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LIMIT DISTRIBUTION FOR THE WEIGHTED RANK CORRELATION COEFFICIENT, r_W

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Abstract:

- A weighted rank correlation coefficient, inspired by Spearman's rank correlation coefficient, has been proposed recently by Pinto da Costa & Soares [5]. Unlike Spearman's coefficient, which treats all ranks equally, r_W weights the distance between two ranks using a linear function of those ranks, giving more importance to top ranks than lower ones. In this work we prove that r_W has a gaussian limit distribution, using the methodology employed in [7].

Key-Words:

- *ranking; correlation; limit distribution.*

AMS Subject Classification:

- 62H20, 62E20, 62F07, 62F12.

1. INTRODUCTION

The objective of rank correlation methods is to assess the degree of monotonicity between two or more series of paired data. By monotonicity we mean a tendency for the values in the series to increase or decrease together (positive correlation) or for one to increase as the other decreases (negative correlation). They are applicable to paired data, that is to data where there is some connection between corresponding members of the samples. To use these methods, we must first rank the observations in each sample, $\boldsymbol{X}$ and $\boldsymbol{Y}$, from 1 (highest rank) to n (lowest rank), where n is the number of pairs of observations. We, thus obtain, $r(X_i)$ and $r(Y_i)$ where X_i and Y_i are the pair of values corresponding to observation i in each sample and $r(X_i)$ returns the rank of value i in the first series. For sake of simplicity, let us use the ranks directly rather than the values in the series. That is, $R_i = r(X_i)$ and $Q_i = r(Y_i)$.

There has been a growing interest about weighted measures of rank correlation [5, 1, 10, 6]; that is, measures that unlike Spearman's [11] coefficient which treat all ranks equally, weight ranks proportionally to how high they are, although other types of weight functions could be considered.

In 2005 Pinto da Costa & Soares [5] have introduced a weighted rank correlation coefficient, r_W, that weights the distance between two ranks using a linear function of those ranks, giving more importance to higher ranks than lower ones. These authors have also analysed the distribution of r_W in the case of independence between the two vectors of ranks. A table of critical values has been provided in order to test whether a given value of the coefficient is significantly different from zero, and a number of applications for this new measure has also been given.

In this work we start by defining this new measure of correlation in section 2. Then, in section 3 we analyse the asymptotic distribution of r_W for the general case; that is, we make no assumption of independence between the two vectors of ranks. To do so, we use the same notation and analogous arguments of those used by Ruymgaart, Shorack and Van Zwet (1972) in the proof of their Theorem 2.1 (see [7]). We prove that r_W has a normal limit distribution.

2. WEIGHTED RANK CORRELATION COEFFICIENT, r_W

In this section we describe a weighted measure of correlation that has been introduced in [5]. r_S is the value obtained by calculating Pearson's linear correlation coefficient of the paired ranks (R_1, Q_1), (R_2, Q_2), ..., (R_n, Q_n). It is easy to see that in the case of no ties,

$$r_S = 1 - \frac{6 \sum_{i=1}^{n} (R_i - Q_i)^2}{n^3 - n} = 1 - \frac{6 \sum_{i=1}^{n} D_i^2}{n^3 - n} ,$$

where $D_i^2 = (R_i - Q_i)^2$. As it is obvious from this expression, r_S only takes into account the differences between paired ranks and not the values of the ranks themselves. For instance, if $D_1 = 2$, doesn't matter whether the values for (R_1, Q_1) are $(1, 3)$ or $(n-2, n)$. Nevertheless, there are applications where top ranks are much more important than lower ones, and Spearman's rank correlation does not take this into account. For instance, when humans state their preferences, it is obvious that top preferences are more important and accurate than lower ones. Another example might be the evaluation of stock trading support systems. A potential invester would like to have a system which gives a grading of the stocks in question so that he/she can make a decision. In order to evaluate the output of the system, one can for instance calculate Spearman's correlation between the ranking predicted by the system and the true ranking of the stocks at that time. However, the top ranked alternatives are obviously more important than the lower ones, which makes weighted measures of correlation more suitable for this application also.

