q:n} , \qquad 0 \le q < 1 .$$

For these estimators we have the asymptotic distributional representations presented in Theorem 3.1.

Theorem 3.1. *In Hall's class* (1.6), *for intermediate sequences* k_n *that satisfy*

(3.3) $$\log(np_n)/\sqrt{k_n} \to 0 , \qquad \text{as} \quad n \to \infty ,$$

with p_n *such that* (1.1) *holds, then, with* T *denoting either* H *or* M, (c_H, d_H, σ_H) *and* (c_M, d_M, σ_M) *defined in* (2.5) *and* (2.6), *respectively, and for any real* q, $0 \le q < 1$,

$$\frac{\sqrt{k_n}}{\sigma_T \log\big(k_n/(np_n)\big)} \left(\frac{\widehat{\chi}_{p_n}^{T(q)}}{\chi_{p_n}} - 1\right) = P_k^T + \sqrt{k_n}\left(c_T A(n/k) + d_T \frac{\chi_q^*}{U(n/k)}\right)\big(1 + o_p(1)\big) ,$$

where P_k^T *is an asymptotically standard normal r.v.*

Proof: From now on, we denote $a_n := \frac{k_n}{np_n}$. With the underlying conditions in (1.1), a_n tends to infinity, as $n \to \infty$, and the quantile to be estimated can be expressed as

$$\chi_{p_n} = U\left(\frac{1}{p_n}\right) = U\left(\frac{n a_n}{k_n}\right) .$$

We will present the proof for $T = H$, since for $T = M$ the proof follows the same steps.

First notice that

$$\begin{aligned}\widehat{\chi}_{p_n}^{H(q)} &= \left(X_{n-k_n:n} - X_{n_q:n}\right) a_n^{\hat{\gamma}_n^{H(q)}} + X_{n_q:n} \\ &= X_{n-k_n:n}\left[\left(1 - \frac{X_{n_q:n}}{X_{n-k_n:n}}\right) a_n^{\hat{\gamma}_n^{H(q)}} + \frac{X_{n_q:n}}{X_{n-k_n:n}}\right].\end{aligned}$$

Now, since $X_{n_q:n} \xrightarrow{p} \chi_q^*$, we have $\frac{X_{n_q:n}}{X_{n-k_n:n}} = o_p(1)$. Then

$$\widehat{\chi}_{p_n}^{H(q)} = X_{n-k_n:n}\left[a_n^{\hat{\gamma}_n^{H(q)}}\left(1 + o_p(1)\right)\right],$$

which means that the proposed estimator $\widehat{\chi}_{p_n}^{H(q)}$ is asymptotically equivalent to the Weissman type estimator (1.7), whenever we use the consistent estimator $\hat{\gamma}_n \equiv \hat{\gamma}_n^{H(q)}$.

Consider now a convenient representation for the difference,

$$\widehat{\chi}_{p_n}^{H(q)} - \chi_{p_n} = X_{n-k_n:n}\left\{a_n^{\hat{\gamma}_n^{H(q)}} - a_n^{\hat{\gamma}_n^{H(q)}}\left(\frac{X_{n_q:n}}{X_{n-k_n:n}}\right) + \frac{X_{n_q:n}}{X_{n-k_n:n}} - \frac{\chi_{p_n}}{X_{n-k_n:n}}\right\},$$

and recall that we may write

$$\frac{\chi_{p_n}}{X_{n-k_n:n}} = \frac{U\left(\frac{n}{k_n}\,a_n\right)}{U\left(\frac{n}{k_n}\right)}\,\frac{U\left(\frac{n}{k_n}\right)}{U(Y_{n-k_n:n})}.$$

According to (1.5), for $\rho<0$, $U\left(\frac{n}{k_n}\,a_n\right)/U\left(\frac{n}{k_n}\right) = a_n^{\gamma}\left(1-A(n/k_n)/\rho\right)\left(1+o_p(1)\right)$.

