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# MEAN NUMBER OF THE CUSTOMERS SERVED DURING A BUSY PERIOD IN THE M|G|∞ QUEUEING SYSTEM

# NÚMERO MÉDIO DE CLIENTES SERVIDOS DURANTE UM PERÍODO DE OCUPAÇÃO NO SISTEMA DE FILA DE ESPERA M|G|∞

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***ABSTRACT:***

- We deduce an exact expression that gives the mean number of the customers served during a busy period in the M|M|∞ queueing system and conjecture an upper bound of this quantity for other M|G|∞ systems, with service times not exponential, based on the results of a Markov Renewal Process.

***KEY-WORDS:***

- *M|G|∞, Mean, Busy Period, Markov Renewal Process.*

***RESUMO:***

- Deduzimos uma expressão exacta para o número médio de clientes servidos durante um período de ocupação no sistema de fila de espera M|M|∞ e conjecturamos um extremo superior dessa quantidade para outros sistemas M|G|∞, com tempos de serviço não exponenciais, com base nos resultados de um Processo Markoviano de Renovamento

***PALAVRAS-CHAVE:***

- *M|G|∞ , Média, Período de Ocupação, Processo Markoviano de Renovamento.*

In a M|G|∞ queueing system $\lambda$ is the Poisson process arrival rate, $\alpha$ is the mean service time, $G(\cdot)$ is the service time distribution function and there are infinite servers.

In general we do not want necessarily the physical presence of infinite servers. But we only guarantee that when a customer arrives at the system it always finds immediately a server available.

Looking at the labour of the system as a sequence of idle and busy periods, it is important to know the number of the customers served during a busy period in order to manage the servers that are indicated to work in the system. We will give here some results about the mean number of the customers served during a busy period in the M|G|$\infty$ queue system. We will call it $N_B$.

Be $\nu_n$, $n = 0,1,\ldots$ the mean number of entries in state $n$ between two entries in state 0. We say that the system is in state $n$, $n = 0,1,\ldots$ when there are $n$ customers being served in the queue.

The number of customers served in a busy period is equal to number of arrivals at the system in it. But, counting all entries in the various states during a busy period we are counting the arrivals and the exits. The number of arrivals equals the number of exits. And between two entries in state 0 there is only one busy period. So

$$N_B = \frac{\sum_{n=0}^{\infty} \nu_n}{2} \qquad \textbf{(1)}$$

For the M|M|$\infty$ system (exponential service time), see (Ramalhoto, 1986),

$$\nu_n = (n+\rho)\frac{\rho^{n-1}}{n!},\ n = 0,1,... \qquad \textbf{(2)}$$

where $\rho = \lambda\alpha$ is the traffic intensity.

So,

$$N_B = \frac{1}{2}\sum_{n=0}^{\infty}(n+\rho)\frac{\rho^{n-1}}{n!} = \frac{1}{2}\left(1 + \sum_{n=1}^{\infty}\frac{\rho^{n-1}}{(n-1)!} + \sum_{n=1}^{\infty}\frac{\rho^n}{n!}\right) = \frac{1}{2}\left(1 + e^{\rho} + e^{\rho} - 1\right) = e^{\rho}$$

and

$$N_B^M(\rho) = e^{\rho} \qquad \textbf{(3)}$$

Note that $N_B^M(0) = 1$, naturally, because when $\lambda \to 0$ or $\alpha \to 0$ it is difficult to have more than one customer in the system.

The mean busy period length, for any M|G|∞ system, is $\frac{e^{\rho}-1}{\lambda}$, (see, for instance, (Takács, 1962)). Let us put $\frac{e^{\rho}-1}{\lambda e^{\rho}} = A(\rho,\lambda)$. So

- $\lim\limits_{\lambda\to 0}\frac{e^{\rho}-1}{\lambda e^{\rho}} = \lim\limits_{\lambda\to 0}\frac{e^{\rho}\alpha}{e^{\rho}+e^{\rho}\rho} = \lim\limits_{\lambda\to 0}\frac{\alpha}{1+\rho} = \alpha$, using l'Hospital's rule,

- $\lim\limits_{\lambda\to \infty}\frac{e^{\rho}-1}{\lambda e^{\rho}} = \lim\limits_{\lambda\to \infty}\frac{\alpha}{1+\rho} = 0$,

- $A_{\lambda}^{'}(\rho,\lambda) = \frac{e^{\rho}\alpha\lambda e^{\rho} - \left(e^{\rho}+e^{\rho}\alpha\lambda\right)\left(e^{\rho}-1\right)}{\lambda^2 e^{2\rho}} = \frac{e^{2\rho}\rho - e^{2\rho} - e^{2\rho}\rho + e^{\rho} + e^{\rho}\rho}{\lambda^2 e^{2\rho}} =$
  $= \frac{-e^{\rho}+1+\rho}{\lambda^2 e^{\rho}} < 0, \quad \rho > 0,$

- $\lim\limits_{\alpha\to 0}\frac{e^{\rho}-1}{\lambda e^{\rho}} = 0$,

- $\lim\limits_{\alpha\to \infty}\frac{e^{\rho}-1}{\lambda e^{\rho}} = \frac{1}{\lambda}\lim\limits_{\alpha\to \infty}\frac{e^{\rho}-1}{e^{\rho}} = \frac{1}{\lambda}$,

- $A_{\alpha}^{'}(\rho,\lambda) = \frac{e^{\rho}\lambda\lambda e^{\rho} - \left(e^{\rho}-1\right)e^{\rho}\lambda^2}{\lambda^2 e^{2\rho}} = \frac{e^{2\rho}\lambda^2 - e^{2\rho}\lambda^2 + e^{\rho}\lambda^2}{\lambda^2 e^{2\rho}} = \frac{1}{e^{\rho}} > 0,$ .
  $\alpha > 0$

Then

- with $\alpha$ constant $A(\rho,\lambda)$ decreases with $\lambda$ varying between $\alpha$ and $0$,
- with $\lambda$ constant $A(\rho,\lambda)$ increases with $\alpha$ varying between 0 and $\frac{1}{\lambda}$.