In [5, 8], Pinto da Costa & Soares propose a measure of correlation — adapted from Spearman's rank correlation coefficient — that weighs ranks proportionally to how high they are. Specifically, they propose the following alternative distance measure:

$$W_i^2 = (R_i - Q_i)^2 \Big((n - R_i + 1) + (n - Q_i + 1) \Big) = D_i^2 (2n + 2 - R_i - Q_i) .$$

The first factor, D_i^2, represents the distance bewteen R_i and Q_i, exactly as in Spearman's; the second factor represents the importance of R_i and Q_i.

The authors then prove that in order to have a coefficient of the form $A + B \sum_{i=1}^{n} W_i^2$ that yields values in the range $[-1, 1]$, A must be 1 and $B = \frac{-6}{n^4+n^3-n^2-n}$. Their weighted measure of correlation is therefore,

$$r_W = 1 - \frac{6 \sum_{i=1}^{n} (R_i - Q_i)^2 \Big((n - R_i + 1) + (n - Q_i + 1) \Big)}{n^4 + n^3 - n^2 - n} .$$

In [5] it is proved that under the hypothesis of independence between the two vectors of ranks, the expected value of r_W is 0, which is a desirable property for a correlation coefficient. Under the same hypothesis, $\mathrm{var}(r_W) = \frac{31n^2+60n+26}{30(n^3+n^2-n-1)}$. In addition, the authors have also conducted an experimental evaluation of the differences between the values obtained by r_W and r_S in various situations, showing that large differences can occur.

3. THE ASYMPTOTIC DISTRIBUTION OF r_W

Let (X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n) represent n i.i.d. random vectors from a continuous distribution. In this section, we show that r_W is asymptotically normal distributed. We start by showing the results of some simulations that indicate that this new statistic convergs to the gaussian curve in a particular case; namely, that the two vectors of ranks are independent. Then, we study formally the asymptotic distribution of r_W for the general case.

We have calculated the exact distribution of r_W for n up to 14. Due to computational limitations, for larger values of n we estimated the distribution based on a random sample of one million permutations. In Figure 1 we plot the distribution for $n = 14$ and $n = 15$, respectively the last exact and the first estimated distributions. In the same figure we also plot the estimated distributions for $n = 20$ and 40, respectively. In all graphs, the values of r_W have been standardized and we plot the Normal curve for comparison. From these graphs it seems clear that at least in this special case, the statistic r_W converges to the gaussian as n increases.