Considering that for the estimator $\hat{\gamma}_n^{H(q)}$, the representation (2.4) holds, we get successively, for sequences k_n that verify (3.3),

$$a_n^{\hat{\gamma}_n^{H(q)}} = a_n^{\gamma}\left(1 + \log a_n\left(\hat{\gamma}_n^{H(q)} - \gamma\right)\right)\left(1 + o_p(1)\right)$$

and

$$\begin{aligned}&\widehat{\chi}_{p_n}^{H(q)} - \chi_{p_n} = \\ &\quad = a_n^{\gamma}\,X_{n-k_n:n}\left\{1 + \log a_n\left(\hat{\gamma}_n^{H(q)} - \gamma\right)\left(1 + o_p(1)\right) - \left(1 - A(n/k_n)/\rho\right)\left(1 + o_p(1)\right)\right\} \\ &\quad = a_n^{\gamma}\,X_{n-k_n:n}\left\{\log a_n\left(\hat{\gamma}_n^{H(q)} - \gamma\right) + A(n/k_n)/\rho\right\}\left(1 + o_p(1)\right).\end{aligned}$$

Now, we consider the following representation for intermediate statistics, proved in Ferreira et al. (2003),

$$X_{n-k_n:n} \overset{d}{=} U\left(\frac{n}{k_n}\right)\left(1 + \frac{\gamma B_k}{\sqrt{k_n}} + o_p\left(\frac{1}{\sqrt{k_n}}\right) + o_p\left(A(n/k_n)\right)\right), \tag{3.4}$$

with B_k an asymptotically standard normal r.v.

Using (2.4) and (3.4), we may write

$$\widehat{\chi}_{p_n}^{H(q)} - \chi_{p_n} \;=\; U\!\left(\frac{n}{k_n}\right) a_n^{\gamma} \Big(1 + O_p\big(1/\sqrt{k_n}\,\big)\Big) \left\{W_n + A\Big(\frac{n}{k_n}\Big)/\rho\right\} \big(1 + o_p(1)\big) \;,$$

where

$$\begin{aligned} W_n \;&=\; \log a_n \big(\hat{\gamma}_n^{H(q)} - \gamma\big) \\ &=\; \log a_n \left(\frac{\sigma_H}{\sqrt{k_n}}\, P_k^H + \left(c_{_H} A(n/k) + d_{_H} \frac{\chi_q^*}{U(n/k)}\right) \big(1 + o_p(1)\big)\right) \;, \end{aligned}$$

with P_k^H independent of the random sequence B_k in (3.4).

Consequently,

$$\frac{\widehat{\chi}_{p_n}^{H(q)} - \chi_{p_n}}{a_n^{\gamma}\, U\big(\frac{n}{k_n}\big)} \;=\; \big\{W_n + A(n/k)/\rho\big\}\, \big(1 + o_p(1)\big)$$

and

$$\frac{\sqrt{k_n}}{\sigma_H \log a_n} \left(\frac{\widehat{\chi}_{p_n}^{H(q)}}{\chi_{p_n}} - 1\right) \;=\; P_k^H + \sqrt{k_n} \left(c_{_H} A(n/k) + d_{_H} \frac{\chi_q^*}{U(n/k)}\right) \big(1 + o_p(1)\big) \,. \quad \square$$

The following result is a direct consequence of Corollary 2.2 and Theorem 3.1.

Corollary 3.1. *Under the same conditions of Theorem 3.1, then, with T replaced by H or M, and $(c_{_H}, d_{_H}, \sigma_{_H})$ and $(c_{_M}, d_{_M}, \sigma_{_M})$ defined in (2.5) and (2.6), respectively, the following results hold.*

i) *For $\gamma > -\rho$,*

$$\frac{\sqrt{k_n}}{\sigma_{_T} \log\big(k_n/(n p_n)\big)} \left(\frac{\widehat{\chi}_{p_n}^{T(q)}}{\chi_{p_n}} - 1\right) \;=\; P_k^T + \sqrt{k_n}\, \big(c_{_T} A(n/k)\big) \big(1 + o_p(1)\big) \;,$$

If $\sqrt{k_n}\, A(n/k_n) \to \mu_1$, finite, as $n \to \infty$, then the mean value is $\mu_1 c_{_T}$.

ii) *For $\gamma < -\rho$,*

$$\frac{\sqrt{k_n}}{\sigma_{_T} \log\big(k_n/(n p_n)\big)} \left(\frac{\widehat{\chi}_{p_n}^{T(q)}}{\chi_{p_n}} - 1\right) \;=\; P_k^T + \sqrt{k_n} \left(d_{_T} \frac{\chi_q^*}{U(n/k_n)}\right) \big(1 + o_p(1)\big) \;,$$