For other M|G|$\infty$ systems, with service time distributions not exponentials, we do not know $v_n$. But in (Ramalhoto, 1986) it is suggested to consider a Markov Renewall Process as an approximation to those systems. For that Markov Renewall Process we have

$$v_n = \lambda^{n-1} \frac{B_1 \ldots B_{n-1}}{(1-\lambda B_1)\ldots(1-\lambda B_n)},\ n = 1,2,\ldots \qquad \textbf{(4)}$$

Where

$$B_n = \int_0^\infty e^{-\lambda t} \left[ \frac{\int_t^\infty [1-G(x)]dx}{\alpha} \right]^n dt,\ n = 0,1,\ldots \qquad \textbf{(5)}$$

And of course,

$$v_0 = 1 \qquad \textbf{(6)}$$

But

- $B_n \le \frac{1}{\lambda},\ n = 1,2,\ldots$ because $\alpha^{-1}\int_t^\infty [1-G(x)]dx \le 1,$

– $B_n \leq \alpha \dfrac{\gamma_s^2 + 1}{n+1}, \ n = 1,2,\ldots$ because $B_n \leq \int_0^\infty \left[ \dfrac{\int_t^\infty [1 - G(x)]dx}{\alpha} \right]^n dt \leq$

$\leq \dfrac{1}{\alpha^n} \dfrac{n\alpha^2}{2} \left(\gamma_s^2 + 1\right) \dfrac{\alpha^{n-1} 2}{n(n+1)} = \alpha \dfrac{\gamma_s^2 + 1}{n+1}$, where $\gamma_s$ is the service time variation coefficient, owing to $\int_0^\infty \left[ \int_t^\infty [1 - G(x)]dx \right]^n dt = \dfrac{n\alpha^2}{2} \left(\gamma_s^2 + 1\right) \dfrac{\alpha^{n-1} b_{n-1}}{n(n+1)}, \quad n \in \mathbb{N}$, with $b_n \leq 2, \ n = 0,1,\ldots$ (see (Sathe, 1985)),

– $\alpha \dfrac{\gamma_s^2 + 1}{n+1} \leq \dfrac{1}{\lambda} \Leftrightarrow n \geq \rho\left(\gamma_s^2 + 1\right) - 1$.

So, if $\rho \leq \dfrac{1}{\gamma_s^2 + 1}$,

$$v_n \leq (n+1) \frac{\rho^{n-1} \left(\gamma_s^2 + 1\right)^{n-1}}{n!}, \ n = 1,2,\ldots \qquad \textbf{(7)}$$

for the Markov Renewal Process.

Then $\displaystyle\sum_{n=0}^\infty v_n = 1 + \sum_{n=1}^\infty \frac{\left(\rho\left(\gamma_s^2 + 1\right)\right)^{n-1}}{(n-1)!} + \sum_{n=1}^\infty \frac{\left(\rho\left(\gamma_s^2 + 1\right)\right)^{n-1}}{n!} = 1 + e^{\rho\left(\gamma_s^2 + 1\right)} + \frac{e^{\rho\left(\gamma_s^2 + 1\right)} - 1}{\rho\left(\gamma_s^2 + 1\right)} =$

$$= \frac{e^{\rho\left(\gamma_s^2+1\right)}\left(\rho\left(\gamma_s^2+1\right)+1\right) + \rho\left(\gamma_s^2+1\right) - 1}{\rho\left(\gamma_s^2+1\right)} . \ \lim_{\rho \to 0} \frac{e^{\rho\left(\gamma_s^2+1\right)}\left(\rho\left(\gamma_s^2+1\right)+1\right) + \rho\left(\gamma_s^2+1\right) - 1}{2\rho\left(\gamma_s^2+1\right)} =$$

$$\lim_{\rho \to 0} \frac{e^{\rho\left(\gamma_s^2+1\right)}\left(\gamma_s^2+1\right)\left(\rho\left(\gamma_s^2+1\right)+1\right) + e^{\rho\left(\gamma_s^2+1\right)}\left(\gamma_s^2+2\right) + \gamma_s^2}{2\left(\gamma_s^2+1\right)} = \frac{\gamma_s^2 + 1 + \gamma_s^2 + 2 + \gamma_s^2}{2\left(\gamma_s^2+1\right)} =$$

$= \dfrac{3\gamma_s^2 + 3}{2\left(\gamma_s^2 + 1\right)} = \dfrac{3}{2} > 1\left(e^0\right)$, using l'Hospital's rule.

So, may be,

$$M_B = \frac{e^{\rho\left(\gamma_s^2+1\right)}\left(\rho\left(\gamma_s^2+1\right)+1\right)+\rho\left(\gamma_s^2+1\right)-1}{2\rho\left(\gamma_s^2+1\right)} \qquad \textbf{(8)}$$

is an upper bound for $N_B$, for M|G|$\infty$ systems whose service times are not exponential. Note that for exponential service times **(4)** is the same that **(2)**.

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