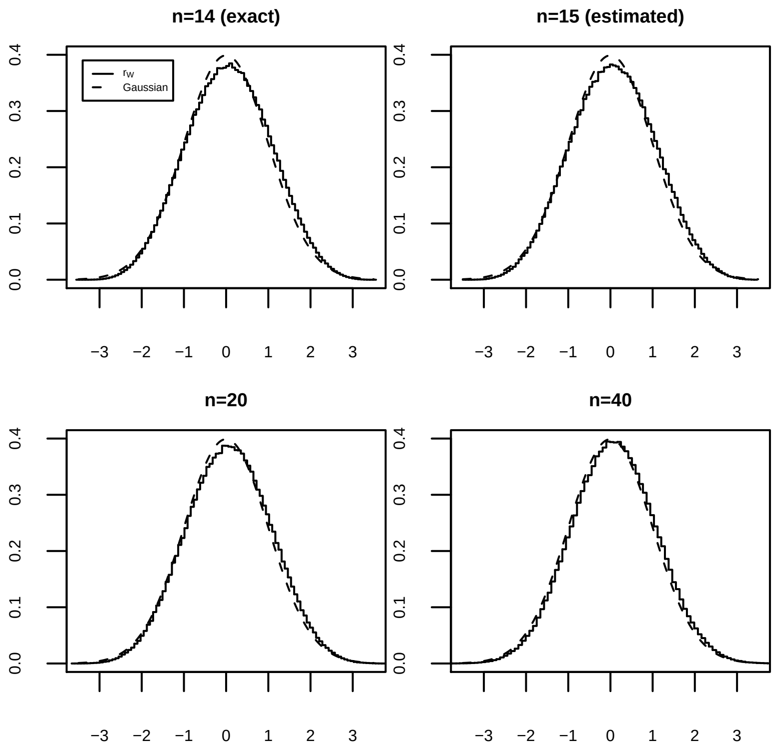

Figure 1: Exact distribution for $n = 14$ and estimated distribution for $n = 15$, 20 and 40, together with the Standard Normal curve.

Now we make no independence assumptions; that is, we study the asymptotic distribution of r_W for the general case. First,

$$\begin{aligned} r_W &= 1 - \frac{6 \sum_{i=1}^{n} (R_i - Q_i)^2 \, (2n + 2 - R_i - Q_i)}{n^4 + n^3 - n^2 - n} \\ &= 1 - \frac{6}{n} \sum_{i=1}^{n} \left(\frac{R_i}{n+1} - \frac{Q_i}{n+1} \right)^2 \left(\frac{2n + 2 - R_i - Q_i}{n-1} \right) . \end{aligned}$$

Therefore, the asymptotic behaviour of r_W is the same as the one of $1 - 6W_n$, where

$$W_n = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{R_i}{n+1} - \frac{Q_i}{n+1} \right)^2 \left(2 - \frac{R_i}{n+1} - \frac{Q_i}{n+1} \right) .$$

W_n is a statistic of the type $\frac{1}{n} \sum_{i=1}^{n} a_n(R_i, Q_i)$, where $a_n(i,j)$ is a real number for $i, j = 1, 2, ..., n$.

If we define $J(s,t) = (s-t)^2 \, (2-s-t)$, $0 \le s, t \le 1$, then $J(s,t)$ is a limit of the score function

$$J_n(s,t) = a_n(i,j) = J\left(\frac{i}{n+1}, \frac{j}{n+1} \right) , \tag{3.1}$$

for i and j such that $\frac{i-1}{n} < s \le \frac{i}{n}$ and $\frac{j-1}{n} < t \le \frac{j}{n}$. Hence, W_n can be written as (see [2]),

$$W_n = \iint J_n(F_n, G_n) \, dH_n , \tag{3.2}$$

where F_n and G_n are the empirical marginal distribution functions of F and G, respectively; H_n is the bivariate empirical distribution function of H. Now, let us define the population moment $\mu = \iint J(F,G) \, dH$. By analogy to r_W, we define the population weighted rank correlation coefficient to be

$$\begin{aligned} \rho_W(X,Y) &= 1 - 6\mu \\ &= 1 - 6 \iint \Big(F(x) - G(y) \Big)^2 \Big(2 - F(x) - G(y) \Big) \, dH(x,y) , \end{aligned}$$

or, by using copulas [4]

$$\rho_W(X,Y) = 1 - 6 \int_{[0,1]^2} (u-v)^2 \, (2-u-v) \, dc(u,v) ,$$

where the copula $c(u,v) = P\big(F(X) \le u, \, G(y) \le v\big)$, $0 \le u, v \le 1$.

Next we present the conclusion that r_W is assymptotically gaussian distributed.