If $\sqrt{k_n}/U(n/k_n) \to \mu_2$, finite, as $n \to \infty$, then the mean values is $\mu_2\, d_{_T} \chi_q^*$.

iii) *For* $\rho = -\gamma$,

$$\frac{\sqrt{k_n}}{\sigma_T \log\big(k_n/(np_n)\big)} \left(\frac{\widehat{\chi}_{p_n}^{T(q)}}{\chi_{p_n}} - 1\right) =$$

$$= P_k^T + \sqrt{k_n}\left(c_T\, A(n/k) + d_T\, \frac{\chi_q^*}{U(n/k_n)}\right) \big(1 + o_p(1)\big) ,$$

If $\sqrt{k_n}\, A(n/k_n) \to \mu_1$, *finite, and* $\sqrt{k_n}/U(n/k_n) \to \mu_2$, *finite, as* $n \to \infty$, *then the mean value is* $\mu_1 c_T + \mu_2\, d_T \chi_q^*$.

4. SIMULATIONS

Here, we compare the finite sample behavior of the proposed high quantile estimators $\widehat{\chi}_{p_n}^{H(q)}$ in (3.1) and $\widehat{\chi}_{p_n}^{M(q)}$ in (3.2) with the classical semi-parametric estimators $\widehat{\chi}_{p_n}^{H}$ and $\widehat{\chi}_{p_n}^{M}$ in (1.11). We have generated $N = 200$ independent replicates of sample size $n = 1000$ from the following models:

- Burr Model: $X \frown \mathit{Burr}(\gamma, \rho)$, $\gamma = 1$, $\rho = -2, -0.5$, with d.f.

$$F(x) = 1 - \big(1 + x^{-\rho/\gamma}\big)^{1/\rho} , \qquad x \geq 0 .$$

- Cauchy Model: $X \frown \mathit{Cauchy}$, $\gamma = 1$, $\rho = -2$, with d.f.

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \operatorname{arctang} x , \qquad x \in \mathbb{R} .$$

At a first stage, we generate samples from the standard models $F_0 := F$. At a second stage, we introduce a positive shift $\lambda = \chi_{0.01}$, i.e., a new location chosen in a comparable basis as the percentile 99% of the starting point distribution F_0. This defines a new model $F_\lambda(x) := F_0(x - \lambda)$ from the same family.

We estimate the high quantile $\chi_{0.001}$, for each model F_0 or F_λ from the referred Burr and Cauchy families, and we present patterns of Mean Values and Root of Mean Squared Errors, plotted against $k = 6, ..., 800$.

The simulations illustrate the dramatic disturbance on the behavior of the classical quantile estimators in (1.11), when a shift is introduced. We, again, enhance that the flat stable zones achieved with these estimators, in the presence of shifts, could lead us to dangerous misleading conclusions, unless we are aware of the suitable threshold k or of specific properties of the underlying model.

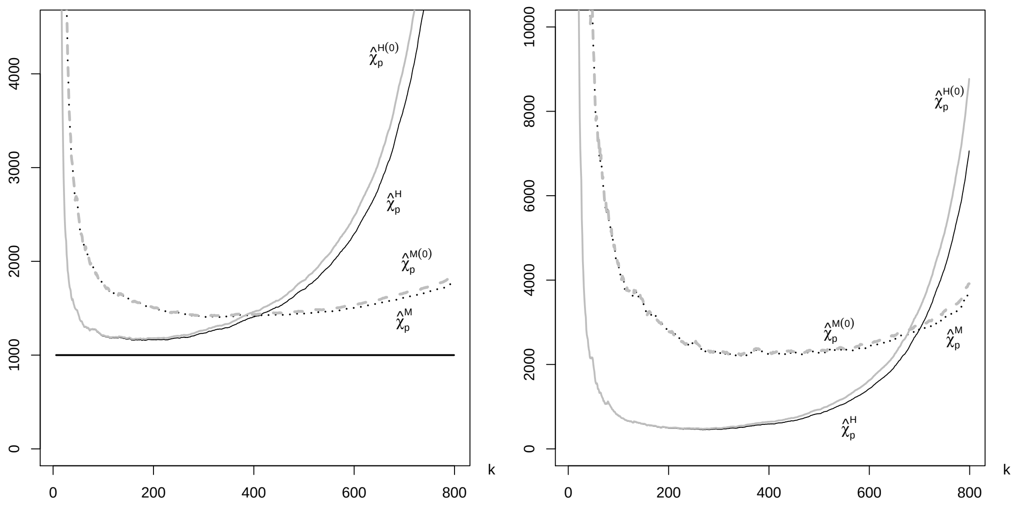

Figure 2: Mean values (left) and root mean squared errors (right), of $\widehat{\chi}^{H(0)}_{p_n}$, $\widehat{\chi}^{M(0)}_{p_n}$, $\widehat{\chi}^{H}_{p_n}$ and $\widehat{\chi}^{M}_{p_n}$, for a sample size $n = 1000$, from a Burr model with $\gamma = 1$, $\rho = -2$ and $\lambda = 0$ (target quantile $\chi_{0.001} = 1000$).

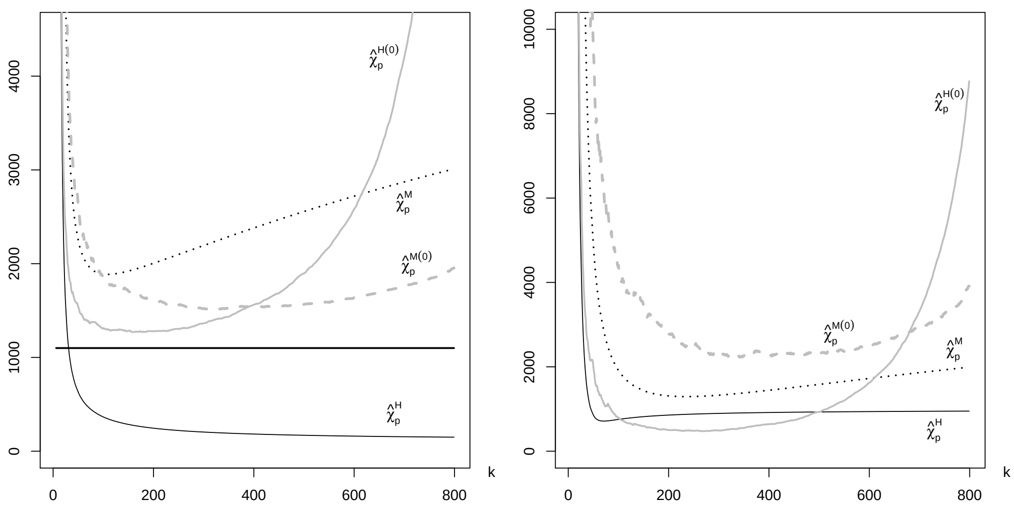

Figure 3: Mean values (left) and root mean squared errors (right), of $\widehat{\chi}^{H(0)}_{p_n}$, $\widehat{\chi}^{M(0)}_{p_n}$, $\widehat{\chi}^{H}_{p_n}$ and $\widehat{\chi}^{M}_{p_n}$, for a sample size $n = 1000$, from a Burr model with $\gamma = 1$, $\rho = -2$ and $\lambda = 99.99$ (target quantile $\chi_{0.001} = 1099.99$).