Theorem 3.1. *r_W is an asymptotic normal and consistent (ANC) estimator of ρ_W.*

Proof: We want to prove that r_W is an asymptotic normal and consistent (ANC) estimator of ρ_W; first,

$$\sqrt{n}\,(r_W - \rho_W) \;=\; -6\sqrt{n}\,(W_n - \mu) \;=\; -6\sqrt{n}\left[\iint J_n(F_n, G_n)\; dH_n \;-\; \mu\right].$$

We start by considering the empirical processes $U_n(F) = \sqrt{n}\,(F_n - F)$, $V_n(G) = \sqrt{n}\,(G_n - G)$, $U_n^*(F) = \sqrt{n}\,(F_n^* - F)$, $V_n^*(G) = \sqrt{n}\,(G_n^* - G)$, where $F_n^* = \left[\frac{n}{n+1}\,F_n\right]$ and $G_n^* = \left[\frac{n}{n+1}\,G_n\right]$. Let now $\bar{\Delta}_n = [X_{1n}, X_{nn}] \times [Y_{1n}, Y_{nn}]$ where X_{in} and Y_{in} denote the i^{th} order statistics and $B_{0n}^* = \sqrt{n}\iint\left[J_n(F_n, G_n) - J(F_n^*, G_n^*)\right] dH_n$.

We will now prove that $J_n(F_n, G_n) = J(F_n^*, G_n^*)$ and so $B_{0n}^* = 0$ for all n. In fact the function F_n, for instance, is a step function and so there is always an $i \in \{0, 1, ..., n\}$ such that $F_n = \frac{i}{n}$; similarly for G_n. This means that by (3.1) $J_n(F_n, G_n) = J\left(\frac{i}{n+1}, \frac{j}{n+1}\right)$ for some i and j. Now, by the definition above, $\frac{i}{n+1} = F_n^*$ and $\frac{j}{n+1} = G_n^*$. So, $B_{0n}^* = 0$ for all n.

Because $B_{0n}^* = 0$ for all n, then an assumption similar to 2.3 b) in [7] (see Appendix A) is satisfied, that is, $B_{0n}^* \to_p 0$. We will now use the same argument of these authors, adapting it to our situation because our score function $a_n(i,j)$ is bivariate and the score functions used in [7], $a_n(i)$ and $b_n(i)$ have just one variable (see Appendix A). Nevertheless, the adaption follows from the same steps of their proof. The asymptotic convergence of r_W to the Normal distribution may be uniform over a class of distribution functions. However in this work we are not interested in proving uniform convergence, but only convergence for a single distribution.

Now we can write,

$$\sqrt{n}\,(W_n - \mu) \;=\; \sum_{i=1}^{3} A_{in} + B_{0n}^* + B_{1n}^* \;,$$

where

$$\begin{aligned}
A_{1n} \;&=\; \sqrt{n}\iint J(F,G)\; d(H_n - H)\;,\\
A_{2n} \;&=\; \iint U_n(F)\,\frac{\partial J}{\partial s}(F,G)\; dH\;,\\
A_{3n} \;&=\; \iint V_n(G)\,\frac{\partial J}{\partial t}(F,G)\; dH\;,\\
B_{0n}^* &\;\text{ is defined above}\;,\\
B_{1n}^* \;&=\; \sqrt{n}\iint\Big[J(F_n^*, G_n^*) - J(F,G)\Big]\, dH_n \;-\; A_{2n} \;-\; A_{3n}\;.
\end{aligned}$$

3.1. $\sum_{i=1}^{3} A_{in}$ is asymptotically normal distributed

As in [7] we can prove the asymptotic normality of A_{1n}, A_{2n} and A_{3n} based on the fact that J is a continuous function and its partial derivatives are continuous and bounded on $(0,1)^2$.

Let us start by noting that $A_{1n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_{1in}$ where $A_{1in} = J\big(F(X_i), G(Y_i)\big) - \mu$. In fact,

$$\begin{aligned} A_{1n} &= \sqrt{n} \iint J(F,G)\; d(H_n - H) \\ &= \sqrt{n} \left(\iint J(F,G)\; dH_n - \iint J(F,G)\; dH \right) . \end{aligned}$$

Now, as in equation 3.2 we get,

$$\begin{aligned} A_{1n} &= \frac{\sqrt{n}}{n} \sum_{i=1}^{n} \Big(J\big(F(X_i), G(Y_i)\big) - \mu \Big) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big(J\big(F(X_i), G(Y_i)\big) - \mu \Big) . \end{aligned}$$

The random variables A_{1in} are i.i.d. with mean zero. If we choose $\delta = \frac{1}{4}$, $D = p_0 = q_0 = 2$, $r(u) = \frac{1}{u(1-u)}$ then we have an assumption similar to assumption 2.1 in the statement of Theorem 2.1 in [7] (See Appendix A), that is,

$$\begin{aligned} J(F,G) &\le D\big(r(F)\big)^a \big(r(G)\big)^b && \text{with } a = \tfrac{\delta - \frac{1}{2}}{po} = -\tfrac{1}{8} \text{ and } b = \tfrac{\delta - \frac{1}{2}}{qo} = -\tfrac{1}{8} , \\ \frac{\partial J}{\partial s}(F,G) &\le D\big(r(F)\big)^{a+1} \big(r(G)\big)^b && \text{with } a = \tfrac{\delta - \frac{1}{2}}{p1} = -\tfrac{1}{8} \text{ and } b = \tfrac{\delta - \frac{1}{2}}{q1} = -\tfrac{1}{8} , \\ \frac{\partial J}{\partial t}(F,G) &\le D\big(r(F)\big)^{b} \big(r(G)\big)^{a+1} && \text{with } a = \tfrac{\delta - \frac{1}{2}}{p2} = -\tfrac{1}{8} \text{ and } b = \tfrac{\delta - \frac{1}{2}}{q2} = -\tfrac{1}{8} . \end{aligned}$$

Taking this assumption into account and by application of Holder's inequality,

$$\iint \big|\phi(F)\, \psi(G)\big|\, dH \le \left[\int |\phi|^{p_0}\, dI \right]^{\frac{1}{p_0}} \left[\int |\psi|^{q_0}\, dI \right]^{\frac{1}{q_0}}, \quad \forall\, p_0 > 0,\ q_o > 0 \colon \frac{1}{p_0} + \frac{1}{q_0} = 1 ,$$

where ϕ and ψ are functions on $(0,1)$, dI denotes Lebesgue measure restricted to the unit interval, we note that A_{1in} has a finite absolute moment of order $2 + \delta_0$ for some $\delta_0 > 0$ (see appendix B).

Let us consider now A_{2n}. As $U_n(F) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \big(I(X_i \le x) - F \big)$ we can write $A_{2n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} A_{2in}$, where $A_{2in} = \iint \big(I(X_i \le x) - F \big) \frac{\partial J}{\partial s}(F,G)\; dH$ are i.i.d. with mean zero. If we choose $\delta = \frac{1}{4}$, $D = p_1 = q_1 = 2$, $r(u) = \frac{1}{u(1-u)}$ then an assumption similar to 2.1 in [7] is satisfied. Again, by applying Holder's inequality and similarly to A_{1in}, it follows that A_{2in} has a finite absolute moment of order $2+\delta_1$ for some $\delta_1 > 0$.

Let us consider now A_{3n}. As $V_n(G) = \frac{1}{\sqrt{n}}\sum_{i=1}^n \big(I(Y_i \le y) - G\big)$ we can write $A_{3n} = \frac{1}{\sqrt{n}}\sum_{i=1}^n A_{3in}$ where $A_{3in} = \iint \big(I(Y_i \le y) - G\big)\,\frac{\partial J}{\partial t}(F,G)\,dH$ are i.i.d. with mean zero. If we choose $\delta = \frac{1}{4}$, $D = p_2 = q_2 = 2$, $r(u) = \frac{1}{u(1-u)}$ then an assumption similar to assumption 2.1 in [7], is satisfied. By application of Holder's inequality and similarly to A_{1in}, it follows that A_{3in} has a finite absolute moment of order $2+\delta_2$ for some $\delta_2 > 0$.