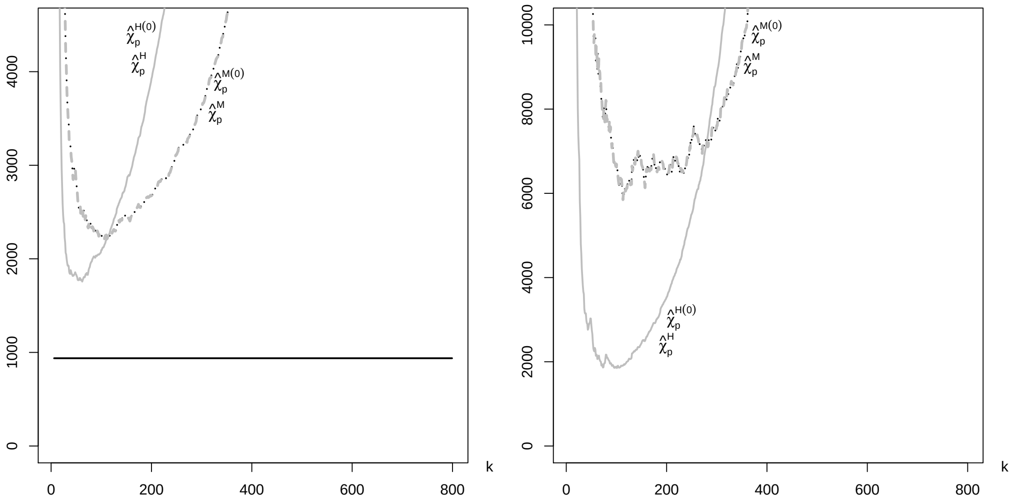

Figure 4: Mean values (left) and root mean squared errors (right), of $\widehat{\chi}_{p_n}^{H(0)}$, $\widehat{\chi}_{p_n}^{M(0)}$, $\widehat{\chi}_{p_n}^{H}$ and $\widehat{\chi}_{p_n}^{M}$, for a sample size $n=1000$, from a Burr model with $\gamma=1$, $\rho=-0.5$ and $\lambda=0$ (target quantile $\chi_{0.001}=937.731$).

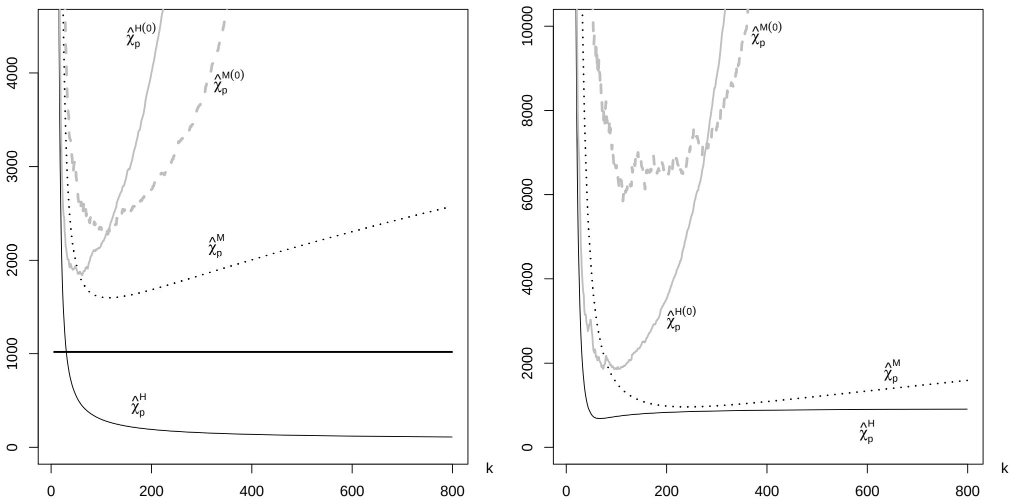

Figure 5: Mean values (left) and root mean squared errors (right), of $\widehat{\chi}_{p_n}^{H(0)}$, $\widehat{\chi}_{p_n}^{M(0)}$, $\widehat{\chi}_{p_n}^{H}$ and $\widehat{\chi}_{p_n}^{M}$, for a sample size $n=1000$, from a Burr model with $\gamma=1$, $\rho=-0.5$ and $\lambda=81.023$ (target quantile $\chi_{0.001}=1018.754$).