From the above conclusions: $A_{1n} = \frac{1}{\sqrt{n}}\sum_{i=1}^n A_{1in}$ where A_{1in} are i.i.d. with mean zero; $A_{2n} = \frac{1}{\sqrt{n}}\sum_{i=1}^n A_{2in}$ where A_{2in} are i.i.d. with mean zero; $A_{3n} = \frac{1}{\sqrt{n}}\sum_{i=1}^n A_{3in}$ where A_{3in} are i.i.d. with mean zero and because A_{1in}, A_{2in}, A_{3in} have a finite absolute moment of order larger than 2, we get $\sum_{i=1}^3 A_{in} \to_d N(0,\sigma^2)$ as $n \to \infty$. The expression for the variance corresponds to equation 3.10 in [7] and is given by

$$\begin{aligned}\sigma^2 \;=\; \operatorname{Var}\Big[J\big(F(X),G(Y)\big) \;&+ \iint \big(I(X \le x) - F\big)\,\frac{\partial J}{\partial s}\big(F(x),G(y)\big)\,dH(x,y)\\ &+ \iint \big(I(Y \le y) - G\big)\,\frac{\partial J}{\partial t}\big(F(x),G(y)\big)\,dH(x,y)\Big]\,.\end{aligned}$$

3.2. B^*_{1n} is asymptotically negligible

We have already seen that an assumption similar to 2.3 b) in [7] is satisfied. If we consider the mean value theorem (see [9]),

$$\sqrt{n}\,J(F^*_n,G^*_n) \;=\; \sqrt{n}\,J(F,G) + U^*_n(F)\,\frac{\partial J}{\partial s}(\phi^*_n,\psi^*_n) + V^*_n(G)\,\frac{\partial J}{\partial t}(\phi^*_n,\psi^*_n)$$

for all (x,y) in $\bar{\Delta}_n$ with $\phi^*_n = F + \alpha_3(F^*_n - F)$ and $\psi^*_n = G + \alpha_4(G^*_n - G)$, where α_3 and α_4 are numbers between 0 and 1, then B^*_{1n} can be decomposed as a sum of seven terms which are all asymptotically negligible by the same arguments used in section 5 of Ruymgaart et al. (1972) [7].

3.3. r_W is asymptotically normal distributed

We have thus that $\sqrt{n}(W_n - \mu) \to N(0,\sigma^2)$ in distribution and it is immediate that r_W is an asymptotic normal and consistent (ANC) estimator of $\rho_W :\ \sqrt{n}(r_W - \rho_W) \to N(0, 36\,\sigma^2)$. □

APPENDIX

A. Asymptotic Normality of Nonparametric Statistics

We present in this appendix Theorem 2.1 of Ruymgaart, Shorack and Van Zwet, 1972 (see [7]) as it is the fundamental tool used in the proof of our Theorem 3.1. We start by introducing some notation. Let $(X_1, Y_1), ..., (X_n, Y_n)$ be a random sample from a continuous bivariate distribution function $H(x, y)$ (bivariate empirical df is denoted by H_n) having marginal dfs $F(x)$ and $G(y)$ and empirical df F_n and G_n, respectively. The rank of X_i is denoted by R_i and the rank of Y_i by Q_i. Let $T_n = \frac{1}{n}\sum_{i=1}^{n} a_n(R_i)\, b_n(Q_i)$, where $a_n(i)$, $b_n(i)$ are real numbers for $i = 1, ..., n$. The standardization of T_n can be written as

$$\sqrt{n}(T_n - \mu) \;=\; \sqrt{n}\left[\iint J_n(F_n)\, K_n(G_n)\; dH_n \;-\; \mu\right] ,$$

where $J_n(s) = a_n(i)$, $K_n(s) = b_n(i)$, for $i = 1, ..., n$ such that $\frac{(i-1)}{n} < s \leq \frac{i}{n}$; $\mu = \iint J(F)\, K(G)\, dH$. The functions J and K can be thought of as limits of the score functions J_n and K_n. $\mathcal{H}$ denote the class of all continuous bivariate dfs H.

Assumption 2.1 (Ruymgaart, Shorack and Van Zwet, 1972)**.** The functions J and K are continuous on $(0, 1)$; each is differentiable except at most at a finite number of points, and in the open intervals between these points the derivatives are continuous. The function J_n, K_n, J, K satisfy $|J_n| \leq Dr^a$, $|K_n| \leq Dr^a$ and $|J^{(i)}| \leq Dr^{a+i}$ and $|K^{(i)}| \leq Dr^{b+i}$ for $i = 0, 1$. Here D is a positive constant, $a = \frac{\left(\frac{1}{2}-\delta\right)}{p}$, $b = \frac{\left(\frac{1}{2}-\delta\right)}{q}$ for some $0 < \delta < \frac{1}{2}$ and some $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Assumption 2.3 b (Ruymgaart, Shorack and Van Zwet, 1972)**.**

$$B^*_{0n} \;=\; \sqrt{n}\iint \Big[J_n(F_n)\, K_n(G_n) - J(F^*_n)\, K(G^*_n)\Big]\, dH_n \underbrace{\;\rightarrow\;}_{p} 0 \qquad \text{as} \quad n \to \infty$$

where $F^*_n = \left[\frac{n}{n+1}\right] F_n$ and $G^*_n = \left[\frac{n}{n+1}\right] G_n$.

Theorem 2.1 of Ruymgaart, Shorack and Van Zwet, 1972 (see [7])**.** *If H is in $\mathcal{H}$ and if assumptions 2.1 and 2.3 b) are satisfied, then*

$$\sqrt{n}(T_n - \mu) \underbrace{\;\rightarrow\;}_{d} N(0, \sigma^2) \qquad \text{as} \quad n \to \infty ,$$

where μ *and* σ^2 *are finite and are given by*

$$\mu = \iint J(F)\,K(G)\;dH \qquad \textit{(expression 1.3 in [7])}$$

and

$$\sigma^2 = \operatorname{Var}\Bigg[J\big(F(X)\big)\,K\big(G(Y)\big) + \iint (\phi_X - F)\,J'(F)\,K(G)\;dH + \iint (\phi_Y - G)\,J(F)\,K'(G)\;dH\Bigg] \qquad \textit{(expression 3.10 in [7])}$$

with $\phi_{X_i}(x) = 0$ *if* $x < X_i$ *and* $\phi_{X_i}(x) = 1$ *if* $x \geq X_i$.

B. A_{1in} has a finite absolute moment of order greater than 2

We show here that there exist $\delta_0 > 0$ and $\delta_0 < \delta = \frac{1}{4}$ such that $E\,|A_{1in}|^{2+\delta_0}$ is bounded. Using Assumption 2.1 in the appendix above we can prove that

$$\iint \Big|J\big(F(X_i), G(Y_i)\big)\Big|^{2+\delta_0}\;dH \;\leq\; D \iint \big|r(F)\big|^{a(2+\delta_0)}\,\big|r(G)\big|^{b(2+\delta_0)}\;dH\;.$$

By using now Holder's Inequality this quantity is

$$\begin{aligned}
&\leq\; D\,\frac{1}{n}\sum_{i=1}^{n}\left\{r^{(2+\delta_0)(\delta-\frac{1}{2})}\left(\frac{i}{n+1}\right)\right\}^{\frac{1}{p_0}}\left\{\frac{1}{n}\sum_{i=1}^{n} r^{(2+\delta_0)(\delta-\frac{1}{2})}\left(\frac{i}{n+1}\right)\right\}^{\frac{1}{q_0}}\\
&=\; \frac{D}{n}\sum r^{(2+\delta_0)(\delta-\frac{1}{2})}\left(\frac{i}{n+1}\right)\\
&\leq\; D\int_0^1 \frac{1}{\big(u(1-u)\big)^{(2+\delta_0)(\frac{1}{2}-\delta)}}\;du
\end{aligned}$$

that is finite for $0 < \delta_0 < \delta = \frac{1}{4}$.

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