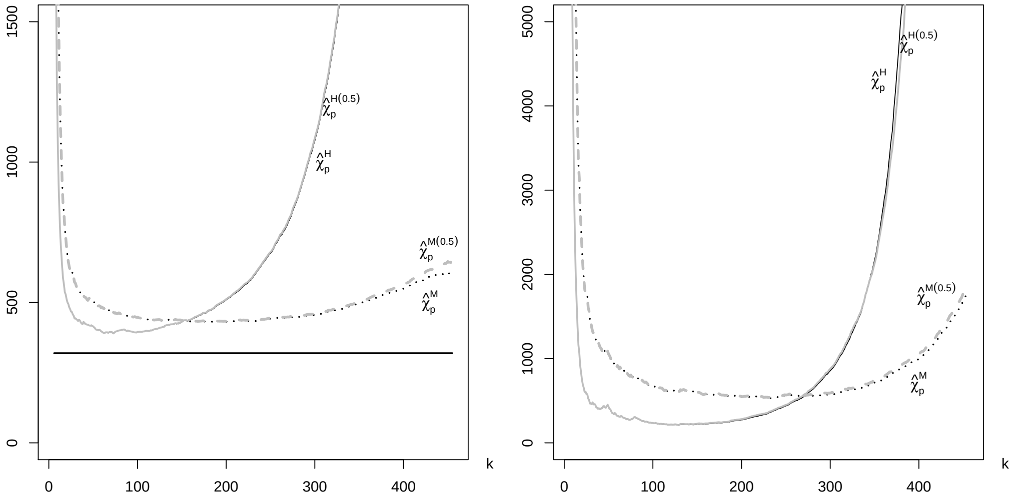

Figure 6: Mean values (left) and root mean squared errors (right), of $\widehat{\chi}_{p_n}^{H(0.5)}$, $\widehat{\chi}_{p_n}^{M(0.5)}$, $\widehat{\chi}_{p_n}^{H}$ and $\widehat{\chi}_{p_n}^{M}$, for a sample size $n=1000$, from a Cauchy model with $\gamma=1$, $\rho=-2$ and $\lambda=0$ (target quantile $\chi_{0.001}=319.309$).

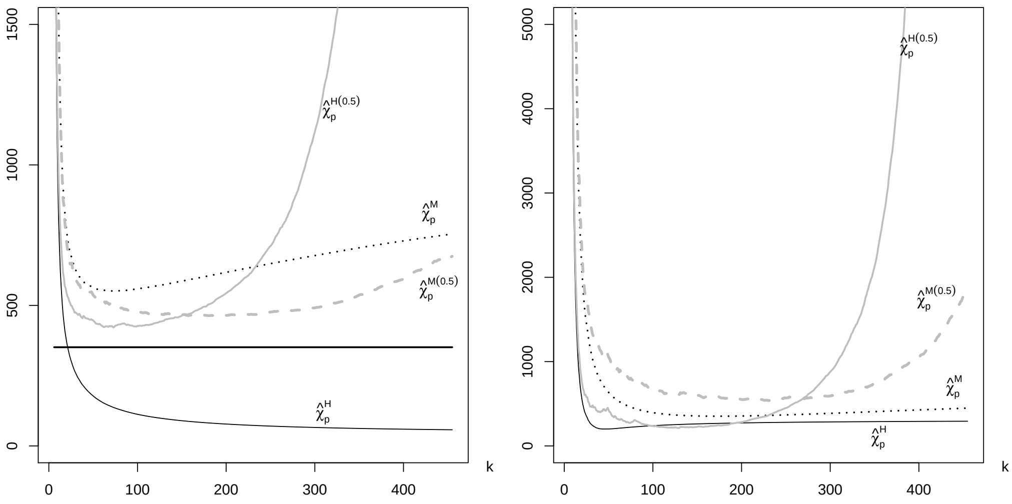

Figure 7: Mean values (left) and root mean squared errors (right), of $\widehat{\chi}_{p_n}^{H(0.5)}$, $\widehat{\chi}_{p_n}^{M(0.5)}$, $\widehat{\chi}_{p_n}^{H}$ and $\widehat{\chi}_{p_n}^{M}$, for a sample size $n=1000$, from a Cauchy model with $\gamma=1$, $\rho=-2$ and $\lambda=31.821$ (target quantile $\chi_{0.001}=351.13$).

From the figures, in this section, we observe that the classical quantile estimators diverge a lot from the important linear property (1.13). On the other hand, the estimators we propose, (3.1) and (3.2), enjoy exactly this property.

5. CONCLUDING REMARKS

- The *PORT* tail index and quantile estimators, based on the sample of excesses, $\underline{X}^{(q)}$, in (1.15), provide us with interesting classes of estimators, invariant for changes in location, as well as scale, a property also common to the classical estimators.
- In practice, whenever we use a *tuning parameter* q in $(0, 1)$, we are always safe. Indeed, in such a case, the new estimators may or may not behave better than the classical ones, but they are consistent and asymptotically normal for the same type of k-values.
- A *tuning parameter* $q = 0$ is appealing but should be used carefully. Indeed, if the underlying parent has not a finite left endpoint, we are led to non-consistent estimators, with sample paths that may be erroneously flat around a value quite far away from the real target.

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